THE LORENZ SYSTEM DOES NOT HAVE A POLYNOMIAL FLOW

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Abstract. One question that has not been answered using the results in [BM] is the following: Do the Lorenz equations have a polynomial flow? By inspecting their p-symmetries we show they do not.

Key words. differential equations, dynamical systems, Lorenz equations, polynomial flows, symmetries

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1. Introduction.

Numerical evidence suggests that the solutions of the Lorenz equations

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}\right\} \quad \begin{aligned} &(x,y,z) \in \mathbb{R}^3 \\ &\sigma,\ \rho,\ \beta > 0 \end{aligned} \tag{1.1}$$

behave chaotically even though their behavior is deterministic. Recently this has aroused the curiosity of many mathematicians and physicists. While much has been gleaned from computer experiments (see, for example, Lorenz [L], Sparrow [S], and Guckenheimer and Holmes [GH]), few results about the Lorenz system currently have rigorous mathematical proofs.

In this paper we consider only rigorously proved mathematical results.

MAIN THEOREM. *The Lorenz system (1.1) is complete and has a polynomial vector field with constant divergence. However, the Lorenz system does not have a polynomial flow.*

The proof that (1.1) does not have a polynomial flow involves showing that its only polynomial symmetries (of any degree) are the identity and the well known symmetry $(-x,-y,z) \mapsto (x,y,z)$.

In section 2 we define polynomial flows and describe some of their properties. A more thorough discussion of the main theorem is left to the end of section 2. Then in section 3 we define symmetries and prove the main theorem.

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2. Polynomial Flows. Consider the initial value problem

$$\dot{y}\ (\equiv \frac{dy}{dt}) = \mathbf{V}(y), \qquad y(0) = x \in \mathbb{R}^n \tag{2.1}$$

where $\mathbf{V}$ is a C^1 vector field on $\mathbb{R}^n$. Let $\phi\colon \Omega \to \mathbb{R}^n$ be the (local) flow associated with (2.1) where Ω, an open subset of $\mathbb{R} \times \mathbb{R}^n$, is the natural domain of ϕ. For each t in $\mathbb{R}$ let U^t be the set of all x in $\mathbb{R}^n$ such that (t, x) is in Ω.

DEFINITION 2.1. The flow ϕ is said to be a *polynomial flow* and $\mathbf{V}$ is said to be a *p-f vector field* if for each t in $\mathbb{R}$ the t-advance map $\phi^t\colon U^t \to \mathbb{R}^n$ is polynomial. That is, if $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the projection map onto the ith coordinate then $\pi_i \circ \phi^t$ is polynomial for $i = 1, \dots, n$.

Define the degree of ϕ^t [in x_j] to be the maximum of the degrees of the $\pi_i \circ \phi^t$, $1 \leq i \leq n$ [in x_j]. Take the degree of 0 to be $-\infty$. We say a vector field $\mathbf{V}$ is *complete* if all solutions of (2.1) extend to solutions defined on $\mathbb{R}$.

If $\mathbf{V}$ is a linear vector field, $\mathbf{V}(y) = Ay$ where A is an $n \times n$ matrix of real numbers, then $\phi(t, x) = e^{tA}x$. Hence linear vector fields are p-f vector fields. However, p-f vector fields are not restricted to linear ones. For example, even on $\mathbb{R}^2$ (hence also on $\mathbb{R}^n$ for any $n \geq 2$) there are p-f vector fields of all degrees (see table 1).

This leads us to the question: Which vector fields have polynomial flows? This question was first asked by Meisters [M1] and investigated more thoroughly by Bass and Meisters [BM]. The question is easy to answer in the case $n = 1$; the p-f vector fields on $\mathbb{R}$ are exactly those of the form $\mathbf{V}(y) = \alpha y + \beta$ where α and β are constants. Bass and Meisters also show that on $\mathbb{R}^n$ (for any n)

(1) p-f vector fields are polynomial;

(2) p-f vector fields are complete;

(3) p-f vector fields have constant divergence; and

(4) if ϕ is a polynomial flow then there exist an integer d and real analytic functions $a_r\colon \mathbb{R} \to \mathbb{R}^n$ such that

$$\phi(t, x) = \sum_{|r| \leq d} a_r(t) x^r.$$

(Here r is an n-tuple of nonnegative integers, $r = (r_1, \dots, r_n)$, $|r| = r_1 + \cdots + r_n$, $x = (x_1, \dots, x_n)$, and $x^r = x_1^{r_1} \cdots x_n^{r_n}$.)

Call a polynomial map $P\colon \mathbb{R}^n \to \mathbb{R}^n$ a *polyomorphism* (short for *poly*nomial auto*morphism*) if P has a polynomial (two-sided) inverse. Let $\mathcal{P}(\mathbb{R}^n)$ denote the group of polyomorphisms of $\mathbb{R}^n$. When it is clear from context we will suppress $\mathbb{R}^n$ and write $\mathcal{P}$ instead of $\mathcal{P}(\mathbb{R}^n)$.

