APPROXIMATION OF HANKEL OPERATORS: TRUNCATION ERROR IN AN H^∞ DESIGN METHOD

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Introduction

Many fashionable methods of designing filters, particularly in the context of control systems, make use of theorems giving the best H^∞ approximation to an L^∞ function in terms of singular vectors of a Hankel operator: such results may be found in [AAK], [C], [G]. Instances are the "H^∞ control" solutions to the problems of sensitivity minimization and robust stabilization [F]. Another example is the H^∞ disc method, which is applied to control problems in [H1] and to gain equalization in circuits in [H2]. To implement these methods it is necessary to represent Hankel operators numerically, a task for which several methods have been proposed [GR, SB, Y3, T, Gl]. Two of these have gained currency. The most straightforward has been developed by Trefethen [T], and christened by him the *Caratheodory-Fejér method*. Mathematically a Hankel operator, which acts between two infinite-dimensional Hilbert spaces, can be represented by an infinite matrix

$$\begin{bmatrix} G_{-1} & G_{-2} & G_{-3} & \cdots \\ G_{-2} & G_{-3} & G_{-4} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \qquad (1)$$

where

$$G(z) \sim \sum_{n=-\infty}^{\infty} G_n z^n \qquad (2)$$

is the Fourier series of an L^∞ function G on the unit circle $\partial\mathbb{D}$. G is called a *symbol* of the operator. The Caratheodory-Fejér method simply approximates the Hankel operator by taking the $N \times N$ truncation of the matrix (1) for some large integer N.

The second popular method uses a state space representation. The requisite formulae were derived by Glover [Gl], and numerous authors have reported favourably on their implementation [F]. However, the state space method is not directly comparable with the Caratheodory-Fejér method as it has a different starting point: it presupposes that a state space model of the symbol G (or "plant") is available. In some applications (for instance, the H^∞ disc method) the data of the problem (i.e. G) are supplied as function values on a grid, and so Glover's representation can only be used after a preliminary stage of identification and realization. This adds to the computational burden and introduces error. The Caratheodory–Fejér method replaces the identification/ realization stage by some procedure for extracting Fourier coefficients from data – typically the fast Fourier transform.

In this paper we propose and analyse a hybrid method which is close in spirit to the Caratheodory–Fejér method but does also involve an element of identification. The reason for our innovation is that in practice the Caratheodory–Fejér method has proved inadequate for the design application under study. We can illustrate simply why this should be so by looking at the case of rational functions G. If all the unstable poles of G are well away from the unit circle then the negative Fourier coefficients of G decay quickly to zero. In this case a modestly-sized truncation of (1) affords a reasonable approximation to the true Hankel operator, and the Caratheodory–Fejér method works well. However poles close to the circle, manifesting themselves in "spikes" in the graph of $G(e^{i\theta})$, give rise to slowly decaying Fourier coefficients. In such circumstances it will require an inordinately large truncation of the Hankel matrix (1) to achieve a satisfactory representation. We propose a modified form of truncation which is intended to cope better with rapidly-varying G.

To address the problem of bad poles of the symbol G (or slow modes of the plant, in system-theoretic language) we begin with an identification stage. From the data we derive estimates $\beta_1,\ldots,\beta_m$ for the true poles $\alpha_1,\ldots,\alpha_m$ of G. Using the β_j we construct, for $N \in \mathbb{N}$, an $m + N$-dimensional subspace E of H^2 with the property that if the β_j are close to the α_j and N is large, then the restriction to E of the Hankel operator H_G is a good approximation to H_G for the purpose of computing the best H^∞ approximation to G.

There are two ways to assess the potential of our algorithm – numerical experimentation and mathematical analysis. The subject of this paper is a truncation error analysis. Our main result (Theorem 3) takes a fixed rational G and estimates the L^∞-distance from the true best H^∞ approximation $\hat{G}$ of G to the function $\tilde{G}$ computed by our modified truncation algorithm. We show:

Suppose that the highest singular value of the Hankel operator corresponding to G is simple. Suppose further that the only poles of G in the annulus $\{z \in \mathbb{C}: r < |z| < 1\}$ *are* $\alpha_1,\ldots,\alpha_m$. *Then there exist A, B > 0 such that if* $\tilde{G}$ *is the computed approximation obtained from the modified truncation algorithm based on* $\beta_1,\ldots,\beta_m$ *and* $N \in \mathbb{N}$ *then, for N sufficiently large, and for* β_j *sufficiently close to* α_j

$$\|\hat{G} - \tilde{G}\|_\infty < A \sum_{j=1}^{m} |\alpha_j - \beta_j| + Br^{N/2}.$$

This provides some evidence in favour of modified truncation as a way of computing with Hankel operators. Further evidence comes from numerical testing in joint work with P G Spain [HSY]. We have tried the method with the "candidate poles" β_j at varying distances from the true poles α_j. Good guesses for the β_j improve substantially on the Caratheodory–Fejér method. The closer the α_j to the circle, the better the β_j

must be guessed for the algorithm to work well. This is consistent with the estimate in Theorem 3; it is unfortunate, but we believe it is intrinsic to the problem*.

In the event that $\alpha_1, \ldots, \alpha_m$ are *all* the unstable poles of G and $\beta_j = \alpha_j$ (the poles are known exactly) Theorem 3 shows that the truncation error is zero. One can view this as the case of perfect knowledge, or alternatively as a truly rational plant generating data uncontaminated by noise, from which the denominator of the plant may be found precisely. Computational methods for this case have been given before [SB, Y3, Gr]: the case of limited or noisy data seems to us to be at least as important. In particular it occurs in the H^∞ disc method. For our purpose there is no point in trying to guess *all* the unstable poles of G: poles of magnitude less than, say, 0.9 cause no difficulties for the Caratheodory–Fejér method, and so we handle them in the same way. This is the reason for the integer N in our theorem above. The poles close to the unit circle are the ones which cause trouble, but they are also, paradoxically, the ones which are most easily detected from the inspection of $G(e^{i\theta})$.

The modified truncation approach will only be advantageous in conjunction with a good way of tracking the bad poles of G directly from data (i.e. of finding reasonable approximations β_j to the true poles of G near the unit circle). This will be the subject of a future article [HSY]. Even quite crude tracking methods enable us to improve substantially on Caratheodory–Fejér. This is not surprising, as the latter is equivalent to choosing the β_j all to be zero.

For some applications it will be more important to obtain near–optimal performance rather than to get close to the true optimum (in other words, to find $\tilde{G} \in H^\infty$ such that $\|G - \tilde{G}\|_\infty$ is near its minimum rather than such that $\|\hat{G} - \tilde{G}\|_\infty$ is small). Nevertheless the issue of representing Hankel operators satisfactorily for computation remains, and the present results may show the way for convergence theorems for related computational tasks.

