RAPIDLY STRETCHING PLASTIC JETS: THE LINEARIZED PROBLEM

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Introduction. When studying the stability of rapidly stretchting plastic jets, one is faced with a free boundary problem for a nonlinear system of differential equations. In this paper we study the linearized version of this problem which constitutes a nonstandard system of linear differential equations. Our purpose is to prove existence and uniqueness, and to establish some properties of the solution.

A typical example of a rapidly stretching jet is furnished by the jet produced by a shaped-charge [1][2][9][13]. A shaped-charge consists of an explosive with a conical cavity lined with a thin metal sheet (see Fig.1). The explosion will cause the metal to collapse toward the axis where an extremely high velocity jet will instantly be formed. The velocity of the particles in these jets increases linearly with the distance from the rear end, so that the jet experiences significant stretching.

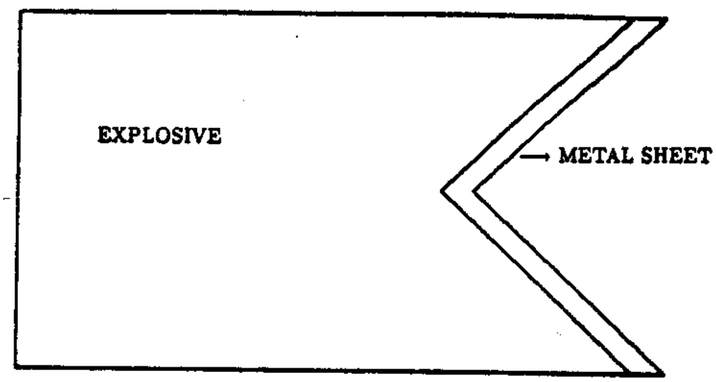

Fig.1

However, beyond a certain distance the jet breaks into a series of segments. The instability leading to the breakup of the jet is similar to that arising in the necking of

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bars [10]; but, if these results were applied to these jets, the conclusion would be that the breaking takes place long before it actually does.

In a recent paper *Romero* [15] analyzes the stability of these jets using the *Levy-von Mises equations* for an incompressible perfectly plastic material (see Section 1 below). He finds that a rapidly stretching plastic jet can be initially stable due to inertial effects, a result that had been anticipated by *Frankel* and *Weihs* [4] based on their study of the stability of a capillary jet of an ideal fluid (see also e.g. *Rayleigh* [14], *Weber* [16], *Levich* [12], *Goldin et al.* [8], *Bogy* [3]).

Assuming that the surface of the jet is stress-free, Romero finds a particular solution ("the undisturbed flow"), for which the axial velocity is linearly increasing. He then linearizes the equations about it, introducing scaled space variables (σ, ξ) $(0 < \sigma < 1, -\infty < \xi < \infty)$ and a scaled time variable s $(0 \leq s \leq T)$, to reduce this linear system to:

(0.1) $$\frac{\partial \phi}{\partial s} + \phi = -\alpha^2 \frac{\partial \Lambda}{\partial \xi} + \Gamma^{-2}\left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial \phi}{\partial \sigma}\right) - \alpha^2 \frac{\partial^2 \phi}{\partial \xi^2}\right) \quad ,$$

(0.2) $$\frac{\partial \psi}{\partial s} - 2\psi = -\frac{\partial \Lambda}{\partial \sigma} + \alpha^2 \Gamma^{-2} \frac{\partial^2 \psi}{\partial \xi^2} \quad ,$$

(0.3) $$\frac{\partial \phi}{\partial \xi} + \frac{1}{\sigma}\frac{\partial}{\partial \sigma}(\sigma \psi) = 0$$

for $(s, \sigma, \xi) \in [0, T] \times (0, 1] \times \mathbf{R}$ $(T > 0)$, and

(0.4) $$2\left(\frac{\partial \psi}{\partial \sigma} + \frac{1}{2}\frac{\partial \phi}{\partial \xi}\right) - \Gamma^2 \Lambda + \frac{3}{4}\Omega \Gamma^2 = 0 \qquad \text{at } \sigma = 1 \quad ,$$

(0.5) $$\frac{\partial \phi}{\partial \sigma} + \alpha^2 \frac{\partial \psi}{\partial \xi} = 3\alpha^2 \frac{\partial \Omega}{\partial \xi} \qquad \text{at } \sigma = 1 \quad ,$$

(0.6) $$\frac{\partial \Omega}{\partial s} = \psi \qquad \text{at } \sigma = 1 \quad ,$$

(0.7) ψ is finite as σ approaches zero.

Here α and Γ are functions of s given by $\alpha(s)^2 = \alpha(0)^2 e^{-3s}$, $\Gamma(s)^2 = \Gamma(0)^2 e^{-3s}$, and $\phi, \psi, \Lambda, \Omega$ are the unknown functions to be found, which correspond to scaled versions of the perturbations in the axial velocity, radial velocity, pressure and radius of the jet, respectively. The problem is three-dimensional: ξ is the variable along the axis of the jet and σ is the distance from the axis,i.e. $\sigma = \sqrt{x^2 + y^2}$.

While the main purpose of [15] is to study the stability of (0.1)-(0.7) (where this is done assuming a dependence in ξ of the form $e^{i\xi}$ for the unknown functions and studying the resulting system), the main result of the present paper is an existence and uniqueness theorem for (0.1)-(0.7); we need, of course, to impose initial conditions, namely

(0.8a) $$\phi(0, \sigma, \xi) = \phi_0(\sigma, \xi) \quad ,$$

(0.8b) $$\Omega(0, 0) = \Omega_0 \quad ,$$

where Ω_0, ϕ_0 are given. More precisely, we show that if $\phi_0(x, y, \xi) = \phi_0((x^2 + y^2)^{1/2}, \xi)$ is $C^{2+\alpha}$ in (x, y) (for some $\alpha, 0 < \alpha < 1$) and if $\phi_0(x, y, .)$ extends to an entire function of ξ in the complex plane of order 1 and sufficiently small type, then there exists a unique solution of (0.1)-(0.8).

In Section 1 we briefly describe the linearization procedure that leads to (0.1)-(0.7). In Section 2 we write down a system which is shown to be equivalent to (0.1)-(0.7) but is easier to study; in particular ψ and Ω will not appear in the new system. In Section 3 we state the main existence and uniqueness result (Theorem 3.1) and we prove a preliminary lemma (Lemma 3.2). Section 4 is devoted to the proof of Theorem 3.1. Finally, in Section 5 we establish properties of the solution regarding periodicity and growth (in ξ).

Section 1: The linearized system. We shall assume that the flow obeys the laws for an incompressible perfectly plastic material satisfying the Levy-von Mises equations and that the surface of the jet is stress-free. The Levy-von Mises equations are

$$\rho\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}.\nabla\bar{u}\right)=-\nabla p+div\bar{\bar{T}} \tag{1.1}$$

$$\nabla.\bar{u}=0 \tag{1.2}$$

where ρ is the constant material density, p is the pressure and $\bar{\bar{T}}$ is the deviatoric stress. In Cartesian coordinates $\bar{\bar{T}}$ is given by

$$T_{ij}=2\mu\dot{\epsilon}_{ij} \tag{1.3}$$

where $\dot{\epsilon}_{ij}$ is the rate of strain tensor,

$$\dot{\epsilon}_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{1.4}$$

μ is the effective viscosity,

$$\mu=Y(2\dot{\epsilon}_{kl}\dot{\epsilon}_{kl})^{-1/2} \tag{1.5}$$

and Y is the yield stress of the material.

Assuming axial symmetry we can write (1.1) and (1.2) in cylindrical coordinates as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial x}+\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{rx}}{\partial r}+\frac{T_{rx}}{r} \tag{1.6}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r}+\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{rx}}{\partial x}+\frac{(T_{rr}-T_{\theta\theta})}{r} \tag{1.7}$$

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv)=0 \tag{1.8}$$

where (x,r) are the cylindrical coordinates and (u,v) their corresponding velocities. Equations (1.3)-(1.5) remain valid (with $i=1,2,3$ replaced by $i=r,\theta,x$) provided we set

$$\dot{\epsilon}_{rr}=\frac{\partial v}{\partial r} \tag{1.9}$$

(1.10)
$$\dot{\epsilon}_{\theta\theta} = \frac{v}{r}$$

(1.11)
$$\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x}$$

(1.12)
$$\dot{\epsilon}_{xr} = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right).$$

We shall now write the boundary conditions that express the fact that the surface of the jet is stress-free and the relation of the time rate of change of the free surface to the velocity field at the boundary. These are given by

(1.13)
$$(T_{rx}, T_{rr} - p).(-\frac{\partial R}{\partial x}, 1) = 0 \qquad \text{on } r = R(x,t)\,,$$

(1.14)
$$(T_{xx} - p, T_{rx}).(-\frac{\partial R}{\partial x}, 1) = 0 \qquad \text{on } r = R(x,t)\,,$$

(1.15)
$$\frac{dR}{dt} = \frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} = v \qquad \text{on } r = R(x,t).$$

Following [15] define

(1.16)
$$u_0(x,r,t) = \frac{\beta(0)x}{q(t)}$$

(1.17)
$$v_0(x,r,t) = \frac{-\beta(0)r}{2q(t)}$$

(1.18)
$$p_0(x,r,t) = \frac{3}{8}\rho\beta(t)^2(r^2 - R_0(t)^2) - \frac{Y}{3^{1/2}}$$

(1.19)
$$R_0(x,t) = \frac{a_0(0)}{q(t)^{1/2}}$$

where $a_0(0)$ is the initial radius of the jet, $\beta(0)$ is the initial strain rate, $\beta(t) = \beta(0)/q(t)$ and $q(t) = \beta(0)t + 1$.

It is easily checked that (u_0, v_0, p_0, R_0) is a solution of (1.6)-(1.8) subject to the boundary conditions (1.13)-(1.15). To linearize about this solution we set

(1.20a) $$u(x,r,t) = u_0(x,r,t) + \delta u(x,r,t)$$

(1.20b) $$v(x,r,t) = v_0(x,r,t) + \delta v(x,r,t)$$

(1.20c) $$p(x,r,t) = p_0(x,r,t) + \delta p(x,r,t)$$

(1.20d) $$R(x,t) = R_0(x,t) + \delta R(x,t)$$

with $\delta u, \delta v, \delta p, \delta R$ "small". From equations (1.20) one can obtain the perturbations to the deviatoric stress tensor; we then substitute these expressions into (1.6)-(1.8) and (1.13)-(1.15) thus obtaining, after dropping quadratically small terms, the equations for the evolution of small disturbances:

(1.21) $$\rho\left(\frac{\partial \delta u}{\partial t} + u_0\frac{\partial \delta u}{\partial x} + v_0\frac{\partial \delta u}{\partial r} + \delta u\frac{\partial u_0}{\partial x}\right) = -\frac{\partial \delta p}{\partial x} + \mu\nabla^2\delta u - 2\mu\frac{\partial^2 \delta u}{\partial x^2}$$

(1.22) $$\rho\left(\frac{\partial \delta v}{\partial t} + u_0\frac{\partial \delta v}{\partial x} + v_0\frac{\partial \delta v}{\partial r} + \delta v\frac{\partial v_0}{\partial r}\right) = -\frac{\partial \delta p}{\partial r} + \mu\frac{\partial^2 \delta v}{\partial x^2}$$

(1.23) $$\frac{\partial \delta u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\delta v) = 0$$

with boundary conditions

(1.24) $$2\mu\left(\frac{\partial \delta v}{\partial r} + \frac{1}{2}\frac{\partial \delta u}{\partial x}\right) - \delta p - \delta R\frac{\partial p_0}{\partial r} = 0 \qquad \text{at } r = R_0\,,$$

$$\mu\left(\frac{\partial \delta u}{\partial r}+\frac{\partial \delta v}{\partial x}\right)=3^{1/2}Y\frac{\partial \delta R}{\partial x} \qquad \text{at } r=R_0\,, \tag{1.25}$$

$$\frac{\partial \delta R}{\partial t}+x\beta(t)\frac{\partial \delta R}{\partial x}=\delta v-\frac{1}{2}\beta(t)\delta R \qquad \text{at } r=R_0. \tag{1.26}$$

Finally we must require

$$\delta v \text{ is finite as r approaches 0.} \tag{1.27}$$

The final step in the derivation of the linearized system as done in [15] is to rescale the variables in order to obtain system (0.1)-(0.7).