Let $\mathcal{S}^n$ be the set of all p-f vector fields on $\mathbb{R}^n$. For $\mathbf{X}$, $\mathbf{V} \in \mathcal{S}^n$ define $\mathbf{X} \sim \mathbf{V}$ if there exists a polyomorphism $P\colon \mathbb{R}^n \to \mathbb{R}^n$ such that under the change of coordinates $y = P(v)$ the differential equation $\dot{v} = \mathbf{X}(v)$ becomes $\dot{y} = \mathbf{V}(y)$. Notice that this defines an equivalence relation on $\mathcal{S}^n$.

Bass and Meisters show that any p-f vector field on $\mathbb{R}^2$ is $\sim$ equivalent to exactly one of the normal forms listed in table 1 (a and b are constants).

TABLE 1. *Normal forms for p-f vector fields on $\mathbb{R}^2$.*

	Vector Field		Flow
(i)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}ax+by\\ ay-bx\end{pmatrix}$,	$b>0$	$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}e^{at}(x_0\cos bt + y_0 \sin bt)\\ e^{at}(y_0 \cos bt - x_0 \sin bt)\end{pmatrix}$
(ii)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}ax\\ by\end{pmatrix}$,	$0<\lvert a\rvert \le \lvert b\rvert$	$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x_0 e^{at}\\ y_0 e^{bt}\end{pmatrix}$
(iii)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0\\ by\end{pmatrix}$		$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x_0\\ y_0 e^{bt}\end{pmatrix}$
(iv)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1\\ by\end{pmatrix}$		$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x_0+t\\ y_0 e^{bt}\end{pmatrix}$
(v)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}ax\\ amy+x^m\end{pmatrix}$,	$a \neq 0$ $m = 1,2,3,\ldots$	$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x_0 e^{at}\\ e^{amt}(y_0 + x_0^m t)\end{pmatrix}$
(vi)	$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0\\ h(x)\end{pmatrix}$,	$h \in \mathbb{R}[x]$ h monic, $\deg h \ge 1$	$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x_0\\ y_0 + h(x_0)t\end{pmatrix}$

Since any p-f vector field on $\mathbb{R}^2$ is, under a change of coordinates, exactly one of those listed in table 1, the dynamics on $\mathbb{R}^2$ can be completely classified. However, there is no known normal form theorem on $\mathbb{R}^n$ for $n \ge 3$.

This illustrates the relevance of the main theorem. P-f vector fields are polynomial, are complete, and have constant divergence. The vector field of the Lorenz system has these three properties (see section 3 of this paper). And even though the dynamics of this system are exotic, this does not rule out the Lorenz system having a polynomial flow.

3. Proof of the main theorem. Part of the main theorem is quite easy to prove. The Lorenz system has a polynomial vector field with constant divergence $-\sigma - 1 - \beta$. We prove the Lorenz system is complete in the following lemma. Then the rest of the paper is devoted to proving that the Lorenz system does not have a polynomial flow.

LEMMA 3.1. *The Lorenz system (1.1) is complete.*

Proof. Let $(x(t), y(t), z(t))$ be a solution of (1.1). To prove that (1.1) is complete we use the Liapunov function $V = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2$ from Sparrow [S].

We calculate

$$\begin{aligned}\dot V &= 2\rho x\dot x + 2\sigma y\dot y + 2\sigma(z-2\rho)\dot z \\ &= 2\sigma\big(-\rho x^2 - y^2 - \beta(z-\rho)^2\big) + 2\sigma\beta\rho^2.\end{aligned}$$

Thus

$$\dot V \le 2\sigma\beta\rho^2. \tag{3.1}$$

Next we notice that $\big(2(z-2\rho)^2+2\rho^2\big) - (z-\rho)^2 = (z-3\rho)^2 \ge 0$. Thus $(z-\rho)^2 \le \big(2(z-2\rho)^2+2\rho^2\big)$ and we have

$$\begin{aligned}\dot V &\ge 2\sigma(-\rho x^2) + 2(-\sigma y^2) - 2\beta\sigma\big(2(z-2\rho)^2+2\rho^2\big) + 2\sigma\beta\rho^2 \\ &= 2\sigma(-\rho x^2) + 2(-\sigma y^2) + 4\beta\big(-\sigma(z-2\rho)^2\big) - 2\sigma\beta\rho^2 \\ &\ge 2\sigma(-V) + 2(-V) + 4\beta(-V) - 2\sigma\beta\rho^2.\end{aligned}$$

Let $b = 2\sigma + 2 + 4\beta$, $c = 2\sigma\beta\rho^2$ and we have

$$\dot V + bV \ge -c. \tag{3.2}$$

From (3.1) we have $V(t) \le V(0) + 2\sigma\beta\rho^2 t$ for $t \ge 0$. Since V cannot go to infinity in finite positive time, $\big(x(t), y(t), z(t)\big)$ cannot go to infinity in finite positive time. Thus the Lorenz system (1.1) is complete in positive time.