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* It would be very interesting to compare our estimates here to comparable estimates which one could do on Glover's algorithm. Suppose the state equations input to Glover's algorithm are not perfectly determined; we have nominal values for the state space operators and are given bounds δ_j on the error in λ_j, the system eigenvalues near the circle. How much error does this produce in the solution $\hat{c}$ to the approximation problem? It would be interesting to compute this error and compare it to that in Corollary 2. That would give an idea of how our algorithm would compare to Glover's if his state space coefficients are actually estimated from data on a grid.

1. Hankel operators and their restrictions

We consider the Nehari problem corresponding to discrete-time systems. That is, we suppose given a function G in the space L^∞ of essentially bounded Lebesgue measurable complex-valued functions on the unit circle $\partial\mathbb{D}$. The problem is to find a function $\hat{G} \in H^\infty$, the space of bounded analytical functions in the open unit disc $\mathbb{D}$, such that the L^∞ norm $\|G - \hat{G}\|_\infty$ is minimized. One of the Adamyan-Arov-Krein theorems gives the solution in terms of the Hankel operator $H_G : H^2 \longrightarrow L^2 \ominus H^2$ defined by

$$H_G x = P_-(Gx), \qquad x \in H^2.$$

Here L^2 is the space of square-integrable functions on $\partial\mathbb{D}$ with its usual inner product and H^2 is the Hardy space on the unit disc (see [F] or [Ho]). P_- is the orthogonal projection operator from L^2 to $L^2 \ominus H^2$. We say that v is a *maximizing vector* for an operator T if $v \neq 0$ and $\|Tv\| = \|T\|\,\|v\|$. The AAK theorem in question asserts that if H_G has a maximizing vector (and so in particular if G is continuous on $\partial\mathbb{D}$) then the best approximation $\hat{G}$ is unique and is given by

$$\hat{G} = G - \frac{H_G v}{v} \qquad (1.1)$$

for any maximizing vector v of H_G. An exposition of this theorem may be found in [Y1].

The *standard bases* in H^2, $L^2 \ominus H^2$ are the orthonormal sequences $1, z, z^2, \ldots$ and $z^{-1}, z^{-2}, z^{-3}, \ldots$, respectively, or when expressed in terms of their restrictions to $\partial\mathbb{D}$, $1, e^{i\theta}, e^{i2\theta}, \ldots$ and $e^{-i\theta}, e^{-i2\theta}, \ldots$. It is easy to check that the matrix of H_G with respect to these bases is given by (1) - (2). Finding a maximizing vector is a singular value problem: to solve this problem computationally we need to represent H_G, or at least an approximation to H_G, by means of a finite matrix. The Caratheodory-Fejér method works with the matrix of the compression of H_G acting from span $\{1, z, \ldots, z^{N-1}\}$ to span $\{z^{-1}, z^{-2}, \ldots, z^{-N}\}$. This is the first approximation one would think of, but since the two subspaces are not tuned in any way to the characteristics of G it is not likely to be the best one can do. A simple example will make this plain.

Consider the function

$$G(z) = \frac{1}{z - a}$$

where $a \in \mathbb{D}$. It is easy to show that H_G is a rank 1 operator. Its cokernel is

$$(\mathrm{Ker}\ H_G)^\perp = \mathrm{span}\ \{ (1 - \bar{a}z)^{-1} \}.$$

Since the maximizing vectors of any operator lie in its cokernel we should ideally like to use the restriction of H_G to a space containing $(\mathrm{Ker}\ H_G)^\perp$: we could then find the maximizing vector v without any truncation error. Failing this we should look for a finite-dimensional restriction to a subspace which is in some way close to $(\mathrm{Ker}\ H_G)^\perp$.

In the example we can readily calculate that the cosine of the angle between the 1-dimensional space (Ker $H_G)^\perp$ and span $\{1, z, \ldots, z^{N-1}\}$ is $1 - |a|^{2N}$, so that if $|a|$ is very close to 1 the spaces are nearly perpendicular until N becomes large. Experience confirms that it requires enormously large truncations to get satisfactory results if G has a pole of modulus, say, 0.98.

We can do better by attempting to guess the space (Ker $H_G)^\perp$ and working with the restriction of H_G to this guess. Now (Ker $H_G)^\perp$ is determined by the poles of G. That is, if G is expressible in the form f/p where $f \in H^\infty$ and p is a polynomial whose zeros lie in $\mathbb{D}$ then (Ker $H_G)^\perp \subseteq H^2 \ominus pH^2$. For if $x \in pH^2$, say $x = py$ with $y \in H^2$, then

$$H_G x = P_-(Gpy) = P_-(fy) = 0,$$

i.e. $x \in$ Ker H_G. This motivates the algorithm which we call the

Modified truncation algorithm.

1. Guess a polynomial q whose zeros are close to the poles of G in $\mathbb{D}$ close to $\partial\mathbb{D}$.
2. Pick a positive integer N.
3. Form

   (a) the restriction operator

   $$\dot{H}_G = H_G|H^2 \ominus z^N qH^2$$

   (b) the compression operator

   $$\ddot{H}_G = P_{(1/z^N q)H^2 \ominus H^2} \; H_G|H^2 \ominus z^N qH^2$$

4. Compute a maximizing vector v of $\dot{H}_G$ or $\ddot{H}_G$ and form

   (a) $\tilde{G} = G - \dfrac{\dot{H}_G v}{v}$, or

   (b) $\tilde{G} = G - \dfrac{\ddot{H}_G v}{v}$

Then $\tilde{G}$ is an approximation to the closest $\hat{G}$ in H^∞ to G with respect to the L^∞ norm.

While this algorithm is stated at a high level, the (b) track of it is finite-dimensional and so can be implemented. One performs Gram-Schmidt orthogonalization to the basis

$$1, z, z^2, \ldots, z^{N-1}, \frac{1}{1-\bar{\beta}_1 z}, \ldots, \frac{1}{1-\bar{\beta}_m z}$$

to obtain an orthonormal basis for $H^2 \ominus z^N qH^2$, and thereby obtains a matrix representation of $\ddot{H}_G$. The details will be published separately.

If the zeros of q are close to the zeros of p, then it is plausible to expect that $H^2 \ominus qH^2$ should come close to containing (Ker $H_G)^\perp$, and hence that the restriction $\dot{H}_G$ and compression $\ddot{H}_G$ should be satisfactory approximations to H_G for the purpose of computing $\hat{G}$. Our goal is to give these imprecise notions a solid analytical foundation. We consider only the restriction procedure, since we believe that it gives a good indication of what happens with compression as well. The key problem is

Error analysis problem

Given $G \in L^\infty$ of the form f/p, where $f \in H^\infty$ and p is a polynomial whose zeros lie in $\mathbb{D}$, and given a polynomial q whose zeros are close to those of p, estimate the error (in the L^∞ norm) incurred in using $H_G | H^2 \ominus qH^2$ instead of H_G to compute the best H^∞ approximation $\hat{G}$ to G.