All axial lengths will be scaled according to

$$L(t)=\frac{q(t)}{k(0)}=q(t)L(0) \tag{1.28}$$

where the parameter $L(0)$ is the initial length scale, and all radial lengths will be scaled according to

$$R_0(t)=\frac{a_0(0)}{q(t)^{1/2}}\,. \tag{1.29}$$

This means that we shall work with spatial coordinates (ξ,σ) given by

$$\xi=\frac{x}{L(t)} \tag{1.31}$$

$$\sigma=\frac{r}{R_0(t)}\,. \tag{1.32}$$

We also introduce a new variable s by

$$s=\ln(q(t)) \tag{1.30}$$

and we set

$$\phi=\frac{\delta u}{L(t)\beta(t)} \tag{1.33}$$

$$\psi = \frac{\delta v}{R_0(t)\beta(t)} \tag{1.34}$$

$$\Lambda = \frac{\delta p}{\rho R_0(t)^2\beta(t)^2} \tag{1.35}$$

$$\Omega = \frac{\delta R}{R_0(t)}. \tag{1.36}$$

Equations (1.21)-(1.27) now become (0.1)-(0.7) where $\alpha(0)^2 = (a_0(0)/L(0))^2$ and $\Gamma(0)^2 = 3^{1/2}\rho a_0(0)^2\beta(0)^2/Y$.

Notice that (0.1)-(0.7) have the advantage that there is no explicit dependence on the axial coordinate.

Section 2: An equivalent linear system. The purpose of this section is to derive a system equivalent to (0.1)-(0.8) which is easier to work with. The new system will not involve Ω and ψ and it will be amenable to the theory of elliptic and parabolic differential equations.

The first step is to notice that if $(\phi, \psi, \Lambda, \Omega)$ is a solution of (0.1)-(0.8), then $(\phi, \psi, \Lambda + (3/4)(\omega_0 - \Omega_0), \Omega + \omega_0 - \Omega_0)$ is a solution of (0.1)-(0.8a) satisfying $\Omega(0,0) = \omega_0$. Thus, without loss of generality, we may assume that

$$\Omega(0,0) = -\frac{1}{3}\int_0^1 r\frac{\partial\phi_0}{\partial\xi}(r,0)\,dr. \tag{2.1}$$

LEMMA 2.1. *Assume that*

$$\frac{\partial\phi_0}{\partial\sigma}(1,\xi) = 0 \qquad ,\xi \in \mathbf{R}. \tag{2.2}$$

Then, $(\phi, \psi, \Lambda, \Omega)$ is a solution of (0.1)-(0.8a) satisfying (2.1) if and only if

(i) (ϕ, Λ) satisfy

$$\frac{\partial\phi}{\partial s} + \phi = -\alpha^2\frac{\partial\Lambda}{\partial\xi} + \Gamma^{-2}\left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma\frac{\partial\phi}{\partial\sigma}) - \alpha^2\frac{\partial^2\phi}{\partial\xi^2}\right) \tag{2.3a}$$

(2.3b)
$$\frac{1}{\sigma}\left(\frac{\partial}{\partial\sigma}(\sigma\frac{\partial\Lambda}{\partial\sigma})\right)=\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\xi}\right)-2\frac{\partial\phi}{\partial\xi}-\alpha^2\Gamma^{-2}\frac{\partial^3\phi}{\partial\xi^3}$$

for $(s,\sigma,\xi)\in[0,T]\times(0,1]\times\mathbf{R}$, *with initial-boundary conditions given by*

(2.4)
$$\frac{\partial\phi}{\partial\sigma}=-3\alpha^2\int_0^s\int_0^1 r\frac{\partial^2\phi}{\partial\xi^2}(t,r,\xi)\,dr\,dt-\alpha^2\int_0^1 r\frac{\partial^2\phi_0}{\partial\xi^2}(r,\xi)\,dr$$
$$+\alpha^2\int_0^1 r\frac{\partial^2\phi}{\partial\xi^2}(s,r,\xi)\,dr\qquad \text{at }\sigma=1\,,$$

(2.5)
$$\Lambda=-\frac{3}{4}\int_0^s\int_0^1 r\frac{\partial\phi}{\partial\xi}(t,r,\xi)\,dr\,dt-\Gamma^{-2}\frac{\partial\phi}{\partial\xi}-\frac{1}{4}\int_0^1 r\frac{\partial\phi_0}{\partial\xi}(r,\xi)\,dr$$
$$+2\Gamma^{-2}\int_0^1 r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr\qquad \text{at }\sigma=1\,,$$

(2.6)
$$\phi(0,\sigma,\xi)=\phi_0(\sigma,\xi)$$

and

(ii) ψ *and* Ω *are given by*

(2.7)
$$\psi(s,\sigma,\xi)=-\frac{1}{\sigma}\int_0^\sigma r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr\,,$$

(2.8)
$$\Omega(s,\xi)=-\int_0^s\int_0^1 r\frac{\partial\phi}{\partial\xi}(t,r,\xi)\,dr\,dt-\frac{1}{3}\int_0^1 r\frac{\partial\phi_0}{\partial\xi}(r,\xi)\,dr.$$

Proof. Suppose that $(\phi,\psi,\Lambda,\Omega)$ is a solution of (0.1)-(0.8a) satisfying (2.1). Multiplying (0.3) by σ and integrating with respect to σ we get

$$\psi(s,1,\xi)-\sigma\psi(s,\sigma,\xi)=-\int_\sigma^1 r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr.$$

Using (0.7) we conclude that

$$\psi(s,1,\xi)=-\int_0^1 r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr\quad,$$

so that

$$-\sigma\psi(s,\sigma,\xi) = -\int_\sigma^1 r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr + \int_0^1 r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr$$
$$= \int_0^\sigma r\frac{\partial\phi}{\partial\xi}(s,r,\xi)\,dr \quad ,$$

and (2.7) follows.

Now multiply (0.2) by σ and differentiate with respect to σ to obtain (using (0.3))

$$-\sigma\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\xi}\right) + 2\sigma\frac{\partial\phi}{\partial\xi} = -\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\Lambda}{\partial\sigma}\right) - c\sigma\frac{\partial^3\phi}{\partial\xi^3}$$

or, equivalently,

$$-\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\xi}\right) + 2\frac{\partial\phi}{\partial\xi} = -\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\Lambda}{\partial\sigma}\right) - c\frac{\partial^3\phi}{\partial\xi^3}$$

which gives (2.3b). Here $c = \alpha(s)^2\Gamma(s)^{-2} = \alpha_0(0)^2\left(L(0)\Gamma(0)\right)^{-2} \geq 0$.

Using (0.6) and (2.7) we have

$$\Omega(s,\xi) = \int_0^s \psi(t,1,\xi)\,dt + \Omega(0,\xi) \tag{2.9}$$
$$= -\int_0^s\int_0^1 r\frac{\partial\phi}{\partial\xi}(t,r,\xi)\,dr dt + \Omega(0,\xi).$$

On the other hand, specializing equation (0.5) at $s = 0$ and using (2.7)

$$\frac{\partial\phi_0}{\partial\sigma}(1,\xi) - \alpha(0)^2\left(\int_0^1 r\frac{\partial^2\phi_0}{\partial\xi^2}(r,\xi)\,dr\right) = 3\alpha(0)^2\frac{\partial\Omega}{\partial\xi}(0,\xi)$$

which, due to (2.2), is

$$\frac{\partial\Omega}{\partial\xi}(0,\xi) = -\frac{1}{3}\int_0^1 r\frac{\partial^2\phi_0}{\partial\xi^2}(r,\xi)\,dr. \tag{2.10}$$

But (2.1) and (2.10) imply

$$\Omega(0,\xi) = -\frac{1}{3}\int_0^1 r\frac{\partial\phi_0}{\partial\xi}(r,\xi)\,dr \tag{2.11}$$

which, together with (2.9), gives (2.8).

Since (2.3a) is exactly (0.1), it only remains to show that (2.4) and (2.5) hold.

First notice that with the aid of (2.7) and (2.8) we can rewrite the boundary condition (0.5) as

$$\frac{\partial\phi}{\partial\sigma} - \alpha^2 \int_0^1 r\frac{\partial^2\phi}{\partial\xi^2}(s,r,\xi)\,dr = -3\alpha^2 \int_0^s \int_0^1 r\frac{\partial^2\phi}{\partial\xi^2}(t,r,\xi)\,dr dt - \alpha^2 \int_0^1 r\frac{\partial^2\phi_0}{\partial\xi^2}(r,\xi) \qquad \text{at } \sigma = 1 \quad ,$$

which is (2.4).

Finally, using (0.3) we can rewrite (0.4) as

$$2\left(-\psi - \frac{\partial\phi}{\partial\xi} + \frac{1}{2}\frac{\partial\phi}{\partial\xi}\right) - \Gamma^2\Lambda + \frac{3}{4}\Omega\Gamma^2 = 0 \qquad \text{at } \sigma = 1 \quad ,$$

or

(2.12) $$-2\psi - \frac{\partial\phi}{\partial\xi} - \Gamma^2\Lambda + \frac{3}{4}\Omega\Gamma^2 = 0 \qquad \text{at } \sigma = 1 \quad ,$$

and (2.5) follows from (2.7),(2.8) and (2.12).

From the above computations it is clear how to proceed in proving the converse statement, that is: if ϕ, Λ satisfy (2.3)-(2.6) and ψ, Ω are given by (2.7) and (2.8) then $(\phi, \psi, \Lambda, \Omega)$ is a solution of (0.1)-(0.8a) satisfying(2.1).