From (3.2) we have

$$\begin{aligned}\frac{d}{dt}(e^{bt}V) &= e^{bt}\dot V + be^{bt}V \\ &\ge -e^{bt}c.\end{aligned}$$

Thus for $t \le 0$ we have

$$V(0) - e^{bt}V(t) \ge (ce^{bt} - c)/b$$

or

$$V(t) \le e^{-bt}V(0) + (ce^{-bt} - c)/b.$$

Thus V cannot go to infinity in finite negative time. Hence $\big(x(t), y(t), z(t)\big)$ cannot go to infinity in finite negative time and the system is complete in negative time. Thus the system is complete.[1] □

Thus the Lorenz system is a complete system and has a polynomial vector field with constant divergence. Thus it satisfies the three criteria of Bass and Meisters that are easiest to check.

We now develop some tools for showing that the Lorenz system does not have a polynomial flow. Following Arnold [A] we make the following

[1] The proof that the Lorenz system is complete in forward time comes from Sparrow [S]. The proof that the Lorenz system is complete in backward time is due to Meisters [M2].

DEFINITION 3.1. A diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *symmetry* of $\mathbf{V}$ if

$$Df(x)\mathbf{V}(x) = \mathbf{V}\big(f(x)\big), \qquad x \in \mathbb{R}^n.$$

The idea of a symmetry of a differential equation was introduced by Lie [LE] who used n-parameter families of symmetries to solve differential equations (see Olver [O] for a modern treatment). His work lead to the development of the theory of Lie groups. However, we will need very little from this theory. In the following we present some basic facts about symmetries. While we do not claim that all of these facts are new, the applications of symmetries to questions concerning polynomial flows do appear to be new.

Let $\mathcal{D}(\mathbf{R}^n)$ denote the group of diffeomorphisms of $\mathbf{R}^n$. When it is clear from context we will suppress $\mathbf{R}^n$ and write $\mathcal{D}$ instead of $\mathcal{D}(\mathbf{R}^n)$. Let $\mathcal{D}_{\mathbf{V}}$ denote the symmetries of $\mathbf{V}$ and let $\mathcal{P}_{\mathbf{V}} = \mathcal{P} \cap \mathcal{D}_{\mathbf{V}}$ denote the *p-symmetries* of $\mathbf{V}$.

THEOREM 3.1. *The set $\mathcal{D}_{\mathbf{V}}$ of symmetries of $\mathbf{V}$ is a subgroup of $\mathcal{D}$.*

Proof. Note that the identity map is a symmetry of $\mathbf{V}$. Hence $\mathcal{D}_{\mathbf{V}}$ is not empty. Next, $f \in \mathcal{D}_{\mathbf{V}}$ implies that for every $x \in \mathbb{R}^n$

$$\begin{aligned} Df\big(f^{-1}(x)\big)\mathbf{V}\big(f^{-1}(x)\big) &= \mathbf{V}\Big(f\big(f^{-1}(x)\big)\Big) \\ &= \mathbf{V}(x). \end{aligned}$$

If $\mathbf{I}$ denotes the $n \times n$ identity matrix then from

$$\begin{aligned} \mathbf{I} &= D(f \circ f^{-1})(x) \\ &= Df\big(f^{-1}(x)\big)Df^{-1}(x) \end{aligned}$$

we see that $Df\big(f^{-1}(x)\big) = [Df^{-1}(x)]^{-1}$. Thus

$$\begin{aligned} \mathbf{V}(x) &= Df\big(f^{-1}(x)\big)\mathbf{V}\big(f^{-1}(x)\big) \\ &= [Df^{-1}(x)]^{-1}\mathbf{V}\big(f^{-1}(x)\big) \end{aligned}$$

or

$$Df^{-1}(x)\mathbf{V}(x) = \mathbf{V}\big(f^{-1}(x)\big).$$

Hence $f^{-1} \in \mathcal{D}_{\mathbf{V}}$. Next suppose that $f, g \in \mathcal{D}_{\mathbf{V}}$. Then for every $x \in \mathbb{R}^n$

$$\begin{aligned} D(f \circ g)(x)\mathbf{V}(x) &= Df\big(g(x)\big)Dg(x)\mathbf{V}(x) \\ &= Df\big(g(x)\big)\mathbf{V}\big(g(x)\big) \\ &= \mathbf{V}\Big(f\big(g(x)\big)\Big) \\ &= \mathbf{V}\big((f \circ g)(x)\big). \end{aligned}$$

Hence $(f \circ g) \in \mathcal{D}_{\mathbf{V}}$. Thus $\mathcal{D}_{\mathbf{V}}$ is a subgroup of $\mathcal{D}$. $\square$

COROLLARY 3.1. *The set $\mathcal{P}_{\mathbf{V}}$ is a subgroup of $\mathcal{P}$.*

Proof. This follows from the facts that $\mathcal{P}$ and $\mathcal{D}_{\mathbf{V}}$ are subgroups of $\mathcal{D}$ and that $\mathcal{P}_{\mathbf{V}} = \mathcal{P} \cap \mathcal{D}_{\mathbf{V}}$. □

For each x in $\mathbb{R}^n$ let $\phi_x \colon I_x \to \mathbb{R}^n$ denote the solution of (2.1) where I_x, a subset of $\mathbb{R}$, is its maximal interval of existence.