In other words, we seek a bound for $\|\hat{G} - \tilde{G}\|_\infty$ where

$$\tilde{G} = G - \frac{H_G v}{v}$$

and v is a maximizing vector for $H_G | H^2 \ominus qH^2$.

We shall obtain such a bound, albeit one involving quantities which cannot be computed explicitly. Our result is thus a convergence theorem rather than a true error estimate.

2. Singular values of projection operators

One ingredient of our analysis is to show that if p is close to q then the spaces $H^2 \ominus pH^2$ and $H^2 \ominus qH^2$ are so close that any maximizing vector for $H_G | H^2 \ominus qH^2$ is near-maximizing for $H_G | H^2 \ominus pH^2$. The relevant sense of closeness of the two spaces is that the orthogonal projection

$$P_{H^2 \ominus pH^2} | H^2 \ominus qH^2$$

from $H^2 \ominus qH^2$ onto $H^2 \ominus pH^2$ have all singular values close to 1.

Theorem 1 Let $n \geq m \geq 1$ and let

$$p(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_m),$$
$$q(z) = (z - \beta_1)(z - \beta_2) \ldots (z - \beta_n)$$

where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathbb{D}$. Then

$$\underline{\sigma}(P_{H^2 \ominus pH^2} | H^2 \ominus qH^2)^2 \geq 1 - \left\{ \sum_{i=1}^{m} \left| \frac{\alpha_i - \beta_i}{1 - \bar{\beta}_i \alpha_i} \right| \right\}^2$$

where $\underline{\sigma}(.)$ denotes the smallest singular value of an operator.

Proof. For brevity let

$$R = P_{H^2 \ominus pH^2} | H^2 \ominus qH^2 .$$

By definition $\underline{\sigma}(R)$ is the smallest eigenvalue of $(RR^*)^{\frac{1}{2}}$. Let

$$\upsilon(z) = \prod_{i=1}^{n} \frac{z - \beta_i}{1 - \bar{\beta}_i z}$$

and let

$$P: H^2 \longrightarrow H^2 \ominus pH^2, \qquad Q: H^2 \longrightarrow H^2 \ominus qH^2$$

be orthogonal projection operators, so that P^*, Q^* are the corresponding injections into H^2. Then $R = PQ^*$ and so

$$RR^* = PQ^*QP^*.$$

Now Q^*Q is the Hermitian projection on H^2 with range $H^2 \ominus qH^2$, and we can give an alternative expression for this operator in terms of υ. Let S denote the forward shift operator on H^2 (i.e. $(Sf)(z) = zf(z)$), so that $\upsilon(S)$ is multiplication by the inner function υ and is an isometry. Thus $\upsilon(S)^*\upsilon(S) = I$ and so $\upsilon(S)\upsilon(S)^*$ is a Hermitian projection with range $\upsilon H^2 = qH^2$. Thus $I - \upsilon(S)\upsilon(S)^*$ is the Hermitian projection with range $H^2 \ominus qH^2$, i.e.

$$Q^*Q = I - \upsilon(S)\upsilon(S)^*,$$

and so

$$RR^* = P(I - \upsilon(S)\upsilon(S)^*)P^*.$$

PP^* is the identity operator on $H^2 \ominus pH^2$, and since $H^2 \ominus pH^2$ is invariant under $\upsilon(S)^*$ we have $PP^*\upsilon(S)^*P^* = \upsilon(S)^*P^*$ and so

$$RR^* = I_{H^2 \ominus pH^2} - P\upsilon(S)P^*P\upsilon(S)^*P^*.$$

Now

$$P\upsilon(S)P^* = \upsilon(PSP^*) = \upsilon(S_p)$$

where

$$S_p = PSP^*$$

is the compression of S to $H^2 \ominus pH^2$, known as the Sarason operator after [S]. Hence

$$R^*R = I - \upsilon(S_p)\upsilon(S_p)^* \in \mathcal{L}(H^2 \ominus pH^2).$$

It follows that

$$\underline{\sigma}(R)^2 = 1 - \|\upsilon(S_p)\|^2 \qquad (2.2)$$

To estimate $\|\upsilon(S_p)\|$ we use an ingenious observation due to the late Constantin Apostol in an oral communication. This useful result was published (with his consent) in [P–Y] and extended in [Y2]. The proof is so short we repeat it here.

Lemma 1 Let $T_1, \dots, T_m$ be $m \times m$ upper triangular matrices with $\|T_i\| < 1$, $1 < i < m$. Let the ith diagonal entry of T_i have modulus r_i, $1 < i < m$. Then

$$\|T_1 T_2 \dots T_m\| < \sum_{i=1}^{m} r_i.$$

Proof. The assertion is true when $m = 1$. Suppose it holds for $m - 1$. Partition T_i as

$$T_i = \begin{bmatrix} R_i & * \\ 0 & \lambda_i \end{bmatrix},$$

so that $|\lambda_m| = r_m$. Note that

$$T_1 T_2 \dots T_m = \begin{bmatrix} R_1 \dots R_{m-1} & 0 \\ 0 & 0 \end{bmatrix} T_m + T_1 \dots T_{m-1} \begin{bmatrix} 0 & 0 \\ 0 & \lambda_m \end{bmatrix}.$$

On taking norms and using the triangle inequality we have

$$\begin{aligned} \|T_1 \dots T_m\| &\leq \|R_i \dots R_{m-1}\| \, \|T_m\| + \|T_1 \dots T_{m-1}\| \, |\lambda_m| \\ &\leq \sum_1^{m-1} r_i \,.1 + 1.r_m = \sum_1^m r_i \,, \end{aligned}$$

establishing the result by induction. In fact the proof shows that the inequality even holds with the operator norm replaced by the trace norm.

To use the lemma we write

$$\upsilon(S_p) = T_1 \dots T_n$$

where

$$T_i = (S_p - \beta_i I)(I - \bar{\beta}_i S_p)^{-1}.$$

It is easy to check that the eigenvalues of S_p are the zeros of p, so that for a suitable choice of orthonormal basis in $H^2 \ominus pH^2$ S_p will be represented by an upper triangular matrix with diagonal entries $\alpha_1, \dots, \alpha_m$. Thus T_i is upper triangular with ith diagonal entry $(\alpha_i - \beta_i)(1 - \bar{\beta}_i\alpha_i)^{-1}$. Since S_p is a contraction T_i is also, and so Apostol's Lemma gives

$$\|\upsilon(S_p)\| \leq \|T_1 \dots T_m\| \leq \sum_{i=1}^m \left| \frac{\alpha_i - \beta_i}{1 - \bar{\beta}_i \alpha_i} \right|$$

Combining this inequality with (2.2) gives the desired result (2.1).