This completes the proof of lemma 2.1. □

It will be convenient to rewrite the system in Cartesian coordinates. Taking σ to be the radial variable:

(2.13a) $$\frac{\partial\phi}{\partial s} - \Gamma(s)^{-2}\Delta\phi + c\frac{\partial^2\phi}{\partial\xi^2} + \phi = -\alpha(s)^2\frac{\partial\Lambda}{\partial\xi} \qquad \text{in } G,$$

(2.13b) $$\Delta\Lambda = \frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\xi}\right) - 2\frac{\partial\phi}{\partial\xi} - c\frac{\partial^3\phi}{\partial\xi^3} \qquad \text{in } G,$$

(2.13c) $$\phi(0,x,y,\xi) = \phi_0(x,y,\xi)$$

and , onΓ_0,

(2.13d)

$$\frac{\partial\phi}{\partial\nu} = -3\frac{\alpha(s)^2}{2\pi}\int_0^s\int_{B_1}\frac{\partial^2\phi}{\partial\xi^2}(t,x,y,\xi)\,dxdydt + \frac{\alpha(s)^2}{2\pi}\int_{B_1}\frac{\partial^2\phi}{\partial\xi^2}(s,x,y,\xi)\,dxdy - \frac{\alpha(s)^2}{2\pi}\int_{B_1}\frac{\partial^2\phi_0}{\partial\xi^2}(x,y,\xi)\,dxdy$$

(2.13e)

$$\Lambda = -\frac{3}{8\pi}\int_0^s\int_{B_1}\frac{\partial\phi}{\partial\xi}(t,x,y,\xi)\,dxdydt - \Gamma(s)^{-2}\frac{\partial\phi}{\partial\xi} - \frac{1}{8\pi}\int_{B_1}\frac{\partial\phi_0}{\partial\xi}(x,y,\xi)\,dxdy + \frac{\Gamma(s)^{-2}}{2\pi}\int_{B_1}\frac{\partial\phi}{\partial\xi}(s,x,y,\xi)\,dxdy$$

where: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, B_1 =unit disc in $\mathbf{R}^2$, $G = [0,T]\times B_1\times\mathbf{R}$, $\Gamma_0 = [0,T]\times\partial B_1\times\mathbf{R}$, ν =exterior unit normal to ∂B_1 and $c = \alpha(s)^2\Gamma(s)^{-2} = \left(\frac{a_0(0)}{L(0)\Gamma(0)}\right)^2 \quad (c \geq 0)$.

Section 3: A preliminary result. Before stating our main result, which will be proved in the next section, we need to define the class of "admissible" initial data.

DEFINITION. *A function $f : \overline{B_1}\times\mathbf{R} \longrightarrow \mathbf{R}$ is said to belong to the class A_η^α $(0 < \alpha < 1, 0 < \eta)$ if f satisfies the following conditions:*

(i) *For each $(x,y) \in \overline{B_1}$, $f(x,y,.)$ extends to an entire function in* $\mathbf{C}$.

(ii) *For each $z \in \mathbf{C}$, $f(.,z) \in C^{2+\alpha}(\overline{B_1})$.*

(iii) *For each $\epsilon > 0$, there exists a constant $c_\epsilon \geq 0$ such that*

(3.1)
$$\|f(.,z)\|_{2+\alpha,\overline{B_1}} \leq c_\epsilon e^{(\eta+\epsilon)|z|} \qquad \forall z \in \mathbf{C}.$$

(iv) *$\partial_\nu f(x,y,z) = 0$ for $(x,y) \in \partial B_1, z \in \mathbf{C}$.*

Examples of functions in A_η^α can be constructed as follows:

1. Let $h \in C^{2+\alpha}(\overline{B_1})$ with $\partial_\nu h = 0$ on ∂B_1, and let $g(z)$ be an entire function of order 1 and type η. Then $f(x,y,z) = h(x,y)g(z) \in A_\eta^\alpha$.

2.Let $\varphi : B_1 \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfy:

(a) $\partial_\nu \varphi = 0 \qquad ,(x,y) \in \partial B_1$.

(b) $\|\varphi(.,\mu)\|_{2+\alpha} \leq M \qquad$ for $|\mu| \leq \eta/(2\pi)$, for some constant $M \geq 0$.

(c) $\varphi(x,y,\mu) = 0 \qquad$ for $|\mu| > \eta/(2\pi)$.

Then, if we set

$$\phi(x,y,\xi) = \varphi(x,y,.)\hat{}(\xi) = \int_{\mathbf{R}} \varphi(x,y,\mu) e^{-2\pi i \mu \xi}\, d\mu$$

we have $f \in A^\alpha_\eta$.

THEOREM 3.1. *Let $T > 0$, $0 < \alpha < 1$. Then, there exists a number $\eta_0 < 1$ such that, for every $\phi_0 \in A^\alpha_{\eta_0}$, there exists a unique solution (ϕ, Λ) of (2.13) satisfying:*

(i) $\phi \equiv \phi_0$ at $s = 0$.

(ii) *For each $(x,y) \in \overline{B_1}$, $s \in [0,T]$, $\phi(s,x,y,.)$ and $\Lambda(s,x,y,.)$ extend to entire functions in $\mathbf{C}$.*

(iii)

$$\phi(.,z) \in \overline{C}_{2+\alpha}([0,T] \times \overline{B_1}), \quad z \in \mathbf{C}.$$

$$\Lambda(.,z) \in \overline{C}_\alpha([0,T] \times \overline{B_1}), \quad z \in \mathbf{C}.$$

$$\Lambda(s,.,z) \in C^{2+\alpha}(\overline{B_1}), \quad z \in \mathbf{C}, \quad s \in [0,T],$$

where $\overline{C}_{2+\alpha}$, $\overline{C}_\alpha$ are the parabolic Hölder spaces (see e.g. *Friedman* [6], pp. 61–63).

(iv) *For every $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

(3.2)

$$\|\phi(.,z)\|_{2+\alpha,[0,T]\times\overline{B_1}} + \|\Lambda(.,z)\|_{\alpha,[0,T]\times\overline{B_1}} + \sup_{s\in[0,T]} \|\Lambda(s,.,z)\|_{2+\alpha,\overline{B_1}} \leq c_\epsilon e^{(\eta_0+\epsilon)|z|}, \quad \forall z \in \mathbf{C}.$$

REMARK. *It will be evident from the proof of Theorem 3.1 that, for each α fixed, $\eta_0 \longrightarrow 0$ as $T \longrightarrow \infty$.*

NOTATION. *We shall write*

$$\|f\|_{2+\alpha,\overline{B_1}} = \|f\|^0_{2+\alpha} \quad ,$$

$$\|g\|_{2+\alpha,[0,T]\times\overline{B_1}} = \|g\|_{2+\alpha}$$

and

$$\|h\|_{\alpha,[0,T]\times\overline{B_1}} + \sup_{s\in[0,T]} \|h(s,.)\|_{2+\alpha,\overline{B_1}} = |||h|||.$$

The underlying approach for proving Theorem 3.1 is by superposition: we expand the initial data into a power series in ξ, $\phi_0(x,y,\xi) = \sum_{n=0}^{\infty} a_n(x,y)\xi^n$, solve the problem with initial data $\phi_0^n(x,y,\xi) = a_n(x,y)\xi^n$, and then sum over n. In order to solve for ϕ_0^n we take the Fourier transform in the ξ variable, solve the resulting problem (using a fixed point argument) and then take the inverse transform. Finally, using the growth condition for ϕ_0, we shall show that the resulting series actually converges to a solution of (2.13).

By Fourier transforming (2.13) in ξ and setting $\phi(s,x,y,.)\hat{}(\mu) = u(s,x,y,\mu)$, $\Lambda(s,x,y,.)\hat{}(\mu) = v(s,x,y,\mu)$, $\phi_0(x,y,.)\hat{}(\mu) = u_0(x,y,\mu)$, we are led to the following system:

(3.3a)
$$\frac{\partial u}{\partial s} - \Gamma^{-2}\Delta u + c(2\pi i\mu)^2 + u = -\alpha^2 2\pi i\mu v \quad \text{in } G,$$

(3.3b)
$$\Delta v = 2\pi i\mu\frac{\partial u}{\partial s} - 2(2\pi i\mu)u - c(2\pi i\mu)^3 u \quad \text{in } G,$$

(3.3c)
$$u(0,x,y,\mu) = u_0(x,y,\mu)$$

and, on Γ_0,

(3.3d)
$$\frac{\partial u}{\partial \nu} = -\frac{3\alpha^2}{2\pi}(2\pi i\mu)^2\int_0^s\int_{B_1} u(t,x,y,\mu)\,dxdydt + \frac{\alpha^2}{2\pi}(2\pi i\mu)^2\int_{B_1} u(s,x,y,\mu)\,dxdy - \frac{\alpha^2}{2\pi}(2\pi i\mu)^2\int_{B_1} u_0(x,y,\mu)\,dxdy$$

(3.3e)
$$v = \frac{3}{8\pi}(2\pi i\mu)\int_0^s\int_{B_1} u(t,x,y,\mu)\,dxdydt - \Gamma^{-2}(2\pi i\mu)u - \frac{(2\pi i\mu)}{8\pi}\int_{B_1} u_0(x,y,\mu)\,dxdy + \frac{\Gamma^{-2}}{2\pi}(2\pi i\mu)\int_{B_1} u(s,x,y,\mu)\,dxdy.$$

Replacing iu by u (and iu_0 by u_0) we may rewrite (3.3) as

$$\frac{\partial u}{\partial s} - \Gamma^{-2}\Delta u + (1 - 4\pi^2 c\mu^2)u = \alpha^2 2\pi\mu v \quad \text{in } G, \tag{3.4a}$$

$$\Delta v = 2\pi\mu\frac{\partial u}{\partial s} + (8\pi^3 c\mu^3 - 4\pi\mu)u \quad \text{in } G, \tag{3.4b}$$

$$u(0, x, y, \mu) = u_0(x, y, \mu) \tag{3.4c}$$

and, on Γ_0,

$$\frac{\partial u}{\partial \nu} = 6\pi\alpha^2\mu^2 \int_0^s \int_{B_1} u - 2\pi\alpha^2\mu^2 \int_{B_1} u + 2\pi\alpha^2\mu^2 \int_{B_1} u_0 \tag{3.4d}$$

$$v = -\frac{3}{4}\mu \int_0^s \int_{B_1} u - \Gamma^{-2}2\pi\mu u - \frac{\mu}{4}\int_{B_1} u_0 + \Gamma^{-2}\mu \int_{B_1} u. \tag{3.4e}$$

We shall now give the proof of existence and uniqueness of a solution for a a particular case of (3.4); namely, the case where u_0 is independent of μ. In fact, this constitutes the main step towards proving theorem 3.1. We state it in the following

LEMMA 3.2. *Let $u_0 : \overline{B_1} \longrightarrow \mathbf{R}$ satisfy*

$$u_0 \in C^{2+\alpha}(\overline{B_1}) \quad , \tag{3.5}$$

$$\frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial B_1. \tag{3.6}$$

Then, there exists a unique solution of (3.4) satisfying:

$u(s, x, y, .), v(s, x, y, .) \in C^\infty(\mathbf{R})$, $\partial_\mu^l u(., \mu) \in \overline{C}_{2+\alpha}([0, T] \times \overline{B_1})$, $\partial_\mu^l v(., \mu) \in \overline{C}_\alpha([0, T] \times \overline{B_1})$, $\partial_\mu^l v(s, ., \mu) \in C^{2+\alpha}(\overline{B_1})$, *and* $\partial_s \partial_\mu^l(s, ., \mu) \in C^{2+\alpha}(B_1) \cap C^0(\overline{B_1}) \quad (l \geq 0)$.