LEMMA 3.2. *Let $f \in \mathcal{D}$. Then $f \in \mathcal{D}_{\mathbf{V}}$ if and only if for every $x \in \mathbb{R}^n$ we have $f \circ \phi_x$ satisfies*

$$\dot{y} = \mathbf{V}(y) \tag{3.3}$$

on I_x.

Proof. First notice that for every $x \in \mathbb{R}^n$

$$\begin{aligned} \frac{d}{dt}\big(f \circ \phi_x(t)\big) &= Df\big(\phi_x(t)\big)\dot{\phi}_x(t) \\ &= Df\big(\phi_x(t)\big)\mathbf{V}\big(\phi_x(t)\big), \qquad t \in I_x. \end{aligned}$$

($\Rightarrow$) If $f \in \mathcal{D}_{\mathbf{V}}$ then

$$\begin{aligned} \frac{d}{dt}\big(f \circ \phi_x(t)\big) &= Df\big(\phi_x(t)\big)\mathbf{V}\big(\phi_x(t)\big) \\ &= \mathbf{V}\Big(f\big(\phi_x(t)\big)\Big), \qquad t \in I_x. \end{aligned}$$

so $f \circ \phi_x$ satisfies (3.3) on I_x.

($\Leftarrow$) If for every $x \in \mathbb{R}^n$ we have $f \circ \phi_x$ satisfies (3.3) on I_x then

$$\begin{aligned} Df\big(\phi_x(t)\big)\mathbf{V}\big(\phi_x(t)\big) &= \frac{d}{dt}\big(f \circ \phi_x(t)\big) \\ &= \mathbf{V}\Big(f\big(\phi_x(t)\big)\Big), \qquad t \in I_x. \end{aligned}$$

In particular at $t = 0$

$$Df(x)\mathbf{V}(x) = \mathbf{V}\big(f(x)\big).$$

Thus $f \in \mathcal{D}_{\mathbf{V}}$. □

THEOREM 3.2. *Let $f \in \mathcal{D}$. Then $f \in \mathcal{D}_{\mathbf{V}}$ if and only if $f \circ \phi_x = \phi_{f(x)}$ for all $x \in \mathbb{R}^n$. That is, for each x in $\mathbb{R}^n$ we have $I_x = I_{f(x)}$ and $(f \circ \phi_x)(t) = \phi_{f(x)}(t)$ for all $t \in I_x$.*

Proof. ($\Leftarrow$) If $f \circ \phi_x = \phi_{f(x)}$ then $f \circ \phi_x$ satisfies (3.3) on I_x. Hence by lemma 3.2, $f \in \mathcal{D}_{\mathbf{V}}$.

($\Rightarrow$) If $f \in \mathcal{D}_{\mathbf{V}}$ then by lemma 3.2 both $f \circ \phi_x$ and $\phi_{f(x)}$ are solutions of the initial value problem

$$\dot{y} = \mathbf{V}(y), \qquad y(0) = f(x). \tag{3.4}$$

We must therefore have $f \circ \phi_x(t) = \phi_{f(x)}(t)$ for all $t \in I_x \cap I_{f(x)}$.

Since $I_{f(x)}$ contains any interval on which a solution to (3.4) exists, we must have $I_x \subset I_{f(x)}$. This argument applies for any $x \in \mathbb{R}^n$ and for any $f \in \mathcal{D}_{\mathbf{V}}$. In particular, since $f^{-1} \in \mathcal{D}_{\mathbf{V}}$, we have

$$\begin{aligned} I_{f(x)} &\subset I_{f^{-1}(f(x))} \\ &= I_x. \end{aligned}$$

Thus $I_{f(x)} = I_x$ and therefore $f \circ \phi_x = \phi_{f(x)}$ for all $x \in \mathbb{R}^n$. □

If $\mathbf{V}$ is complete then we write $\mathcal{F}_{\mathbf{V}} = \{\phi^t : t \in \mathbb{R}\}$. Since for each t and s in $\mathbb{R}$ we have $\phi^t \circ \phi^s = \phi^{t+s} = \phi^t \circ \phi^s$, $\mathcal{F}_{\mathbf{V}}$ is an abelian subgroup of $\mathcal{D}$. Furthermore, we see that the map $t \mapsto \phi^t$ is a endomorphism of the reals under addition to $\mathcal{F}_{\mathbf{V}}$.