Lemma 2 Let p, q be as in Theorem 1 and let $G \in p^{-1}H^\infty$. Then

$$\|H_G | H^2 \ominus qH^2\| \geq \underline{\sigma}(P_{H^2 \ominus pH^2} | H^2 \ominus qH^2) \, \|H_G\|.$$

Proof. Since H_G has finite rank it has a maximizing vector $x \neq 0$. Necessarily $x \in (\text{Ker } H_G)^\perp \subseteq H^2 \ominus pH^2$. $P_{H^2 \ominus pH^2} | H^2 \ominus qH^2$ maps its domain surjectively onto its range, and hence there exists $y \neq 0$ in $H^2 \ominus qH^2$ such that $Py = x$ (P being as in the proof of Lemma 1). Then

$$\begin{aligned} \|H_G y\| &= \|H_G P y\| = \|H_G x\| = \|H_G\| \, \|x\| \\ &= \|H_G\| \, \|Py\| \geq \underline{\sigma}(P_{H^2 \ominus pH^2} | H^2 \ominus qH^2) \, \|H_G\| \, \|y\|. \end{aligned}$$

3. Computing with near-maximizing vectors

The upshot of the preceding section is that, for polynomials q sufficiently close to the denominator of G, $\|H_G | H^2 \ominus qH^2\|$ is close to $\|H_G\|$. It follows that a maximizing vector for the former operator is nearly maximizing for H_G. We shall make this observation precise and examine its implications for the purpose of computing $\hat{G}$. Let us say that a vector x is *ϵ-maximizing* for an operator T if $x \neq 0$ and

$$\|Tx\| \geq (1 - \epsilon)\|T\|\,\|x\|.$$

Lemma 3 Let p, q and G be as in Lemma 2 and write

$$M = \sum_{i=1}^{m} \left|\frac{\alpha_i - \beta_i}{1 - \bar{\beta}_i \alpha_i}\right| .$$

Let v be a maximizing vector for $H_G | H^2 \ominus qH^2$. Then v is ϵ-maximizing for H_G, where

$$\epsilon = 1 - \surd(1 - M^2).$$

Proof. Using Theorem 1 and Lemma 2 we have $v \neq 0$ and

$$\begin{aligned} \|H_G v\| &= \|H_G | H^2 \ominus qH^2\|\,\|v\| \\ &\geq \underline{\sigma}(P_{H^2 \ominus qH^2} | H^2 \ominus pH^2)\,\|H_G\|\,\|v\| \\ &\geq \surd(1 - M^2)\,\|H_G\|\,\|v\| \\ &= (1 - \epsilon)\,\|H_G\|\,\|v\| \end{aligned}$$

where $1 - \epsilon = \surd(1 - M^2)$.

Lemma 4 Let $G \in L^\infty$, $0 < \epsilon < 1$ and let v be an ϵ-maximizing vector for H_G. Let $\hat{G}$ be a best approximation in H^∞ to G. Then

$$(G - \hat{G})v = H_G v + \eta$$

where $\eta \in H^2$ and

$$\|\eta\| \leq \surd(2\epsilon - \epsilon^2)\|H_G\|\,\|v\|. \qquad (3.1)$$

Proof. Let $P_+ \colon L^2 \longrightarrow H^2$ be the orthogonal projection operator. We take

$$\eta = P_+((G - \hat{G})v).$$

Then

$$\begin{aligned} (G - \hat{G})v &= P_-(G - \hat{G})v + (G - \hat{G})v \\ &= H_G v + \eta. \end{aligned}$$

The important point is that η has small norm. We may assume $\|v\| = 1$. Then

$$(1 - \epsilon)^2\|H_G\|^2 \leq \|H_G v\|^2 = \|P_-(G - \hat{G})v\|^2$$

$$< \|(G - \hat{G})v\|^2 < \|G - \hat{G}\|_\infty^2 \|v\|^2 = \|H_G\|^2,$$

the last step by Nehari's theorem. Thus

$$\|(G - \hat{G})v\|^2 - \|P_-(G - \hat{G})v\|^2 < \{1 - (1 - \epsilon)^2\}\|H_G\|^2.$$

That is,

$$\|\eta\|^2 < (2\epsilon - \epsilon^2)\|H_G\|^2.$$

Lemma 4 gives us an L^2 bound on the error vector η, whereas we are aiming for an L^∞ bound. To succeed we must pin η down to a finite-dimensional subspace of H^2 for which we can estimate the constant relating the two norms.

Recall from [AAK] that $G - \hat{G}$ has constant modulus $\|H_G\|$ a.e. on the unit circle. Thus, in the case that H_G has finite rank, when $G - \hat{G}$ is rational, we can write

$$G - \hat{G} = \|H_G\|\theta\bar{\varphi} \tag{3.2}$$

for some relatively prime finite Blaschke products θ and φ.

Lemma 5 With the assumptions of Lemma 4, suppose further that $v \in H^2 \ominus qH^2$ for some polynomial q and that $G - \hat{G}$ is expressible in the form (3.2). Then η in Lemma 4 lies in the space $H^2 \ominus \theta qH^2$.

Proof. We have

$$\begin{aligned}(G - \hat{G})v &= \|H_G\|\theta\bar{\varphi}v \\ &\in \theta\bar{\varphi}(H^2 \ominus qH^2) \\ &= \bar{\varphi}(\theta H^2 \ominus \theta qH^2) \\ &\subseteq \bar{\varphi}(H^2 \ominus \theta qH^2).\end{aligned}$$

Hence

$$\begin{aligned}\eta &= P_+(G - \hat{G})v \in P_+\bar{\varphi}(H^2 \ominus \theta qH^2) \\ &= \varphi(S)^*(H^2 \ominus \theta qH^2)\end{aligned}$$

where, as before, S is the forward shift on H^2. Since $H^2 \ominus \theta qH^2$ is invariant under S^* it follows that $\eta \in H^2 \ominus \theta qH^2$.

Lemma 6 Let $\rho_1, \ldots, \rho_k \in \mathbb{D}$ and let

$$\chi(z) = \prod_{j=1}^{k} \frac{z - \rho_j}{1 - \bar{\rho}_j z}.$$

For any $x \in H^2 \ominus \chi H^2$, $x \in H^\infty$ and

$$\|x\|_\infty < \sum_{j=1}^{k} \left(\frac{1 + |\rho_j|}{1 - |\rho_j|}\right)^{\frac{1}{2}} \|x\|$$

Proof. Let

$$K(z,\ w) = \frac{1 - \chi(\bar{w})\chi(z)}{1 - \bar{w}z}, \qquad z,\ w \in \mathbb{D}.$$

K is the reproducing kernel for $H^2 \ominus \chi H^2$ - that is $K(.,\ w) \in H^2 \ominus \chi H^2$ for all $w \in \mathbb{D}$ and, for any $x \in H^2 \ominus \chi H^2$, $w \in \mathbb{D}$,

$$x(w) = (x,\ K(.,\ w)).$$

By the Cauchy-Schwarz inequality,

$$|x(w)| \leq \|x\|\ \|K(.,\ w)\|_{H^2} \tag{3.3}$$

Let

$$\chi_j(z) = \frac{z - \rho_j}{1 - \bar{\rho}_j z}, \qquad 1 \leq j \leq k,$$

so that $\chi = \chi_1 \ldots \chi_k$. Fixing $w \in \mathbb{D}$ and regarding $z \in \partial\mathbb{D}$ as the independent variable we have

$$\begin{aligned} \|K(z,\ w)\| &= \left\|\frac{1 - \chi(\bar{w})\chi(z)}{1 - \bar{w}z}\right\| \\ &= \left\|\frac{\chi(z) - \chi(w)}{z - w}\right\| . \end{aligned}$$