Furthermore, for each $l \geq 0$, we have

$$\|\frac{\partial^l u}{\partial \mu^l}(., 0)\|_{2+\alpha} \leq c_1 c_2^l \, l! \, \|u_0\|^0_{2+\alpha} \quad , \tag{3.7}$$

$$|||\frac{\partial^l v}{\partial \mu^l}(., 0)||| \leq c_1 c_2^l \, l! \, \|u_0\|^0_{2+\alpha} \tag{3.8}$$

for some constants $c_1, c_2 > 0$.

Proof. We shall use c_μ to denote a generic constant (not always the same), depending only on μ. We divide the proof into five steps.

STEP I. *(Transformation of the system)*

First we transform (3.4) into a more convenient form. Using (3.4a), we may replace (3.4b) by

$$\Delta v - 4\pi^2\mu^2\alpha^2 v = 2\pi\mu\Gamma^{-2}\Delta u + 2\pi\mu(8\pi^2 c\mu^2 - 3)u$$

and setting $w = v - 2\pi\mu\Gamma^{-2}u$, we are led to

(3.9a) $$\frac{\partial u}{\partial s} - \Gamma^{-2}\Delta u + (1 - 8\pi^2 c\mu^2)u = \alpha^2 2\pi\mu w \quad \text{in } G,$$

(3.9b) $$\Delta w - 4\pi^2\alpha^2\mu^2 w = (24\pi^3 c\mu^3 - 6\pi\mu)u \quad \text{in } G,$$

with initial-boundary conditions given by

(3.9c) $$u(0, x, y, \mu) = u_0(x, y) \quad \text{on } B_1,$$

(3.9d) $$\frac{\partial u}{\partial \nu} = 6\pi\alpha^2\mu^2 \int_0^s \int_{B_1} u - 2\pi\alpha^2\mu^2 \int_{B_1} u + 2\pi\alpha^2\mu^2 \int_{B_1} u_0 \quad \text{on } [0, T] \times \partial B_1 \times \mathbf{R},$$

(3.9e) $$w = -\frac{3}{4}\mu \int_0^s \int_{B_1} u - \Gamma^{-2}4\pi\mu u - \frac{\mu}{4}\int_{B_1} u_0 + \Gamma^{-2}\mu \int_{B_1} u \quad \text{on } [0, T] \times \partial B_1 \times \mathbf{R}.$$

Finally, we want to write the right hand side of (3.9d) in terms of u_0 and w.

Integrating (3.9a) over B_1 we get

(3.10) $$f'(s) + (1 - 8\pi^2 c\mu^2)f(s) - \Gamma^{-2}2\pi\frac{\partial u}{\partial \nu} = g(s)$$

where

$$f(s) = \int_{B_1} u\, dx dy, \quad g(s) = 2\pi\alpha^2 \int_{B_1} w\, dx dy,$$

and using (3.9d) we can replace $\partial_\nu u$ in (3.10), to get

$$f'(s) + (1 - 4\pi^2 c\mu^2) f(s) - 12\pi^2 c\mu^2 \int_0^s f(t)\, dt - 4\pi^2 c\mu^2 \int_{B_1} u_0 = g(s). \tag{3.11}$$

Since $f(0) = \int_{B_1} u_0(x,y)\, dxdy$, we may solve (3.11) for f in terms of u_0 and w. Replacing in (3.9d), we see that, on ∂B_1,

$$\frac{\partial u}{\partial \nu} = E_2(u_0, w)$$

where $E_2(u_0,w)$ is an expression in (u_0,w) (which can be explicitly written) depending only on s and with the property that, if $w \in \overline{C}_\alpha([0,T] \times \overline{B_1})$ and $u_0 \in C^{2+\alpha}(\overline{B_1})$, we have $\partial_s E_2(u_0,w) \in \overline{C}_\alpha([0,T] \times \overline{B_1})$.

Thus, if we set

$$E_1(u_0, u) = -\frac{3}{4}\mu \int_0^s \int_{B_1} u - \Gamma^{-2} 4\pi\mu u - \frac{\mu}{4} \int_{B_1} u_0 + \Gamma^{-2}\mu \int_{B_1} u$$

then we need to study system (3.9a,b,c) subject to the boundary conditions

$$\frac{\partial u}{\partial \nu} = E_2(u_0, w) \qquad \text{on } \partial B_1, \tag{3.12a}$$

$$w = E_1(u_0, u) \qquad \text{on } \partial B_1. \tag{3.12b}$$

STEP II. *(Local existence)*

We now show that there exists a solution of (3.9a,b,c),(3.12) if T is small enough.

Fix $\mu \in \mathbf{R}$. Given $F \in \overline{C}_{2+\alpha}$ such that $F(x,y,0) = u_0(x,y)$, let $\mathcal{R}F = L$ be the unique solution of

$$\Delta L - 4\pi^2\alpha^2\mu^2 L = (24\pi^3 c\mu^3 - 6\pi\mu) F \quad \text{on } B_1 \times [0,T], \tag{3.13a}$$

$$L = E_1(u_0, F) \quad \text{on } \partial B_1 \times [0,T]. \tag{3.13b}$$

It is then clear that $L = \mathcal{R}F$ satisfies:

$L(.,s) \in C^{2+\alpha}(\overline{B_1})$, $\partial_s L \in C^{2+\alpha}(B_1)$ and $L \in \overline{C}_\alpha$.

Furthermore:

If $F_1, F_2 \in \overline{C}_{2+\alpha}$, $F_1(x,y,0) = F_2(x,y,0) = u_0(x,y)$, and $L_i = \mathcal{R}F_i \quad (i = 1,2)$, then, for some constant c_μ we have

$$\|L_1 - L_2\|_\alpha \leq c_\mu \|F_1 - F_2\|_E \tag{3.14}$$

where

$$\|h\|_E = \|h\|_{\infty,D} + \|\frac{\partial h}{\partial x}\|_{\infty,D} + \|\frac{\partial h}{\partial y}\|_{\infty,D} + \|\frac{\partial^2 h}{\partial x^2}\|_{\infty,D} + \|\frac{\partial^2 h}{\partial y^2}\|_{\infty,D} + \|\frac{\partial^2 h}{\partial x \partial y}\|_{\infty,D} + \|\frac{\partial h}{\partial t}\|_{\infty,D}$$

and $D = \overline{B_1} \times [0,T]$.

Indeed this is a straightforward consequence of the maximum principle and the following result

LEMMA 3.3. *Let* $U, W \in C^{2+\alpha}(\overline{B_1})$, $V \in C^{1+\alpha}(\overline{B_1})$ *satisfy*

$$\begin{cases} \Delta U - \lambda^2 U = V & \text{on } B_1 \quad (B_1 \subset \mathbb{R}^2) \\ U = W & \text{on } \partial B_1. \end{cases}$$

Then

$$\|\nabla U\|_0 \leq 6[\|V\|_1 + \lambda^2 \|W\|_0 + \|W\|_2]$$

where $\|h\|_k = \sum_{|\alpha| \leq k} \|\frac{\partial^\alpha h}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\|_{\infty,B_1}$.

(See e.g. *Gilbarg-Trudinger* [7], p. 48).

Now, given $L \in \overline{C}_\alpha$ let $\mathcal{S}L = F$ be the unique solution of

$$\frac{\partial F}{\partial s} - \Gamma^{-2}\Delta F + (1 - 8\pi^2 c\mu^2)F = \alpha^2 2\pi\mu L \quad \text{on } B_1 \times [0,T], \tag{3.15a}$$

(3.15b) $$\frac{\partial F}{\partial \nu} = E_2(u_0, L) \quad \text{on } \partial B_1 \times [0,T],$$

(3.15c) $$F(x,y,0) = u_0(x,y) \quad \text{on } B_1.$$

Notice that, from the hypotheses on u_0 ((3.5) and (3.6)), the compatibility condition

(3.16) $$\frac{\partial u_0}{\partial \nu} = E_2(u_0, L) \text{ on } \partial B_1 \text{ at } s = 0$$

is satisfied. Thus, $F = \mathcal{S}L \in \overline{C}_{2+\alpha}$ and the parabolic Schauder estimates (see e.g. *Ladyženskaja-Solonnikov-Ural'ceva* [11], pp. 320–321) imply that:

If $L_1, L_2 \in \overline{C}_\alpha$, $\quad F_i = \mathcal{S}L_i\,(i = 1,2)$, then

(3.17) $$\|F_1 - F_2\|_{2+\alpha} \le c_\mu \|L_1 - L_2\|_\alpha.$$

Let $\mathcal{Q} = \mathcal{S}\mathcal{R}$; then, combining the above results we obtain

(3.18) $$\|\mathcal{Q}(F_1) - \mathcal{Q}(F_2)\|_{2+\alpha} \le c_\mu T^{\alpha/2} \|F_1 - F_2\|_{2+\alpha}$$

for every $F_1, F_2 \in \overline{C}_{2+\alpha}$ such that $F_i(x,y,0) = u_0(x,y)\,(i = 1,2)$. (Notice that $(\mathcal{R}F_1 - \mathcal{R}F_2)(x,y,0) \equiv 0$).

Let $X = \{F \in \overline{C}_{2+\alpha} \mid F(x,y,0) = u_0(x,y)\} \subset \overline{C}_{2+\alpha}$. Then, choosing $T = T_0 = T_0(\mu)$ small enough, we may apply the Contraction Mapping Theorem to conclude that there exists a unique $F_1 \in X$ such that $\mathcal{Q}(F_1) = F_1$. Hence, if we set $L_1 = \mathcal{R}F_1$, then (F_1, L_1) satisfy:

(3.19a) $$\frac{\partial F_1}{\partial s} - \Gamma^{-2}\Delta F_1 + (1 - 8\pi^2 c\mu^2)F_1 = \alpha^2 2\pi\mu L_1 \quad \text{on } B_1 \times [0, T_0],$$

(3.19b) $$\Delta L_1 - 4\pi^2\alpha^2\mu^2 L_1 = (24\pi^3 c\mu^3 - 6\pi\mu)F_1 \quad \text{on } B_1 \times [0, T_0]$$

and

(3.19c) $$F_1(x,y,0) = u_0(x,y) \quad \text{on } B_1,$$

(3.19d) $$\frac{\partial F_1}{\partial \nu} = E_2(u_0, L_1) \quad \text{on } \partial B_1 \times [0, T_0],$$

(3.19e) $$L_1 = E_1(u_0, F_1) \quad \text{on } \partial B_1 \times [0, T_0]$$

with $F_1 \in \overline{C}_{2+\alpha}$, $L_1 \in \overline{C}_\alpha$, $L_1(.,s) \in C^{2+\alpha}(\overline{B_1})$ and $\partial_s L_1(.,s) \in C^{2+\alpha}(B_1) \cap C^0(\overline{B_1})$.

Now let us show that

(3.20) $$\|F_1\|_{2+\alpha} \leq c_\mu \|u_0\|^0_{2+\alpha},$$

(3.21) $$\sup_{s \in [0,T_0]} \|L_1(.,s)\|^0_{2+\alpha} \leq c_\mu \|u_0\|^0_{2+\alpha}$$

and

(3.22) $$\|\frac{\partial L_1}{\partial s}\|_0 = \|\frac{\partial L_1}{\partial s}\|_{\infty, \overline{B_1} \times [0,T_0]} \leq c_\mu \|u_0\|^0_{2+\alpha}.$$

Using the elliptic Schauder estimates and the maximum principle (in (3.13)), (3.21) and (3.22) easily follow from (3.20). Thus, it remains to show that (3.20) holds.