If H is a subgroup of a group G then we denote the *centralizer* of H in G by

$$C_G(H) = \{g \in G : gh = hg \text{ for all } h \in H\}.$$

Notice that $C_G(H)$ is a subgroup of G.

THEOREM 3.3. *If $\mathbf{V}$ is a complete vector field then $\mathcal{D}_{\mathbf{V}} = C_{\mathcal{D}}(\mathcal{F}_{\mathbf{V}})$. In particular, since $\mathcal{F}_{\mathbf{V}}$ is abelian, we have $\mathcal{F}_{\mathbf{V}} \subset \mathcal{D}_{\mathbf{V}}$.*

Proof. Since $\mathbf{V}$ is complete

$$\begin{aligned} f \in \mathcal{D}_{\mathbf{V}} &\Leftrightarrow f \circ \phi_x(t) = \phi_{f(x)}(t), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n \\ &\Leftrightarrow f \circ \phi^t(x) = \phi^t(f(x)), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n \\ &\Leftrightarrow f \circ \phi^t = \phi^t \circ f, \qquad t \in \mathbb{R} \\ &\Leftrightarrow f \in C_{\mathcal{D}}(\mathcal{F}_{\mathbf{V}}). \ \square \end{aligned}$$

If $\mathbf{V}$ is a p-f vector field then $\mathcal{F}_{\mathbf{V}} \subset \mathcal{P}$. In this case $\mathcal{P}_{\mathbf{V}} = C_{\mathcal{P}}(\mathcal{F}_{\mathbf{V}})$.

THEOREM 3.4. *Let $\mathbf{V}$ be a p-f vector field on $\mathbb{R}^n$, $n \geq 1$. Then $\mathcal{P}_{\mathbf{V}}$ has a subgroup isomorphic to either the reals under addition or the circle. In particular $\mathcal{P}_{\mathbf{V}}$ is uncountable.*

Proof. Suppose that $\mathbf{V}$ is the zero vector field. Then for each t in $\mathbb{R}$ the map $x \mapsto e^t x$ is in $\mathcal{P}_{\mathbf{V}}$. These maps form a group isomorphic to the reals under addition.

Suppose that $\mathbf{V}$ is not the zero vector field. Notice that $\mathbf{V}$ is complete. Let K be the kernel of the homomorphism $t \mapsto \phi^t$ of the reals under addition to $\mathcal{F}_{\mathbf{V}} \subset \mathcal{P}_{\mathbf{V}}$. Since $\mathbf{V}$ is not zero, $K \neq \mathbb{R}$. We claim that K is closed. To see this, let $\{t_n\}_{n=1}^{\infty}$ be a sequence in K which converges to $t \in \mathbb{R}$ and let $x \in \mathbb{R}^n$. Then since $t_n \in K$, we have $\phi^{t_n}(x) = x$. Since $\phi^{t_n}(x) = \phi_x(t_n)$ and since solutions of (2.1) are continuous, $\phi_x(t_n) \to \phi_x(t)$ as $n \to \infty$. Thus $\phi_x(t) = x$ or $\phi^t(x) = x$. Since x was arbitrary, $t \in K$. Thus K is closed and hence is either zero or an infinite cyclic group. Therefore $\mathbb{R}/K$ is isomorphic to either $\mathbb{R}$ or the circle.

In either case $\mathbb{R}/K$ is uncountable which implies $\mathcal{F}_{\mathrm{V}}$ and $\mathcal{P}_{\mathrm{V}}$ are uncountable. □

LEMMA 3.3. *If*

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

is a polynomial symmetry of the Lorenz system

$$\left.\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}\right\} \quad \sigma \neq 0 \tag{3.5}$$

then $\deg p = 1$, $\deg q \leq 2$, *and* $\deg r \leq 2$.

Notice that in (3.5) we have more allowable parameter values than in (1.1).

Proof. Let

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

be a polynomial symmetry of (3.5). Then

$$\begin{pmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{pmatrix} \begin{pmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - \beta z \end{pmatrix} = \begin{pmatrix} \sigma(q - p) \\ \rho p - q - pr \\ pq - \beta r \end{pmatrix}. \tag{3.6}$$

We rewrite (3.6) as the three equations

$$\sigma(y - x)p_x + (\rho x - y - xz)p_y + (xy - \beta z)p_z + \sigma p = \sigma q \tag{3.7}$$

$$\sigma(y - x)q_x + (\rho x - y - xz)q_y + (xy - \beta z)q_z + q = p(\rho - r) \tag{3.8}$$

$$\sigma(y - x)r_x + (\rho x - y - xz)r_y + (xy - \beta z)r_z + \beta r = pq. \tag{3.9}$$

Notice that p, q, and r are all nonconstant (else P is not a diffeomorphism). Thus the degree of $(\rho - r)$ is the same as that of r.