Now

$$\frac{\chi(z) - \chi(w)}{z - w} = \chi_1 \ldots \chi_{k-1}(z)\frac{\chi_k(z) - \chi_k(w)}{z - w} + \chi_1 \ldots \chi_{k-2}(z)\frac{\chi_{k-1}(z) - \chi_{k-1}(w)}{z - w}\chi_k(w)$$

$$+ \ \ldots \ + \frac{\chi_1(z) - \chi_1(w)}{z - w}\chi_2 \ldots \chi_k(w).$$

It follows that

$$\begin{aligned} \|K(.,\ w)\| &\leq \sum_{j=1}^{k} \left\|\frac{\chi_j(z) - \chi_j(w)}{z - w}\right\| \\ &= \sum_{j=1}^{k} \frac{1 - |\rho_j|^2}{|1 - \bar{\rho}_j w|} \left\|\frac{1}{1 - \bar{\rho}_j z}\right\| \\ &= \sum_{j=1}^{k} \frac{1 - |\rho_j|^2}{|1 - \bar{\rho}_j w|} \cdot \frac{1}{[1 - |\rho_j|^2]^{\frac{1}{2}}} \\ &= \sum_{j=1}^{k} \frac{[1 - |\rho_j|^2]^{\frac{1}{2}}}{|1 - \bar{\rho}_j w|} \\ &\leq \sum_{j=1}^{k} \frac{[1 - |\rho_j|^2]^{\frac{1}{2}}}{|1 - \rho_j|} \end{aligned}$$

$$= \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}$$

Combining this inequality with (3.3) we have

$$|x(w)| \leq \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}} \cdot \|x\| .$$

As this holds for all $w \in \mathbb{D}$, the Lemma follows.

Lemmas 3 to 6 taken together suffice to give an L^∞ bound on the error vector $\eta = (G - \hat{G})v - H_G v$ where v is maximizing for $H^2 \ominus qH^2$ and q is close to the denominator of G. Indeed, by Lemma 3 v is ϵ-maximizing, where $\epsilon = 1 - \surd(1 - M^2)$, and so by Lemma 4

$$\|\eta\| \leq \sqrt{1 - (1 - \epsilon)^2}\, \|H_G\|\, \|v\|.$$

$$= M\, \|H_G\|\, \|v\|. \tag{3.4}$$

Lemma 5 tells us that $\eta \in H^2 \ominus \theta qH^2$, where $G - \hat{G}$ is expressible in the form (3.2), and so Lemma 6 enables us to convert (3.4) to an L^∞-estimate:

$$\|\eta\|_\infty \leq \left\{\sum_{j=1}^{m} \left[\frac{1 + |\beta_j|}{1 - |\beta_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}\right\} M\|H_G\|\, \|v\| \tag{3.5}$$

where β_j are the zeros of q and the ρ_j are the zeros of θ. The ρ_j's are unknown in the situation we envisage, but the inequality is enough to give us a rate of convergence.

4. A convergence theorem

Theorem 2 Let $G \in L^\infty$ be such that $(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_m)G(z) \in H^\infty$ for some $\alpha_1,\ldots \alpha_m \in \mathbb{D}$ and suppose that $\|H_G\|$ is a simple singular value of H_G. There exist a constant $K > 0$ and neighbourhoods U_i of α_i, $1 \leq i \leq m$, with the following property. If $\beta_i \in U_i$, $1 \leq i \leq m$, and v is a maximizing vector of

$$H_G \mid H^2 \ominus (z - \beta_1)(z - \beta_2)\ldots(z - \beta_m)H^2$$

then v is bounded away from zero on $\partial\mathbb{D}$ and the function $\tilde{G}$ defined by

$$\tilde{G} = G - \frac{H_G v}{v}$$

satisfies

$$\|\tilde{G} - \hat{G}\|_\infty \leq K \sum_{j=1}^{m} |\alpha_j - \beta_j|$$

where $\hat{G}$ is the best approximation in H^∞ to G with respect to the L^∞ norm. A constant K is given explicity in (4.9).

Proof. We may assume v is always chosen to be a unit vector. Section 3 (inequality (3.5)) shows that, for any choice of $\beta_1, \ldots, \beta_m \in \mathbb{D}$

$$(\tilde{G} - \hat{G})v = \eta \tag{4.1}$$

where

$$\|\eta\|_\infty \leq \left[\sum_{j=1}^{m} \left[\frac{1 + |\beta_j|}{1 - |\beta_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}\right] \times \sum_{j=1}^{m} \left|\frac{\alpha_j - \beta_j}{1 - \bar{\beta}_j \alpha_j}\right| . \|H_G\|$$

where $\rho_1, \ldots, \rho_k$ are the zeros (with repetition according to multiplicity) of the function θ occurring in the expression

$$G - \hat{G} = \|H_G\| \theta \bar{\varphi}$$

with θ, φ coprime finite Blaschke products (the assumption that a polynomial multiple of G is in H^∞ entails that $G - \hat{G}$ is rational, and since it has constant modulus it does have such an expression). It follows that $\rho_1, \ldots, \rho_k$ are determined by G and are independent of the β_j. As $\beta_j \longrightarrow \alpha_j$ we have

$$\sum_{j=1}^{m} \left[\frac{1 + |\beta_j|}{1 - |\beta_j|}\right]^{\frac{1}{2}} \longrightarrow \sum_{j=1}^{m} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}}$$

and

$$1 - \bar{\beta}_j \alpha_j \longrightarrow 1 - |\alpha_j|^2 .$$

It follows that, for β_j sufficiently close to α_j, $1 \leq j \leq k$, we have

$$\|\eta\|_\infty \leq A \sum_{j=1}^{k} |\alpha_j - \beta_j| \tag{4.2}$$

with

$$A = 2\left[\sum_{j=1}^{m} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}\right] . \left\{\min_{1 \leq j \leq m} (1 - |\alpha_j|^2)\right\}^{-1} \|H_G\| \tag{4.3}$$

Dividing by v in (4.1) gives

$$\begin{aligned} \|\tilde{G} - \hat{G}\|_\infty &= \|\eta / v\|_\infty \\ &\leq \|\eta\|_\infty \ (\inf_{|z|=1} |v(z)|)^{-1} \end{aligned} \tag{4.4}$$

To show that this gives a finite bound we need the singular value condition.