Let $G_1(x,y,s) = E_2(u_0, L_1)(s)\,(x^2+y^2) + u_0(x,y)$. Then, $G_1 \in \overline{C}_{2+\alpha}$ and

$$\frac{\partial F_1}{\partial \nu} = \frac{\partial G_1}{\partial \nu} \quad \text{on } \partial B_1 \times [0, T_0],$$

$$F_1(x,y,0) = G_1(x,y,0).$$

It is clear that

$$\|F_1\|_E \leq T_0^{\alpha/2} \|F_1\|_{2+\alpha} + \|u_0\|^0_{2+\alpha} + \|\frac{\partial F_1}{\partial s}(.,0)\|_0.$$

Also, from (3.19),

$$\|\frac{\partial F_1}{\partial s}(.,0)\|_0 \leq c_\mu(\|u_0\|^0_{2+\alpha} + \|L_1(.,0)\|_0) \leq c_\mu \|u_0\|^0_{2+\alpha}.$$

Thus,

$$\|F_1\|_E \leq T_0^{\alpha/2} \|F_1\|_{2+\alpha} + c_\mu \|u_0\|^0_{2+\alpha},$$

and applying the parabolic Schauder estimates to (3.19a,c,d)

$$\begin{aligned}\|F_1\|_E &\le c_\mu T_0^{\alpha/2}(\|L_1\|_\alpha + \|G_1\|_{2+\alpha}) + c_\mu \|u_0\|^0_{2+\alpha} \\ &\le c_\mu T_0^{\alpha/2}\|L_1\|_\alpha + c_\mu\|u_0\|^0_{2+\alpha}.\end{aligned}$$

But $\|L_1\|_\alpha \le \|L_1\|_0 + \|\partial_x L_1\|_0 + \|\partial_y L_1\|_0 + T_0^{1-\alpha/2}\|\partial_s L_1\|_0$ so that, using the maximum principle and lemma 3.3, we have

$$\|L_1\|_\alpha \le c_\mu(\|F_1\|_E + \|u_0\|^0_{2+\alpha})$$

so

$$\|F_1\|_E \le c_\mu T_0^{\alpha/2}\|F_1\|_E + c_\mu\|u_0\|^0_{2+\alpha}.$$

Hence

$$\|F_1\|_E \le c_\mu\|u_0\|^0_{2+\alpha} \tag{3.23}$$

and therefore also

$$\|L_1\|_\alpha \le c_\mu\|u_0\|^0_{2+\alpha}. \tag{3.24}$$

Finally, since $\|F_1\|_{2+\alpha} \le c_\mu(\|L_1\|_\alpha + \|u_0\|^0_{2+\alpha})$, using (3.24) we get

$$\|F_1\|_{2+\alpha} \le c_\mu\|u_0\|^0_{2+\alpha}$$

which is (3.20).

STEP III. *(Global existence and uniqueness)*

In order to prove global existence of a solution on $[0,T]\times B_1$ we just need to write

$$u_1(x,y) = u_0(x,y) + 3\int_0^{T_0} F_1(x,y,t)\,dt$$

and repeat the above argument for the system

(3.25a)
$$\frac{\partial F}{\partial s} - \Gamma^{-2}\Delta F + (1 - 8\pi^2 c\mu^2)F = \alpha^2 2\pi\mu L \quad \text{on } B_1 \times [T_0, 2T_0],$$

(3.25b)
$$\Delta L - 4\pi^2\alpha^2\mu^2 L = (24\pi^3 c\mu^3 - 6\pi\mu)F \quad \text{on } B_1 \times [T_0, 2T_0]$$

with initial-boundary conditions

(3.25c)
$$F(x, y, T_0) = F_1(x, y, T_0) \quad \text{on } B_1,$$

(3.25d)
$$\begin{aligned}\frac{\partial F}{\partial \nu} &= 6\pi\alpha^2\mu^2 \int_{T_0}^{s}\int_{B_1} F - 2\pi\alpha^2\mu^2 \int_{B_1} F \\ &+ 2\pi\alpha^2\mu^2 \int_{B_1} u_1 \qquad \text{for } (x, y) \in \partial B_1,\end{aligned}$$

(3.25e)
$$\begin{aligned}L &= -\frac{3}{4}\mu \int_{T_0}^{s}\int_{B_1} F - \Gamma^{-2}4\pi\mu F \\ &- \frac{\mu}{4}\int_{B_1} u_1 + \Gamma^{-2}\mu \int_{B_1} F \qquad \text{for } (x, y) \in \partial B_1.\end{aligned}$$

(Here we use the fact that the size of the time interval for local existence depends only on μ, not on the initial data).

Thus, we have proved that there there exists a solution (u, v) of (3.4) satisfying:

$u(., \mu) \in \overline{C}_{2+\alpha}(\overline{B_1} \times [0, T])$, $v(., \mu) \in \overline{C}_{\alpha}(\overline{B_1} \times [0, T])$, $v(s, ., \mu) \in C^{2+\alpha}(\overline{B_1})$ and $\partial_s v \in C^{2+\alpha}(B_1) \cap C^0(\overline{B_1})$.

Also

$$\|u(., \mu)\|_{2+\alpha} + |||v(., \mu)||| \leq c_\mu \|u_0\|^0_{2+\alpha}.$$

The uniqueness of the solution (u, v) easily follows from (3.9),(3.12) and the uniqueness statement in the Contraction Mapping Theorem.

STEP IV. *(Regularity of the solution)*

We want to show:

(a) $u(t,x,y,.), v(t,x,y,.) \in C^\infty(\mathbf{R})$.

(b) $\partial_\mu^l u(.,\mu) \in \overline{C}_{2+\alpha}([0,T]\times\overline{B_1})$, $\partial_\mu^l v(.,\mu) \in \overline{C}_\alpha([0,T]\times\overline{B_1})$, $\partial_\mu^l v(s,.,\mu) \in C^{2+\alpha}(\overline{B_1})$, $\partial_s\partial_\mu^l v(s,.,\mu) \in C^{2+\alpha}(B_1)\cap C^0(\overline{B_1})$ for each $l \geq 0$.

Fix $\mu_0 \in \mathbf{R}$ and let $U_0(x,y,s) = u(s,x,y,\mu_0)$, $U_h(x,y,s) = u(s,x,y,\mu_0+h)$, $V_0(x,y,s) = v(s,x,y,\mu_0)$ and $V_h(x,y,s) = v(s,x,y,\mu_0+h)$ $(h \in \mathbf{R})$.

Then, from (3.4) we see that

$$\text{(3.26a)}\quad \frac{\partial}{\partial s}\left(\frac{U_h-U_0}{h}\right) - \Gamma^{-2}\Delta\left(\frac{U_h-U_0}{h}\right) + (1-4\pi^2c\mu_0^2)\left(\frac{U_h-U_0}{h}\right) = \alpha^2 2\pi\mu_0\left(\frac{V_h-V_0}{h}\right) + (4\pi^2c(2\mu_0+h)U_h + 2\pi\alpha^2V_h) \quad \text{on } B_1\times[0,T],$$

$$\text{(3.26b)}\quad \Delta\left(\frac{V_h-V_0}{h}\right) = 2\pi\mu_0\frac{\partial}{\partial s}\left(\frac{U_h-U_0}{h}\right) + (8\pi^3c\mu_0^3 - 4\pi\mu_0)\left(\frac{U_h-U_0}{h}\right) + [U_h(8\pi^3c(3\mu_0^2+3\mu_0h+h^2)-2\pi)] \quad \text{on } B_1\times[0,T],$$

$$\text{(3.26c)}\quad \left(\frac{U_h-U_0}{h}\right)(x,y,0) = 0,$$

(3.26d)
$$\frac{\partial}{\partial \nu}\left(\frac{U_h-U_0}{h}\right) = 6\pi\alpha^2\mu_0^2\int_0^s\int_{B_1}\left(\frac{U_h-U_0}{h}\right) - 2\pi\alpha^2\mu_0^2\int_{B_1}\left(\frac{U_h-U_0}{h}\right) + \left\{6\pi\alpha^2(2\mu_0+h)\int_0^s\int_{B_1}U_h - 2\pi\alpha^2(2\mu_0+h)\int_{B_1}U_h + 2\pi\alpha^2(2\mu_0+h)\int_{B_1}u_0\right\} \quad \text{on } \partial B_1\times[0,T],$$

(3.26e)
$$\left(\frac{V_h-V_0}{h}\right) = -\frac{3}{4}\mu_0\int_0^s\int_{B_1}\left(\frac{U_h-U_0}{h}\right) - \Gamma^{-2}2\pi\mu_0\left(\frac{U_h-U_0}{h}\right) + \Gamma^{-2}\mu_0\int_{B_1}\left(\frac{U_h-U_0}{h}\right) + \left\{-\frac{1}{4}\int_{B_1}u_0 - \frac{3}{4}\int_0^s\int_{B_1}U_h - 2\pi\Gamma^{-2}U_h + \Gamma^{-2}\int_{B_1}U_h\right\} \quad \text{on } \partial B_1\times[0,T].$$

Now, proceeding as before, we can prove that, for h small,

$$\| \left(\frac{U_h - U_0}{h}\right) \|_{2+\alpha} + ||| \left(\frac{V_h - V_0}{h}\right) ||| \leq$$
$$c_{\mu_0}(\|U_h\|_{2+\alpha} + |||V_h||| + \|u_0\|^0_{2+\alpha}) \leq c_{\mu_0}\|u_0\|^0_{2+\alpha}.$$

Thus we may conclude that $u(s,x,y,.)$ and $v(s,x,y,.)$ are differentiable at μ_0 and that $\partial^l_\mu u(.,\mu_0), \partial^l_\mu v(.,\mu_0)$ satisfy (b) for $l = 1$.

Furthermore, we have

$$\text{(3.27a)} \quad \frac{\partial}{\partial s}\frac{\partial u}{\partial \mu} - \Gamma^{-2}\Delta\frac{\partial u}{\partial \mu} + (1 - 4\pi^2 c\mu_0^2)\frac{\partial u}{\partial \mu} = \alpha^2 2\pi\mu\frac{\partial v}{\partial \mu} + (8\pi^2 c\mu u + 2\pi\alpha^2 v) \quad \text{on } [0,T] \times B_1 \times \mathbf{R},$$

$$\text{(3.27b)} \quad \Delta\frac{\partial v}{\partial \mu} = 2\pi\mu\frac{\partial}{\partial s}\frac{\partial u}{\partial \mu} + (8\pi^3 c\mu^3 - 4\pi\mu)\frac{\partial u}{\partial \mu} + [u(24\pi^3 c\mu^2 - 2\pi)] \quad \text{on } [0,T] \times B_1 \times \mathbf{R},$$

$$\text{(3.27c)} \quad \frac{\partial u}{\partial \mu}(0,x,y,\mu) = 0\,,$$

$$\text{(3.27d)} \quad \frac{\partial}{\partial \nu}\frac{\partial u}{\partial \mu} = 6\pi\alpha^2\mu^2 \int_0^s \int_{B_1} \frac{\partial u}{\partial \mu} - 2\pi\alpha^2\mu^2 \int_{B_1} \frac{\partial u}{\partial \mu} + \left\{12\pi\alpha^2\mu \int_0^s \int_{B_1} u - 4\pi\alpha^2\mu \int_{B_1} u + 4\pi\alpha^2\mu \int_{B_1} u_0\right\} \quad \text{on } [0,T] \times \partial B_1 \times \mathbf{R},$$

$$\text{(3.27e)} \quad \frac{\partial v}{\partial \mu} = -\frac{3}{4}\mu \int_0^s \int_{B_1} \frac{\partial u}{\partial \mu} - \Gamma^{-2} 2\pi\mu\frac{\partial u}{\partial \mu} + \Gamma^{-2}\mu \int_{B_1} \frac{\partial u}{\partial \mu} + \left\{-\frac{1}{4}\int_{B_1} u_0 - \frac{3}{4}\int_0^s \int_{B_1} u - 2\pi\Gamma^{-2} u + \Gamma^{-2}\int_{B_1} u\right\} \quad \text{on } [0,T] \times \partial B_1 \times \mathbf{R}.$$

By repeating this procedure we may conclude that (a) and (b) hold.