Recall that taking a partial derivative of a polynomial reduces its degree by at least one. Let $l = \deg p$, $m = \deg q$, and $n = \deg r$. From (3.8) we have

$$\begin{aligned} l + n &= \deg p(\rho - r) \\ &= \deg\big(\sigma(y-x)q_x + (\rho x - y - xz)q_y + (xy - \beta z)q_z + q\big) \\ &\leq \max\{m, m+1, m+1, m\} \\ &= m + 1. \end{aligned}$$

From (3.9) we have

$$\begin{aligned} l + m &= \deg pq \\ &= \deg\big(\sigma(y-x)r_x + (\rho x - y - xz)r_y + (xy - \beta z)r_z + \beta r\big) \\ &\leq \max\{n, n+1, n+1, n\} \\ &= n + 1. \end{aligned}$$

That is, we have the two inequalities $l + n \leq m + 1$ and $l + m \leq n + 1$. Adding these two together gives $2l + m + n \leq m + n + 2$ which implies $l \leq 1$. Since p is nonconstant, $l = 1$. This implies that $1 + m \leq n + 1$ and $1 + n \leq m + 1$. Hence $n = m$.

We now use the fact that $\sigma \neq 0$. Since $\deg p = 1$, the left hand side of (3.7) has degree at most two. Thus q has degree at most two. Since the degree of r is equal to that of q, the degree of r is at most two. □

THEOREM 3.5. *The set of polynomial symmetries of the Lorenz system (3.5) is*

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} \right\}.$$

This set is also the group of p-symmetries of (3.5).

We are here considering the more general version of the Lorenz system; the version where the only restriction on parameters is $\sigma \neq 0$.

Proof. Let

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

be a polynomial symmetry of (3.5). By lemma 3.3 there exist constants a_i, $1 \leq i \leq 4$ such that

$$p = a_1 x + a_2 y + a_3 z + a_4. \tag{3.10}$$

From equation (3.7) we have that

$$q = (y - x)p_x + (\rho x - y - xz)p_y/\sigma + (xy - \beta z)p_z/\sigma + p. \tag{3.11}$$

This implies that

$$q = b_1 x + b_2 y + b_3 z + b_4 xy + b_5 xz + b_6 \tag{3.12}$$

where $b_1 = a_2\rho/\sigma$, $b_2 = a_1 + a_2(1 - 1/\sigma)$, $b_3 = a_3(1 - \beta/\sigma)$, $b_4 = a_3/\sigma$, $b_5 = -a_2/\sigma$, and $b_6 = a_4$.

By lemma 3.3 there also exist constants c_i, $1 \le i \le 10$ such that

$$r = c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 y^2 + c_6 z^2 + c_7 xy + c_8 xz + c_9 yz + c_{10}. \tag{3.13}$$

The left hand side of (3.9) is

$$\sigma(y - x)r_x + (\rho x - y - xz)r_y + (xy - \beta z)r_z + \beta r \tag{3.14}$$

and the right hand side of (3.9) is

$$pq. \tag{3.15}$$

We expand (3.14) using (3.13) to get

(3.16)

$$\begin{aligned}
&\sigma(y - x)r_x + (\rho x - y - xz)r_y + (xy - \beta z)r_z + \beta r \\
&\quad = x\big(c_1(\beta - \sigma) + c_2\rho\big) + y\big(c_1\sigma + c_2(\beta - 1)\big) + x^2\big(c_4(\beta - 2\sigma) + c_7\rho\big) \\
&\qquad + y^2\big(c_5(\beta - 2) + c_7\sigma\big) + z^2(-c_6\beta) + xy\big(c_3 + 2c_4\sigma + 2c_5\rho + c_7(\beta - \sigma - 1)\big) \\
&\qquad + xz(-c_2 - c_8\sigma + c_9\rho) + yz(c_8\sigma - c_9) + x^2y(c_8) + x^2z(-c_7) \\
&\qquad + xy^2(c_9) + xz^2(-c_9) + xyz(2c_6 - 2c_5) + c_{10}\beta.
\end{aligned}$$

We expand (3.15) using (3.10) and (3.12) to get

(3.17)

$$\begin{aligned}
pq &= x(a_1b_6 + a_4b_1) + y(a_2b_6 + a_4b_2) + z(a_3b_6 + a_4b_3) + x^2(a_1b_1) + y^2(a_2b_2) \\
&\quad + z^2(a_3b_3) + xy(a_1b_2 + a_2b_1 + a_4b_4) + xz(a_1b_3 + a_3b_1 + a_4b_5) \\
&\quad + yz(a_2b_3 + a_3b_2) + x^2y(a_1b_4) + x^2z(a_1b_5) + xy^2(a_2b_4) + xz^2(a_3b_5) \\
&\quad + xyz(a_2b_5 + a_3b_4) + (a_4b_6).
\end{aligned}$$

Rewrite (3.8) as

$$\sigma(y - x)q_x + (\rho x - y - xz)q_y + (xy - \beta z)q_z + q - \rho p = -pr. \tag{3.18}$$