Lemma 7 Let T be a compact operator between Hilbert spaces and let $\|T\|$ be a simple singular value of T. For any ϵ, $0 < \epsilon < 1$, there exists $\delta > 0$ such that, whenever x is a δ-maximizing unit vector for T, there is a unit maximizing vector x' of T such that $\|x - x'\| < \epsilon$. Indeed we may take $\delta = K\epsilon^2$ where $K > 0$ is a constant depending only on T.

Proof. Let E be the 1-dimensional space of maximizing vectors for T. With respect to the direct decompositions $E \oplus E^{\perp}$, $TE \oplus (TE)^{\perp}$ of the domain and codomain of T respectively we can write

$$T \quad \begin{bmatrix} \|T\| & 0 \\ 0 & T_1 \end{bmatrix}$$

where $\|T_1\| < \|T\|$. Consider a δ-maximizing unit vector

$$x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

for T. Then

$$\|Tx\|^2 \geq (1 - \delta)^2 \|T\|^2,$$

i.e.

$$\|T\|^2 |x_0|^2 + \|T_1 x_1\|^2 \geq (1 - \delta)^2 \|T\|^2,$$

and hence

$$\|T\|^2 |x_0|^2 + \|T_1\|^2 (1 - |x_0|^2) \geq (1 - 2\delta) \|T\|^2.$$

On rearranging we obtain

$$|x_0|^2 \geq 1 - D\delta$$

where

$$D = \frac{2\|T\|^2}{\|T\|^2 - \|T_1\|^2} > 0.$$

A fortiori

$$|x_0| \geq 1 - D\delta$$

as long as $\delta \leq 1/D$.

Let x' be the unit maximizing vector

$$x' = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \Big/ |x_0|.$$

Then

$$\begin{aligned} \|x - x'\|^2 &= (|x_0| - 1)^2 + \|x_1\|^2 \\ &= (|x_0| - 1)^2 + 1 - |x_0|^2 \\ &= 2(1 - |x_0|) \\ &\leq 2D\delta \end{aligned}$$

for $\delta \leq 1/D$. Hence we may take $\delta = \epsilon^2/(2D)$.

We resume the demonstration that $|v|$ is bounded away from zero. It is a well-known result of deLeeuw-Rudin, cf [G], that, when $\|H_G\|$ is a simple singular value of H_G, any maximizing vector of H_G is non-zero everywhere. Such maximizing vectors lie in $H^2 \ominus pH^2$, where

$$p(z) = (z - \alpha_1)\ldots(z - \alpha_m),$$

and so are rational. Hence there exists $c > 0$ such that

$$|x(z)| \geq c \tag{4.5}$$

for all $z \in \partial\mathbb{D}$ and all unit maximizing vectors x of H_G.

Next observe that if v is maximizing for $H_G | H^2 \ominus qH^2$, where

$$q(z) = (z - \beta_1)\ldots(z - \beta_m), \tag{4.6}$$

and v' is any maximizing vector for H_G then $v' \in H^2 \ominus qH^2$, $v' \in H^2 \ominus pH^2$ and so $v - v' \in H^2 \ominus pqH^2$. Hence, by Lemma 6

$$\|v - v'\|_\infty \leq \left[\sum_{j=1}^{m} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{m} \left[\frac{1 + |\beta_j|}{1 - |\beta_j|}\right]^{\frac{1}{2}} \right] \|v - v'\|$$

Hence, for β_j sufficiently close to α_j,

$$\|v - v'\| \leq L \|v - v'\| \tag{4.7}$$

where

$$L = 3 \sum_{j=1}^{m} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}}$$

By Lemma 7 there exists $\delta > 0$ such that, if v is a δ-maximizing unit vector for H_G then there exists a unit maximizing vector v' for H_G such that

$$\|v - v'\| < \frac{c}{2L}.$$

By (4.7) there are neighbourhoods V_j of α_j, $1 \leq j \leq m$, such that, if in addition $\beta_j \in V_j$ and $v \in H^2 \ominus qH^2$ with q given by (4.6) then

$$\|v - v'\|_\infty < L \|v - v'\| < \frac{c}{2}.$$

In view of (4.5) this implies that

$$|v(z)| \geq \frac{c}{2}, \qquad \text{all } z \in \partial\mathbb{D}. \tag{4.8}$$

Now by Lemma 3 there exist neighbourhoods W_j of α_j, $1 \leq j \leq m$, such that if $\beta_j \in W_j$ and v is maximizing for $H_G | H^2 \ominus qH^2$ then v is δ-maximizing for H_G: we simply have to make the $\alpha_j - \beta_j$ so small that

$$1 - \sqrt{\left\{1 - \left[\sum_{j=1}^{m} \left|\frac{\alpha_j - \beta_j}{1 - \bar{\beta}_j \alpha_j}\right|\right]^2\right\}} < \delta$$

Thus, for β_j sufficiently close to α_j, (4.8) is satisfied and so (4.4) shows that

$$\|\tilde{G} - \hat{G}\|_\infty \leq \frac{2}{c}\|\eta\|_\infty.$$

It follows from (4.2) that

$$\|\tilde{G} - \hat{G}\|_\infty \leq \frac{2A}{c} \sum_{j=1}^{m} |\alpha_j - \beta_j|$$

whenever β_j is sufficiently close to α_j, $1 \leq j \leq m$. Thus the theorem holds with

$$K = \frac{2A}{c} \tag{4.9}$$

$$= \frac{4}{c}\left[\sum_{j=1}^{m}\left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{k}\left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}\right] \cdot \left\{\min_{1 \leq j \leq m} (1 - |\alpha_j|^2)\right\}^{-1} \|H_G\|$$

where c is the inf of $|v|$ over $\partial\mathbb{D}$ for a unit maximizing vector v for H_G and $\rho_1, \ldots, \rho_k$ are the zeros in $\mathbb{D}$ of the rational function $G - \hat{G}$.

5. An example

In Section 1 we considered the function

$$G(z) = \frac{1}{z - a}$$

where $a \in \mathbb{D}$ is very close to the unit circle. This example is easy to analyse by hand but nevertheless illustrates the issues we are addressing. We have pointed out that the Caratheodory–Fejér method does not work well in such a case: let us investigate the alternatives. It is easy to show that the exact best H^∞ approximation to G is the constant function

$$\hat{G}(z) = \frac{\bar{a}}{1 - |a|^2}.$$

Suppose that the pole is a mis-identified as $a + \delta a \in \mathbb{D}$, with δa small. If we carry out best analytical approximation exactly, by whatsoever method, on the "nominal plant" $(z - a - \delta a)^{-1}$ we shall obtain a computed best approximant

$$\hat{G}(z) = \frac{\bar{a} + \overline{\delta a}}{1 - |a + \delta a|^2}.$$

For a real the error is approximately

$$\frac{d}{da}\frac{a}{1 - a^2} \cdot \delta a = \frac{\delta a}{2(1 - a)^2}.$$

On the other hand, if we use modified truncation and succeed in representing $H_G | H^2 \ominus (z - a - \delta a)H^2$ exactly then an easy calculation shows that we obtain error approximately $\delta a/4(1 - a)^2$. Thus by using the data twice we have halved the error! Of course, in neither case is the idealization realistic: state space methods, for example, typically involve solving Lyapunov equations which become ill-conditioned in the presence of poles close to the unit circle.