STEP V. *(Bounds on the μ-derivatives at $\mu = 0$).*

Consider the system (3.9) and let (for $l \geq 0$)

$$F_l(x,y,s) = \frac{1}{l!}\frac{\partial^l u}{\partial \mu^l}(s,x,y,0) \quad ,$$

$$L_l(x,y,s) = \frac{1}{l!}\frac{\partial^l w}{\partial \mu^l}(s,x,y,0).$$

It is then clear that, for $l \geq 0$,

(3.28a) $$\frac{\partial F_l}{\partial s} - \Gamma^{-2}\Delta F_l + F_l = \alpha^2 2\pi L_{l-1} + 4\pi^2 c F_{l-2} \quad \text{on } B_1 \times [0,T],$$

(3.28b) $$\Delta L_l = 2\pi^2\alpha^2 L_{l-2} + 4\pi^3 c F_{l-3} - 6\pi F_{l-1} \quad \text{on } B_1 \times [0,T],$$

(3.28c) $$F_l(x,y,0) = F_l^0(x,y) \quad \text{on } B_1,$$

(3.28d) $$\frac{\partial F_l}{\partial \nu} = 3\pi\alpha^2 \int_0^s \int_{B_1} F_{l-2} - \pi\alpha^2 \int_{B_1} F_{l-2} + \pi\alpha^2 \int_{B_1} F_{l-2}^0 \quad \text{on } \partial B_1 \times [0,T],$$

(3.28e) $$L_l = -\frac{3}{4}\int_0^s \int_{B_1} F_{l-1} - \Gamma^{-2} 4\pi F_{l-1} - \frac{1}{4}\int_{B_1} F_{l-1}^0 + \Gamma^{-2}\int_{B_1} F_{l-1} \quad \text{on } \partial B_1 \times [0,T],$$

where

(3.29) $$F_l^0(x,y) = \begin{cases} u_0(x,y), & \text{if } l = 0 \\ 0, & \text{if } l \neq 0. \end{cases}$$

First we notice that (3.28),(3.29) imply that

$$F_{2k+1} \equiv L_{2k} \equiv 0 \qquad (k \geq 0).$$

On the other hand, for $l \geq 0$,

$$\|F_l\|_{2+\alpha} \leq \gamma_1(\|L_{l-1}\|_\alpha + \|F_{l-2}\|_{2+\alpha} + \|F_l^0\|^0_{2+\alpha} + \|F_{l-2}^0\|^0_{2+\alpha}) \quad , \tag{3.30a}$$

$$|||L_{l+1}||| \leq \gamma_2(|||L_{l-1}||| + \|F_{l-2}\|_{2+\alpha} + \|F_l\|_{2+\alpha} + \|F_l^0\|^0_{2+\alpha}) \tag{3.30b}$$

for some constants γ_1, γ_2.

But then, from (3.30) we may conclude that

$$|||L_{l+1}||| \leq \gamma_2(|||L_{l-1}||| + \|F_{l-2}\|_{2+\alpha} + \|F_l^0\|^0_{2+\alpha})$$
$$+ \gamma_1\gamma_2(|||L_{l-1}||| + \|F_{l-2}\|_{2+\alpha} + \|F_l^0\|^0_{2+\alpha} + \|F_{l-2}\|^0_{2+\alpha}).$$

so that

$$|||L_{l+1}||| \leq \gamma_3(|||L_{l-1}||| + \|F_{l-2}\|_{2+\alpha} + \|F_l^0\|^0_{2+\alpha} + \|F_{l-2}\|^0_{2+\alpha}) \tag{3.31}$$

for some constant γ_3.

Let $a_l = \|F_l\|_{2+\alpha} + |||L_{l+1}|||$; then, from (3.30a) and (3.31) we get

$$a_l \leq \gamma(a_{l-2} + \|F_l^0\|^0_{2+\alpha} + \|F_{l-2}^0\|^0_{2+\alpha})$$

for some constant γ.

In particular, for $k \geq 2$

$$a_{2k} \leq \gamma a_{2(k-1)}$$

so that

$$a_{2k} \leq \gamma^{k-1} a_2. \tag{3.32}$$

But,

$$a_2 \leq \gamma(a_0 + \|F_0^0\|^0_{2+\alpha})$$
$$\leq \gamma(\gamma\|F_0^0\|^0_{2+\alpha} + \|F_0^0\|^0_{2+\alpha})$$
$$\leq (\gamma + \gamma^2)\|u_0\|^0_{2+\alpha}.$$

Hence, there exists a constant γ_0 such that

$$a_l \leq \gamma_0^l \|u_0\|_{2+\alpha}^0. \tag{3.33}$$

Finally, since $v = w + \Gamma^{-2} 2\pi\mu u$, from (3.33) we conclude that (3.7) and (3.8) hold.

Thus, the proof of lemma 3.2 is complete. □

Section 4: Existence and Uniqueness. Having proved lemma 3.2 we proceed to establish theorem 3.1.

EXISTENCE.

Write $\phi_0(x,y,\xi) = \sum_{n=0}^{\infty} a_n(x,y)\xi^n$. Then, using the fact that $\phi_0 \in A_{\eta_0}^{\alpha}$ and Cauchy's Formula, we get that $a_n \in \overline{C}_{2+\alpha}(\overline{B_1})$ and, for every $\epsilon > 0$

$$\|a_n\|_{2+\alpha}^0 \leq c_\epsilon \frac{(\eta_0+\epsilon)^n e^n}{n^n} \qquad \text{for } n \geq 1. \tag{4.1}$$

Now let $(A_n(s,x,y,\mu), B_n(s,x,y,\mu))$ be the solution of (3.4) given by lemma 3.2 when u_0 is replaced by a_n.

Finally set

$$\phi(s,x,y,\xi) = \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \sum_{k=0}^{n} \binom{n}{k} (2\pi i\xi)^k \frac{\partial^{n-k} A_n}{\partial \mu^{n-k}}(s,x,y,0) \quad , \tag{4.2a}$$

$$\Lambda(s,x,y,\xi) = \sum_{n=0}^{\infty} \frac{i}{(2\pi i)^n} \sum_{k=0}^{n} \binom{n}{k} (2\pi i\xi)^k \frac{\partial^{n-k} B_n}{\partial \mu^{n-k}}(s,x,y,0). \tag{4.2b}$$

We claim:

If η_0 is small enough, then (ϕ,Λ) defined by (4.2) is a solution of (2.13) satisfying (i)-(iv) of theorem 3.1.

Proof. We have

$$\|\phi(.,\xi)\|_{2+\alpha} \leq \sum_{n=0}^{\infty} \frac{1}{(2\pi)^n} \sum_{k=0}^{n} \binom{n}{k} (2\pi|\xi|)^k \|\frac{\partial^{n-k} A_n}{\partial \mu^{n-k}}(.,0)\|_{2+\alpha}.$$

But from (3.7) we know that

$$\|\frac{\partial^{n-k}A_n}{\partial\mu^{n-k}}(.,0)\|_{2+\alpha} \leq c_1 c_2^{n-k}(n-k)!\,\|a_n\|^0_{2+\alpha}$$

so that

$$\|\phi(.,\xi)\|_{2+\alpha} \leq c_1 \sum_{n=0}^{\infty} n! \sum_{k=0}^{n} \frac{|\xi|^k}{k!} c_2^{n-k} \|a_n\|^0_{2+\alpha}. \tag{4.3}$$

Thus, using (4.1),

$$\|\phi(.,\xi)\|_{2+\alpha} \leq c_1 c_\epsilon \left(\sum_{n=1}^{\infty} n! \sum_{k=1}^{n} \frac{|\xi|^k}{k!} c_2^{n-k} \frac{(\eta_0+\epsilon)^n e^n}{n^n} + \sum_{n=1}^{\infty} n!\, c_2^n \frac{(\eta_0+\epsilon)^n e^n}{n^n} + 1 \right)$$

and changing the order of summation,

$$\|\phi(.,\xi)\|_{2+\alpha} \leq c_1 c_\epsilon \left(\sum_{k=1}^{\infty} \frac{[(\eta_0+\epsilon)|\xi|]^k}{k!} \sum_{n\geq k} (c_2(\eta_0+\epsilon))^{n-k} \frac{n!\,e^n}{n^n} + \sum_{n=1}^{\infty} (c_2(\eta_0+\epsilon))^n \frac{n!\,e^n}{n^n} + 1 \right). \tag{4.4}$$

By using the estimate

$$\frac{n!\,e^n}{n^n} \leq c_3 n^{1/2} \quad , \tag{4.5}$$

we get

$$\|\phi(.,\xi)\|_{2+\alpha} \leq c_1 c_3 c_\epsilon \left(\sum_{k=1}^{\infty} \frac{[(\eta_0+\epsilon)|\xi|]^k}{k!} \sum_{n\geq k} (c_2(\eta_0+\epsilon))^{n-k} n^{1/2} + \sum_{n=1}^{\infty} (c_2(\eta_0+\epsilon))^n n^{1/2} + 1 \right)$$

that is

$$\|\phi(.,\xi)\|_{2+\alpha} \leq \tilde{c}_\epsilon \left(\sum_{k=1}^{\infty} \frac{[(\eta_0+\epsilon)|\xi|]^k}{k!} \sum_{n\geq 0} (c_2(\eta_0+\epsilon))^{n} (n+k)^{1/2} + \sum_{n=1}^{\infty} (c_2(\eta_0+\epsilon))^n n^{1/2} + 1 \right) \qquad (\tilde{c}_\epsilon = c_1 c_3 c_\epsilon). \tag{4.6}$$

Therefore, if $\eta_0 < 1/c_2$ and $0 < \epsilon < 1/c_2 - \eta_0$, then (4.6) implies

(4.7a) $$\|\phi(.,\xi)\|_{2+\alpha} \le \hat{c}_\epsilon(1+|\xi|)e^{(\eta_0+\epsilon)|\xi|},$$

for some constant $\hat{c}_\epsilon$.