We expand the left hand side of (3.18) using (3.10) and (3.11) to get

(3.19)

$$\sigma(y-x)q_x + (\rho x - y - xz)q_y + (xy - \beta z)q_z + q - \rho p$$
$$= z\big(a_3(\beta^2/\sigma - \beta/\sigma - \rho - \beta + 1)\big) + x^2(a_3\rho/\sigma) + y^2(a_3) + xy(-a_3\beta/\sigma)$$
$$+ xz(-a_1 + a_2\beta/\sigma) + yz(-a_2) + x^2y(-a_2/\sigma) + x^2z(-a_3/\sigma) + \big(a_4(1-\rho)\big).$$

We expand the right hand side of (3.18) using (3.10) and (3.13) to get

(3.20)

$$-pr = x(-a_1c_{10} - a_4c_1) + y(-a_2c_{10} - a_4c_2) + z(-a_3c_{10} - a_4c_3)$$
$$+ x^2(-a_1c_1 - a_4c_4) + y^2(-a_2c_2 - a_4c_5) + z^2(-a_3c_3 - a_4c_6)$$
$$+ xy(-a_1c_2 - a_2c_1 - a_4c_7) + xz(-a_1c_3 - a_3c_1 - a_4c_8)$$
$$+ yz(-a_2c_3 - a_3c_2 - a_4c_9) + x^2y(-a_1c_7 - a_2c_4) + x^2z(-a_1c_8 - a_3c_4)$$
$$+ xy^2(-a_1c_5 - a_2c_7) + xz^2(-a_1c_6 - a_3c_8) + xyz(-a_1c_9 - a_2c_8 - a_3c_7)$$
$$+ (-a_4c_{10}) + x^3(-a_1c_4) + y^3(-a_2c_5) + z^3(-a_3c_6)$$
$$+ y^2z(-a_2c_9 - a_3c_5) + yz^2(-a_2c_6 - a_3c_9).$$

Suppose that we have shown that a_3 must be zero. Then equating coefficients of xz^2 in (3.19) and (3.20) gives

$$0 = a_1c_6 + a_3c_8$$
$$= a_1c_6;$$

of yz^2 in (3.19) and (3.20) gives

$$0 = a_2c_6 + a_3c_9$$
$$= a_2c_6.$$

Since p is not constant and $a_3 = 0$, at least one of a_1 and a_2 is nonzero. Hence $c_6 = 0$.

Equating coefficients of x^2z in (3.16) and (3.17) gives

$$c_7 = -a_1b_5$$
$$= a_1a_2/\sigma;$$

of x^2y in (3.16) and (3.17) gives

$$c_8 = a_1b_4$$
$$= a_1a_3/\sigma$$
$$= 0;$$

of xy^2 in (3.16) and (3.17) gives

$$\begin{aligned} c_9 &= a_2 b_4 \\ &= a_2 a_3/\sigma \\ &= 0. \end{aligned}$$

Next, equating the coefficients of x^3 in (3.19) and (3.20) gives

$$0 = a_1 c_4;$$

of y^3 in (3.19) and (3.20) gives

$$0 = a_2 c_5.$$

Thus one of the four pairs

$$(a_1, a_2) \qquad (a_1, c_5) \qquad (c_4, a_2) \qquad (c_4, c_5) \tag{3.21}$$

must be $(0, 0)$ (we have actually already ruled out (a_1, a_2) as a possible candidate for "zeroness"). If (a_1, c_5) is zero then equating coefficients of xyz in (3.16) and (3.17) gives

$$2(c_6 - c_5) = a_2 b_5 + a_3 b_4$$

or

$$0 = -a_2^2/\sigma$$

which implies that $a_2 = 0$. If (c_4, c_5) is zero then equating coefficients of xy^2 in (3.19) and (3.20) gives

$$\begin{aligned} 0 &= a_1 c_5 + a_2 c_7 \\ &= a_1 a_2^2/\sigma; \end{aligned} \tag{3.22}$$

of $x^2 y$ in (3.19) and (3.20) gives

$$\begin{aligned} a_2/\sigma &= a_1 c_7 + a_2 c_4 \\ &= a_1^2 a_2/\sigma \end{aligned}$$

or

$$a_2(1 - a_1^2) = 0. \tag{3.23}$$

Equation (3.22) implies $a_1 = 0$ or $a_2 = 0$. But $a_1 = 0$ along with (3.23) implies that $a_2 = 0$. Thus if any pair listed in (3.21) holds then we must have $a_2 = 0$; since one pair must be zero we do have $a_2 = 0$. Since $c_7 = a_1 a_2/\sigma$, we have $c_7 = 0$.

Equating the coefficients of xy^2 in (3.19) and (3.20) gives

$$0 = a_1 c_5 + a_2 c_7$$

or

$$0 = a_1 c_5;$$

of x^3 in (3.19) and (3.20) gives

$$0 = a_1 c_4.$$

Since p cannot be constant, we must have $a_1 \neq 0$. Thus we have $c_4 = c_5 = 0$.