More generally, let us analyse what happens when we guess m candidates poles $\beta_1, \ldots, \beta_m$ in ignorance that G has only one pole. Write

$$\upsilon(z) = \prod_{j=1}^{m} \frac{z - \beta_j}{1 - \bar{\beta}_j z}$$

Since $(\mathrm{Ker}\, H_G)^\perp = \mathrm{span}\,\{(1 - \bar{a}z)^{-1}\}$, a maximizing vector for $H_G | H^2 \ominus \upsilon H^2$ is the projection on this space of $(1 - \bar{a}z)^{-1}$, i.e. the vector

$$v(z) = \frac{1 - \upsilon(\bar{a})\upsilon(z)}{1 - \bar{a}z} .$$

Let $b = \upsilon(a)$. Then

$$H_G v(z) = P_- \frac{1}{z - a} \cdot \frac{1 - \upsilon(\bar{a})\upsilon(z)}{1 - \bar{a}z} .$$

$$= \frac{1 - |b|^2}{1 - |a|^2} \cdot \frac{1}{z - a} ,$$

so that the computed approximation is

$$\tilde{G}(z) = \frac{1}{z - a} - \frac{1 - |b|^2}{1 - |a|^2} \cdot \frac{1}{z - a} \cdot \frac{1 - \bar{a}z}{1 - \bar{b}\upsilon(z)}$$

$$= \frac{1}{(1 - |a|^2)(1 - \bar{b}\upsilon(z))} \left[\bar{a} - \bar{b}\frac{(1 - |a|^2)\upsilon(z) - (1 - \bar{a}z)b}{z - a} \right] .$$

We observe that here $\tilde{G}$ does belong to H^∞, since $|b| < 1$. If one or more candidate poles β_j are close to a then b will be small and so $\tilde{G}$ will be close to $\hat{G}$, as expected. We can also make b small by taking υ to have high degree, compensating for poor pole tracking by brute force. In particular we could take $\upsilon(z) = z^N$ for large N: this would be a half way stage to the Caratheory–Fejér method, in which the Hankel matrix is approximated by its $\infty \times N$ truncation.

6. Convergence of the modified truncation algorithm

Theorem 3 Let $G \in L^\infty$ be rational and suppose that $\|H_G\|$ is a simple singular value of H_G. Let $0 < r < 1$ and let $\alpha_1, \ldots, \alpha_m$ be the poles of G which lie in the annulus $\{z: r \leq |z| < 1\}$. Let $\hat{G}$ be the closest function in H^∞ to G with respect to the L^∞ norm. Then there exist constants $A, B > 0$ and a positive integer N_0 with the following property. For each $N > N_0$ there are neighbourhoods U_j of α_j, $1 \leq j \leq m$, such that if $\beta_j \in U_j$, $1 \leq j \leq m$, and if v is a maximizing vector of

$$H_G|H^2 \ominus z^N(z - \beta_1)\ldots(z - \beta_n)H^2$$

for some $n \geq m$ and $\beta_j \in \mathbb{D}$, $m < j \leq n$, then v is bounded away from zero on $\partial\mathbb{D}$ and the function $\tilde{G}$ defined by

$$\tilde{G} = G - \frac{H_G v}{v}$$

satisfies

$$\|\tilde{G} - \hat{G}\|_\infty \leq A \sum_{j=1}^{m} |\alpha_j - \beta_j| + Br^{N/2}.$$

Proof. Let

$$p(z) = \prod_{j=1}^{m} z - \alpha_j, \qquad q(z) = \prod_{j=1}^{n} z - \beta_j.$$

Let

$$T = H_G|H^2 \ominus z^N q H^2,$$

so that v is maximising for T. Let the Fourier series of pG be

$$(pG)(z) \quad \sum_{-\infty}^{\infty} c_k z^k.$$

Since pG is analytic in the annulus $\{z: r < |z| < 1\}$ it follows from Cauchy's integral formula that

$$c_{-k} = O(r^k), \qquad k \in \mathbb{N}.$$

Hence, if we write

$$g_N(z) = \sum_{k=-N}^{\infty} c_k z^k.$$

we have

$$\|pG - g_N\| \leq Cr^N$$

for some constant C. Thus, if

$$G_N = g_N/p,$$

we have

$$\|G - G_N\|_\infty \leq Cr^N \tag{6.1}$$

for some (possibly different) $C > 0$, and furthermore

$$z^N pG_N = z^N g_N = \sum_{k=0}^{\infty} c_{k-n} z^k \in H^\infty \tag{6.2}$$

We show that v is near-maximizing for the operator

$$\tilde{T} = H_{G_N}| H^2 \ominus z^N q H^2.$$

Indeed,

$$\|T - \tilde{T}\| = \|(H_G - H_{G_N})|H^2 \ominus z^N qH^2\|$$

$$\leq \|H_{G-G_N}\| \leq \|G - G_N\|_\infty$$

$$\leq Cr^N.$$

Since $\|Tv\| = \|T\| \|v\|$ we have

$$\|\tilde{T}v\| = \|Tv + (\tilde{T} - T)v\|$$

$$\geq (\|T\| - Cr^N) \|v\|$$

$$\geq (\|\tilde{T}\| - 2Cr^N) \|v\| \tag{6.3}$$

Next we show that v is near-maximizing for H_G. In view of (6.2) Lemma 2 shows that

$$\|\tilde{T}\| \geq \underline{\sigma}(P_{H^2 \ominus pH^2}|H^2 \ominus qH^2) \|H_{G_N}\|.$$

By Theorem 1,

$$\|\tilde{T}\| \geq \sqrt{1 - M^2} \|H_{G_N}\| \tag{6.4}$$

where

$$M = \sum_{j=1}^{m} \left|\frac{\alpha_j - \beta_j}{1 - \bar{\beta}_j \alpha_j}\right| . \tag{6.5}$$