In a similar way, but using (3.8) instead of (3.7), we can prove that

(4.7b) $$|||\Lambda(.,\xi)||| \le \tilde{\hat{c}}_\epsilon(1+|\xi|)e^{(\eta_0+\epsilon)|\xi|}.$$

Since (4.7) clearly implies (3.2) we see that (ϕ,Λ) satisfies (ii)-(iv) of theorem 3.1, and it only remains to show that (ϕ,Λ) is a solution of (2.13).

First we check (2.13c); we have

$$\begin{aligned}\phi(0,x,y,\xi) &= \sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i\xi)^k\frac{\partial^{n-k}a_n}{\mu^{n-k}}(x,y)\\ &= \sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}(2\pi i\xi)^n a_n(x,y)\\ &= \phi_0(x,y,\xi).\end{aligned}$$

Now set $A_{n,k} = \partial_\mu^{n-k}A_n(.,0)$, $\quad B_{n,k} = \partial_\mu^{n-k}B_n(.,0)$. Then from (3.4) we have

(4.8a)
$$\begin{aligned}\frac{\partial A_{n,k}}{\partial s} - \Gamma^{-2}\Delta A_{n,k} + A_{n,k} &= 4\pi^2 c(n-k)(n-k-1)A_{n,k+2}\\ &\quad + 2\pi\alpha^2(n-k)B_{n,k+1} \qquad \text{on } [0,T]\times B_1,\end{aligned}$$

(4.8b)
$$\begin{aligned}\Delta B_{n,k} &= 2\pi(n-k)\frac{\partial A_{n,k+1}}{\partial s} + 8\pi^3 c(n-k)(n-k-1)(n-k-2)A_{n,k+3}\\ &\quad - 4\pi(n-k)A_{n,k+1} \qquad \text{on } [0,T]\times B_1\end{aligned}$$

and, on $[0,T]\times\partial B_1$,

(4.8c)
$$\begin{aligned}\frac{\partial A_{n,k}}{\partial\nu} &= 6\pi\alpha^2(n-k)(n-k-1)\int_0^s\int_{B_1}A_{n,k+2}\\ &\quad - 2\pi\alpha^2(n-k)(n-k-1)\int_{B_1}A_{n,k+2} + 2\pi\alpha^2(n-k)(n-k-1)\delta_{k,n-2}\int_{B_1}a_n\end{aligned}$$

(4.8d)
$$B_{n,k} = -\frac{3}{4}(n-k)\int_0^s \int_{B_1} A_{n,k+1} - \frac{1}{4}\delta_{k,n-1}\int_{B_1} a_n - \Gamma^{-2}2\pi(n-k)A_{n,k+1} + \Gamma^{-2}(n-k)\int_{B_1} A_{n,k+1}.$$

Using (4.8a),

$$\begin{aligned}
&\frac{\partial\phi}{\partial s} - \Gamma^{-2}\Delta\phi + \phi \\
&= \sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i\xi)^k[\frac{\partial A_{n,k}}{\partial s} - \Gamma^{-2}\Delta A_{n,k} + A_{n,k}] \\
&= \sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i\xi)^k[4\pi^2 c(n-k)(n-k-1)A_{n,k+2} + 2\pi\alpha^2(n-k)B_{n,k+1}] \\
&= -c\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n-2}\binom{n}{k}(2\pi i)^{k+2}(n-k)(n-k-1)A_{n,k+2}\xi^k \\
&- \sum_{n=0}^{\infty}\frac{i}{(2\pi i)^n}\sum_{k=0}^{n-1}\binom{n}{k}(2\pi i)^{k+1}\alpha^2(n-k)B_{n,k+1}\xi^k \\
&= -c\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{j=2}^{n}\binom{n}{j-2}(2\pi i)^j(n-j+2)(n-j+1)A_{n,j}\xi^{j-2} \\
&- \alpha^2\sum_{n=0}^{\infty}\frac{i}{(2\pi i)^n}\sum_{j=1}^{n}\binom{n}{j-1}(2\pi i)^j(n-j+1)B_{n,j}\xi^{j-1} \\
&= -c\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{j=2}^{n}\binom{n}{j}(2\pi i)^j j(j-1)\xi^{j-2}A_{n,j} \\
&- \alpha^2\sum_{n=0}^{\infty}\frac{i}{(2\pi i)^n}\sum_{j=1}^{n}\binom{n}{j}(2\pi i)^j j\xi^{j-1}B_{n,j} \\
&= -c\frac{\partial^2\phi}{\partial\xi^2} - \alpha^2\frac{\partial\Lambda}{\partial\xi}
\end{aligned}$$

and therefore (2.13a) is satisfied.

Similarly, using (4.8d) we see that, on Γ_0,

$$\begin{aligned}
\Lambda &= \sum_{n=0}^{\infty}\frac{i}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i\xi)^k B_{n,k} \\
&= \sum_{n=0}^{\infty}\frac{i}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i\xi)^k[-\frac{3}{4}(n-k)\int_0^s\int_{B_1} A_{n,k+1} \\
&- \frac{1}{4}\delta_{k,n-1}\int_{B_1} a_n - \Gamma^{-2}2\pi(n-k)A_{n,k+1} + \Gamma^{-2}(n-k)\int_{B_1} A_{n,k+1}].
\end{aligned}$$

Hence,

$$\begin{aligned}\Lambda = &-\frac{3}{8\pi}\int_0^s\int_{B_1}\left(\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n-1}\binom{n}{k}(2\pi i)^{k+1}\xi^k(n-k)A_{n,k+1}\right)\\ &-\frac{1}{8\pi}\int_{B_1}\left(\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n}\binom{n}{k}(2\pi i)^{k+1}\xi^k\delta_{k,n-1}a_n\right)\\ &-\Gamma^{-2}\left(\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n-1}\binom{n}{k}(2\pi i)^{k+1}\xi^k(n-k)A_{n,k+1}\right)\\ &-\frac{\Gamma^{-2}}{2\pi}\int_{B_1}\left(\sum_{n=0}^{\infty}\frac{1}{(2\pi i)^n}\sum_{k=0}^{n-1}\binom{n}{k}(2\pi i)^{k+1}\xi^k(n-k)A_{n,k+1}\right)\\ &=\frac{3}{8\pi}\int_0^s\int_{B_1}\frac{\partial\phi}{\partial\xi}-\frac{1}{8\pi}\int_{B_1}\frac{\partial\phi_0}{\partial\xi}-\Gamma^{-2}\frac{\partial\phi}{\partial\xi}+\frac{\Gamma^{-2}}{2\pi}\int_{B_1}\frac{\partial\phi}{\partial\xi}\end{aligned}$$

so that (2.13e) also holds.

In a similar way we can show that (2.13b) and (2.13d) are satisfied, thus completing the existence proof for theorem 3.1. □

UNIQUENESS.

Let (ϕ,Λ) be a solution of (2.13) satisfying (i)-(iv) of theorem 3.1 for $\phi_0\equiv 0$. We wish to show that $\phi\equiv\Lambda\equiv 0$.

Write $\phi(s,x,y,\xi)=\sum_{n=0}^{\infty}\phi^n(x,y,s)\xi^n/n!$ and $\Lambda(s,x,y,\xi)=\sum_{n=0}^{\infty}\Lambda^n(x,y,s)\xi^n/n!$.

Then, for each $n\geq 0$,

(4.9a) $$\frac{\partial\phi^n}{\partial s}-\Gamma^{-2}\Delta\phi^n+\phi^n=-\alpha^2\Lambda^{n+1}-c\phi^{n+2}\quad\text{on } B_1\times[0,T],$$

(4.9b) $$\Delta\Lambda^n=\frac{\partial\phi^{n+1}}{\partial s}-2\phi^{n+1}-\phi^{n+3}\quad\text{on } B_1\times[0,T],$$

(4.9c) $$\phi^n(x,y,0)=0\,,$$

(4.9d) $$\frac{\partial\phi^n}{\partial\nu}=-\frac{3\alpha^2}{2\pi}\int_0^s\int_{B_1}\phi^{n+2}+\frac{\alpha^2}{2\pi}\int_{B_1}\phi^{n+2}\quad\text{on } \partial B_1\times[0,T],$$

(4.9e) $$\Lambda^n = -\frac{3}{8\pi}\int_0^s\int_{B_1}\phi^{n+1} - \Gamma^{-2}\phi^{n+1} + \frac{\Gamma^{-2}}{2\pi}\int_{B_1}\phi^{n+1} \quad \text{on } \partial B_1 \times [0,T].$$

Using (4.9a,c,d) together with the parabolic Schauder estimates we get

(4.10) $$\|\phi^n\|_{2+\alpha} \leq k_1(|||\Lambda^{n+1}||| + \|\phi^{n+2}\|_{2+\alpha})$$

for some constant k_1.

Also, using (4.9b,e) and the maximum principle

$$|||\Lambda^{n-1}||| \leq k_2(\|\phi^n\|_{2+\alpha} + \|\phi^{n+2}\|_{2+\alpha})$$

and using (4.10)

(4.11) $$|||\Lambda^{n-1}||| \leq k_3(|||\Lambda^{n+1}||| + \|\phi^{n+2}\|_{2+\alpha})$$

for some constant k_3.

For $n \geq 0$ set $\gamma_n = \|\phi^n\|_{2+\alpha} + |||\Lambda^{n-1}|||$, where $\Lambda^{-1} \equiv 0$. Then, (4.10) and (4.11) give

(4.12) $$\gamma_n \leq k_0\gamma_{n+2}, \qquad n \geq 0$$

for some constant k_0.

On the other hand, using Cauchy's Formula and part (iv) of theorem 3.1 we see that, for $m \geq 2$

$$\gamma_m \leq c_\epsilon[m!\,\frac{(\eta_0+\epsilon)^m e^m}{m^m} + (m-1)!\,\frac{(\eta_0+\epsilon)^{m-1}e^{m-1}}{(m-1)^{m-1}}]$$

and from (4.5)

(4.13) $$\gamma_m \leq \tilde{c}_\epsilon(\eta_0+\epsilon)^m m^{1/2} \quad \text{for } \epsilon > 0, m \geq 2.$$

Fix $n \geq 0$; then from (4.12)

(4.14) $$\gamma_n \leq k_0^l\,\gamma_{n+2l}, \quad \forall l \geq 0$$

so that from (4.13),(4.14) we obtain

$$\gamma_n \le \tilde{c}_\epsilon k_0^l (\eta_0 + \epsilon)^{n+2l}(n+2l)^{1/2}, \quad l \ge 1. \tag{4.15}$$

Thus, for every $l \ge 1$

$$\gamma_n \le \tilde{c}_\epsilon (\eta_0 + \epsilon)^n [((\eta_0+\epsilon)^2 k_0)^l (n+2l)^{1/2}]. \tag{4.16}$$

Hence, if $\eta_0 < (1/k_0)^{1/2}$ and $0 < \epsilon < (1/k_0)^{1/2} - \eta_0$, letting $l \longrightarrow \infty$ in (4.16) we get

$$\gamma_n = 0, \quad \forall n \ge 0$$

and therefore

$$\phi \equiv \Lambda \equiv 0.$$

Thus, the proof of theorem 3.1 is complete, provided we choose $\eta_0 < \min(1/c_2, (1/k_0)^{1/2})$. (Recall that c_2 is the constant in (3.7),(3.8)). □

REMARK. The analyticity assumption on ϕ_0 as a function of ξ is very restrictive. However, since part of system (2.13) displays the features of a backward parabolic equation it seems quite natural to make this assumption. In fact, we can produce examples for the system (2.13a,b,c), although with somewhat different (and simpler) boundary conditions, for which the analyticity assumption on ϕ_0 in ξ cannot be dropped. For example, consider the boundary conditions

$$\frac{\partial \phi}{\partial \nu} = -\frac{1}{2\pi}\int_{B_1} \frac{\partial^2 \phi}{\partial \xi^2} \qquad \text{for } (x,y) \in \partial B_1, \tag{4.17a}$$

$$\frac{\partial \Lambda}{\partial \xi} = \frac{3}{1-\alpha^2}\left(\frac{\partial^2 \phi}{\partial \xi^2} + \phi\right) - \Gamma^{-2}\frac{\partial^2 \phi}{\partial \xi^2} \qquad \text{for } (x,y) \in \partial B_1. \tag{4.17b}$$

It is easy to check that the proof of theorem 3.1 implies that given $\phi_0 \in A_\eta^\alpha$ with η small enough, ϕ and $\partial_\xi \Lambda$ are uniquely determined among the functions satisfying properties (i)-(iv) of this theorem.

Now assume that ϕ_0 is given as the (unique) rapidly decaying solution of

$$\Delta\phi_0 + \frac{\partial^2\phi_0}{\partial\xi^2} = 0 \qquad (x,y) \in B_1, \quad \xi \in \mathbf{R},$$

$$\phi_0 = g(\xi) \qquad (x,y) \in \partial B_1, \quad \xi \in \mathbf{R}.$$

where g is an even function in $\mathcal{S}$, the Schwartz class, and let $u_0(x,y,\mu) = \phi_0(x,y,.)^\frown(\mu)$.

If we Fourier transform (2.13a,b) we get (3.3a,b) (see Section 3), and then we can easily derive the following equation for $u(s,x,y,\mu) = \phi(s,x,y,.)^\frown(\mu)$:

$$(4.18) \quad \partial_s\Delta u - \Gamma^{-2}\Delta^2 u - 4\pi^2 c\mu^2\Delta u + \Delta u = -\alpha^2(2\pi i\mu)[(2\pi i\mu)\partial_s u - 4\pi i\mu u - c(2\pi i\mu)^3 u].$$

If we look for a solution u of the form $u(s,x,y,\mu) = u_0(x,y,\mu)K(s,\mu)$, we find, using the fact that $\Delta u_0 = 4\pi^2\mu^2 u_0$, that K must satisfy

$$(4.19) \qquad \partial_s K = \frac{1}{1-\alpha^2}(4\pi^2\mu^2\Gamma^{-2} + 4\pi^2 c\mu^2 + 4\pi^2 c\alpha^2\mu^2 - 1 - 2\alpha^2)K.$$

If we assume that $\alpha^2(s) < 1$ then

(4.20)
$$K(s,\mu) = exp\left\{\int_0^s \frac{1}{1-\alpha^2(t)}(4\pi^2\mu^2\Gamma^{-2}(t) + 4\pi^2 c\mu^2 + 4\pi^2 c\alpha^2(t)\mu^2 - 1 - 2\alpha^2(t))\,dt\right\} = \sum_{n=0}^{\infty} a_{2n}(s)\mu^{2n}$$

where the a_n's are the coefficients in the power series expansion of $K(s,.)$, and we get

$$u(s,x,y,\mu) = u_0(x,y,\mu)K(s,\mu).$$

Finally, notice that

$$K(s,.)^\vee(\zeta) = \sum_{n=0}^{\infty} \frac{a_{2n}(s)}{(-4\pi^2)^n}\partial_\zeta^{2n}\delta_0$$

where δ_0 is the Dirac delta, so that

$$\phi(s,x,y,\xi) = \sum_{n=0}^{\infty} \frac{a_{2n}(s)}{(-4\pi^2)^n} \left(\partial_\zeta^{2n}\delta_0\, ,\, \phi_0(x,y,\zeta-\xi)\right) \tag{4.21}$$
$$= \sum_{n=0}^{\infty} \frac{a_{2n}(s)}{(-4\pi^2)^n} \partial_\xi^{2n}\phi_0(x,y,\xi).$$

But in order for the series in (4.21) to converge we *must* require that ϕ_0 be entire in ξ (see e.g. *Friedman* [5], pp. 158–159).

Section 5: Properties of the solution.

THEOREM 5.1. *Let $\phi_0 \in A^{\alpha}_{\eta_0}$ where $0 < \alpha < 1$ and η_0 is as in theorem 3.1. Let (ϕ, Λ) be the solution of (2.13) given in theorem 3.1. Then:*

(a) *If ϕ_0 is periodic in ξ (i.e. $\phi_0(.,\xi+p) = \phi_0(.,\xi)$ for some $p \in \mathbf{R}$ and all $\xi \in \mathbf{R}$) then also ϕ and Λ are periodic in ξ, with the same period p.*

(b) *Let $m \geq 2$ and assume that*

$$\|\phi_0(.,\xi)\|^0_{2+\alpha} \leq \frac{c_0}{(1+|\xi|)^m} \tag{5.1}$$

for some constant $c_0 > 0$ ($\xi \in \mathbf{R}$). Then

$$\|\phi(.,\xi)\|_{2+\alpha} + |||\Lambda(.,\xi)||| = o\big((1+|\xi|)^{2-m}\big) \text{ as } \xi \longrightarrow \infty. \tag{5.2}$$

THEOREM 5.2. *Let ϕ_0, ϕ and Λ be as in theorem 5.1. If we further assume that $\phi_0(x,y,\xi) = \phi_0(x,y,-\xi)$ and*

$$\int_{\mathbf{R}} \|\frac{\partial \phi_0}{\partial \xi}\|^0_{2+\alpha}\, d\xi < \infty \tag{5.3}$$

then,

$$\phi(s,x,y,\xi) = \phi(s,x,y,-\xi), \quad \Lambda(s,x,y,\xi) = -\Lambda(s,x,y,-\xi) \tag{5.4}$$

and

$$\|\frac{\partial \phi}{\partial \xi}(.,\xi)\|_{2+\alpha} + |||\frac{\partial \Lambda}{\partial \xi}(.,\xi)||| = o(1) \text{ as } \xi \longrightarrow \infty. \tag{5.5}$$

In order to prove theorems 5.1 and 5.2 we shall use the following

THEOREM (PALEY-WIENER). *Let K be a compact, convex and balanced subset of $\mathbf{R}^n$, and let $I_K(\eta) = \sup_{x\in K} |\langle \eta, x\rangle|$ (where $|\langle\eta, x\rangle| = \eta.x = \sum_{i=1}^{n} \eta_i x_i$).*

Let $T \in S'$ (i.e. T is a tempered distribution). Then the following are equivalent:

(i) $supp\, T \subset K$.

(ii) $\hat{T}$ *has an entire holomorphic extension to* $\mathbf{C}^n$ *so that, for some* $m \geq 0$

$$\left|\hat{T}(z)\right| \leq c(1+|z|)^m exp\, I_K(\Im(z)).$$

(Actually m can be taken to be the order of T.)

(For a proof see e.g. [5], pp. 145–146).

Proof of theorem 5.1. Notice that (a) follows immediately from the uniqueness of solutions: in fact, if we set $\tilde{\phi}(s,x,y,\xi) = \phi(s,x,y,\xi+p)$, $\tilde{\Lambda}(s,x,y,\xi) = \Lambda(s,x,y,\xi+p)$ it is clear that $(\tilde{\phi}, \tilde{\Lambda})$ is a solution of (2.13) satisfying (i)-(iv) of theorem 3.1, so that, by uniqueness, we must have $\tilde{\phi} \equiv \phi$, $\tilde{\Lambda} \equiv \Lambda$.

We now turn to the proof of (b). Clearly, $u_0(x,y,\mu) = \phi_0(x,y,.)\hat{}(\mu)$ satisfies

(5.6a) $$u_0(.,\mu) \in \overline{C}_{2+\alpha}(\overline{B_1}) \quad ,$$

(5.6b) $$u_0(x,y,.) \in C^{m-2}(\mathbf{R}).$$

Also, an application of the Paley-Wiener theorem shows that $u_0(x,y,.)$ is compactly supported and that, for each $(x,y) \in \overline{B_1}$

(5.6c) $$supp\, u_0(x,y,.) \subset I_0 = \{\, \mu \mid |\mu| \leq \eta_0 \,\}.$$

We may then consider the Fourier transformed system (3.3) with initial data $u_0(x,y,\mu)$. Following the proof of lemma 3.2 we see that there exists a solution (u,v) of (3.3) satisfying:

$$u(s,x,y,.),\ v(s,x,y,.) \in C_0^{m-2}(\mathbf{R}),$$

$$\frac{\partial^l u}{\partial \mu^l}(.,\mu) \in \overline{C}_{2+\alpha}([0,T] \times \overline{B_1}) \qquad (0 \le l \le m-2),$$

$$\frac{\partial^l v}{\partial \mu^l}(.,\mu) \in \overline{C}_{\alpha}([0,T] \times \overline{B_1}) \qquad (0 \le l \le m-2),$$

$$\frac{\partial^l v}{\partial \mu^l}(s,.,\mu) \in C^{2+\alpha}(\overline{B_1}) \qquad (0 \le l \le m-2),$$

$$\frac{\partial}{\partial s}\frac{\partial^l v}{\partial \mu^l}(s,.,\mu) \in C^{2+\alpha}(B_1) \cap C^0(\overline{B_1}) \qquad (0 \le l \le m-2).$$

Furthermore, for each $(x,y) \in \overline{B_1}$, $s \in [0,T]$

$$\text{(5.7)} \qquad supp\, u(s,x,y,.) \subset I_0 \text{ and } supp\, v(s,x,y,.) \subset I_0.$$

Then, if we set

$$\tilde{\phi}(s,x,y,\xi) = u(s,x,y,.)^{\vee}(\xi) \text{ and } \tilde{\Lambda}(s,x,y,\xi) = v(s,x,y,.)^{\vee}(\xi)$$

we see that $(\tilde{\phi}, \tilde{\Lambda})$ is a solution of (2.13) satisfying (5.2). But yet another application of the Paley-Wiener theorem (using (5.7)) implies that $(\tilde{\phi}, \tilde{\Lambda})$ satisfy (i)-(iv) of theorem 3.1. Thus, by uniqueness, $(\tilde{\phi}, \tilde{\Lambda}) = (\phi, \Lambda)$, and this completes the proof of (b). □

Proof of theorem 5.2. If $\phi_0(x,y,\xi) = \phi_0(x,y,-\xi)$, set $\tilde{\phi}(s,x,y,\xi) = \phi(s,x,y,-\xi)$ and $\tilde{\Lambda}(s,x,y,\xi) = -\Lambda(s,x,y,-\xi)$. It is easily checked then that $(\tilde{\phi}, \tilde{\Lambda})$ is a solution of (2.13) satisfying (i)-(iv) of theorem 3.1 and hence, by uniqueness, we obtain $\tilde{\phi} \equiv \phi$ and $\tilde{\Lambda} \equiv \Lambda$. This proves (5.4).

Finally, (5.5) may be proved using an argument similar to that used for proving (5.2). □

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