From (6.1) we have

$$\|H_{G_N}\| \geq \|H_G\| - \|G - G_N\|_\infty$$

$$\geq \|H_G\| - Cr^N,$$

and so (6.4) yields

$$\|\tilde{T}\| \geq \sqrt{1 - M^2} (\|H_G\| - Cr^N)$$

$$\geq \sqrt{1 - M^2} \|H_G\| - Cr^N.$$

Now we invoke (6.4) to obtain

$$\|H_{G_N}v\| = \|\tilde{T}v\|$$

$$\geq (\|\tilde{T}\| - 2Cr^N) \|v\|$$

$$\geq \quad (\sqrt{1 - M^2}\ \|H_G\| - 3Cr^N)\ \|v\|. \tag{6.6}$$

Next it transpires that v is near-maximizing for H_G. We have, using (6.6)

$$\|H_G v\| \geq \|H_{G_N} v\| - \|H_{G-G_N} v\|$$

$$\geq \quad (\sqrt{1 - M^2}\ \|H_G\| - 4Cr^N)\ \|v\|.$$

That is, v is ϵ-maximizing for H_G, where

$$1 - \epsilon = \sqrt{1 - M^2} - \frac{4Cr^N}{\|H_G\|} \tag{6.7}$$

Now we may show as in the proof of Theorem 2 that $(\tilde{G} - \hat{G})v$ is small in the L^2 norm. By Lemma 4,

$$(\tilde{G} - \hat{G})v = \eta \tag{6.8}$$

where $\eta = P_+(G - \hat{G})v$ and

$$\|\eta\| \leq \sqrt{(1 - (1 - \epsilon)^2)}\ \|H_G\|\ \|v\|.$$

Here

$$\sqrt{(1 - (1 - \epsilon)^2)} = \sqrt{\left\{1 - \left[1 - M^2 - \frac{\sqrt{1 - M^2}\ 8Cr^N}{\|H_G\|} + \frac{16C^2r^{2N}}{\|H_G\|^2}\right]\right\}}$$

$$= \sqrt{\left\{ M^2 + \frac{\sqrt{1 - M^2}\ 8Cr^N}{\|H_G\|} - \frac{16C^2r^{2N}}{\|H_G\|^2}\right\}}$$

$$\leq \sqrt{\left\{ M^2 + \frac{8Cr^N}{\|H_G\|} \right\}}$$

$$\leq M + \left[\frac{8Cr^N}{\|H_G\|}\right]^{\frac{1}{2}}.$$

Thus

$$\|\eta\| \leq (M\ \|H_G\| + (8C\ \|H_G\|\ r^N)^{\frac{1}{2}})\ \ \|v\|.$$

Lemma 6 enables us to convert this to an L^∞ bound. Since G is rational, $G - \hat{G}$ is also, and if its zeros in $\mathbb{D}$ are $\rho_1, \ldots, \rho_k$ then we have

$$\|\eta\|_\infty \leq \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}} \|\eta\|$$

$$\leq \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}} (M\|H_G\| + (8C\|H_G\|r^N)^{\frac{1}{2}})\|v\| \tag{6.9}$$

Dividing by ν in (6.8) we obtain

$$\|\tilde{G} - \hat{G}\|_\infty = \|\eta/v\|_\infty$$
$$\leq \|\eta\|_\infty \quad (\inf_{|z|=1} |v(z)|)^{-1} \qquad (6.10)$$

We have to show that, for β_j sufficiently close to α_j and N sufficiently large, $|\nu|/\|\nu\|$ is uniformly bounded away from 0. As in (4.5) there exists $c > 0$ such that

$$|x(z)| \geq c \qquad (6.11)$$

for all $z \in \partial\mathbb{D}$ and all unit maximizing vectors x of H_G. Furthermore, if $\alpha_1, \ldots, \alpha_l$ are all the poles of G in $\mathbb{D}$ and

$$a(z) = \prod_{j=1}^{l} z - \alpha_j$$

then every maximizing vector x of H_G lies in $H^2 \ominus aH^2$. Thus, if ν is a maximizing vector of T and ν' is a maximizing vector H_G we have $\nu - \nu' \in H^2 \ominus z^N qaH^2$ and so, by Lemma 6,

$$\|v - v'\|_\infty \leq \left\{N + \sum_{j=1}^{m} \left[\frac{1 + |\beta_j|}{1 - |\beta_j|}\right]^{\frac{1}{2}} + \sum_{j=1}^{l} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}}\right\} \|v - v'\|$$

Hence, for β_j sufficiently close to α_j, $1 \leq j \leq m$, we have

$$\|v - v'\| \leq (N + L) \|v - v'\| \qquad (6.12)$$

where

$$L = 3\sum_{j=1}^{l} \left[\frac{1 + |\alpha_j|}{1 - |\alpha_j|}\right]^{\frac{1}{2}}.$$

By hypothesis, $\|H_G\|$ is a simple singular value of H_G. Hence, by Lemma 7, there is a constant $K > 0$ such that, if $0 < \epsilon < 1$ and ν is a $K\epsilon^2$–maximizing unit vector for H_G then there is a unit maximizing vector ν' for H_G such that $\|\nu - \nu'\| < \epsilon$. Apply this observation to the ν above (which we now take to be a unit vector). By (6.7) ν is δ–maximizing, where

$$\delta = 1 - \sqrt{1 - M^2} + \frac{4Cr^N}{\|H_G\|},$$

so that there exists a unit maximizing vector ν' for H_G such that

$$\|v - v'\| < \sqrt{\frac{\delta}{K}}$$

and so, by (6.12)

$$\|v - v'\|_\infty < \left\{1 - \sqrt{1 - M^2} + \frac{4Cr^N}{\|H_G\|}\right\}^{\frac{1}{2}} \frac{(N + L)}{\sqrt{K}}. \qquad (6.13)$$

Choose $N_0 \in \mathbb{N}$ so large that $N > N_0$ implies

$$\frac{4Cr^N (N + L)}{\|H_G\| \; K} < \frac{c^2}{8} .$$

Corresponding to $N > N_0$ there are neighbourhoods W_j^N of α_j, $1 \leq j \leq m$, such that $\beta_j \in W_j^N$ implies

$$(1 - \sqrt{1 - M^2} \quad \frac{(N + L)^2}{K} < \frac{c^2}{8} ,$$

M being given by (6.5). Then for $N > N_0$ and $\beta_j \in W_j^N$, $1 \leq j \leq m$, (6.13) shows that

$$\|v - v'\|_\infty < \frac{c}{2} ,$$

v' being a unit maximizing vector for H_G, and hence, in view of (6.11),

$$|v(z)| \geq \frac{c}{2} , \qquad z \in \partial\mathbb{D}.$$

Combining this with inequalities (6.9) and (6.10) we have: $N > N_0$ and $\beta_j \in W_j^N$, $1 \leq j \leq m$, imply

$$\|\tilde{G} - \hat{G}\|_\infty \leq A \sum_{j=1}^{m} |\alpha_j - \beta_j| + Br^{N/2}$$

where

$$A = \frac{4}{c}\left[\sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}}\right] \left\{\min_{1 \leq j \leq m} (1 - |\alpha_j|^2)\right\}^{-1} \|H_G\|$$

and

$$B = \frac{(32\, C\, \|H_G\|)^{\frac{1}{2}}}{c} \sum_{j=1}^{k} \left[\frac{1 + |\rho_j|}{1 - |\rho_j|}\right]^{\frac{1}{2}} .$$

We recall that in these formulae $\rho_1,\ldots,\rho_k$ are the zeros in $\mathbb{D}$ of $G - \hat{G}$, c is the infimum of the modulus of a unit maximizing vector on $\partial\mathbb{D}$ and C is a constant related to the rate of decay of the Fourier coefficients of G (cf. (6.1)).

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