Equating coefficients of x^2 in (3.19) and (3.20) gives

$$a_3 \rho / \sigma = -a_1 c_1 - a_4 c_4$$

or

$$0 = a_1 c_1;$$

of xy in (3.19) and (3.20) gives

$$a_3 \beta / \sigma = a_1 c_2 + a_2 c_1 + a_4 c_7$$

or

$$0 = a_1 c_2.$$

Again since $a_1 \neq 0$, we must have $c_1 = c_2 = 0$.

Equating coefficients of x in (3.19) and (3.20) gives

$$0 = a_1 c_{10} + a_4 c_1$$

or

$$0 = a_1 c_{10}$$

which implies that $c_{10} = 0$. Equating the constant terms of (3.16) and (3.17) gives

$$c_{10} \beta = a_4 b_6$$

or

$$0 = a_4^2.$$

Therefore $a_4 = 0$.

Summarizing, we have shown, assuming $a_3 = 0$, that we must have

$$a_2 = a_3 = a_4 = 0$$
$$b_1 = b_3 = b_4 = b_5 = b_6 = 0$$
$$c_1 = c_2 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0.$$

Equating the coefficients of xy in (3.16) and (3.17) gives

$$c_3 + 2c_4\sigma + 2c_5\rho + c_7(\beta - \sigma - 1) = a_1 b_2 + a_2 b_1 + a_4 b_4$$

or

$$c_3 = a_1^2.$$

Equating the coefficients of xz in (3.19) and (3.20) gives

$$a_1 - a_2\beta/\sigma = a_1 c_3 + a_3 c_1 + a_4 c_8$$

or

$$a_1 = a_1^3.$$

Thus $a_1 = \pm 1$ (a_1 cannot be zero). Hence we have

$$\begin{aligned} a_1 = b_2 &= \pm 1 \\ c_3 &= 1 \end{aligned}$$

and the polynomial symmetries are contained in

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} \right\}.$$

One checks that both maps in the above set are polyomorphisms and both satisfy (3.6). Hence this set is the set of polynomial symmetries of (3.5) and also its group of p-symmetries.

It remains to be shown that $a_3 = 0$. Equating the coefficients of $x^2 z$ in (3.16) and (3.17) gives

$$\begin{aligned} c_7 &= -a_1 b_5 \\ &= a_1 a_2/\sigma; \end{aligned}$$

of $x^2 y$ in (3.16) and (3.17) gives

$$\begin{aligned} c_8 &= a_1 b_4 \\ &= a_1 a_3/\sigma; \end{aligned}$$

of xy^2 in (3.16) and (3.17) gives

$$\begin{aligned} c_9 &= a_2 b_4 \\ &= a_2 a_3/\sigma; \end{aligned}$$

of z^2 in (3.16) and (3.17) gives

$$-c_6\beta = a_3b_3 \\ = a_3^2(1-\beta/\sigma); \tag{3.24}$$

of y^2 in (3.16) and (3.17) gives

$$c_5(\beta-2)+c_7\sigma = a_2b_2$$

or

$$c_5(\beta-2) = a_2b_2 - c_7\sigma \\ = a_1a_2 + a_2^2(1-1/\sigma) - a_1a_2 \\ = a_2^2(1-1/\sigma); \tag{3.25}$$

of yz in (3.16) and (3.17) gives

$$c_8\sigma - c_9 = a_2b_3 + a_3b_2$$

or

$$a_1a_3 - a_2a_3/\sigma = a_2a_3(1-\beta/\sigma) + a_1a_3 + a_2a_3(1-1/\sigma)$$

which implies that

$$0 = a_2a_3(2-\beta/\sigma). \tag{3.26}$$

Equating coefficients of xyz in (3.16) and (3.17) gives

$$2(c_6-c_5) = a_2b_5 + a_3b_4 \\ = (a_3^2 - a_2^2)/\sigma. \tag{3.27}$$

If $\beta = 0$ then (3.24) implies

$$0 = a_3^2.$$

Hence $a_3 = 0$.

If $\beta = 2\sigma$ (so $\beta \neq 0$) then equating the coefficients of xyz in (3.19) and (3.20) gives

$$0 = a_1c_9 + a_2c_8 + a_3c_7 \\ = a_1a_2a_3/\sigma + a_1a_2a_3/\sigma + a_1a_2a_3/\sigma \\ = 3a_1a_2a_3/\sigma$$

which implies $a_1a_2a_3 = 0$. Equating the coefficients of xz in (3.16) and (3.17) gives

$$-c_2 - c_8\sigma + c_9\rho = a_1b_3 + a_3b_1 + a_4b_5$$

which implies

$$\begin{aligned} c_2 &= -c_8\sigma + c_9\rho - a_1b_3 - a_3b_1 - a_4b_5 \\ &= -a_1a_3 + a_2a_3\rho/\sigma - a_1a_3(1-\beta/\sigma) - a_2a_3\rho/\sigma + a_2a_4/\sigma \\ &= a_1a_3(\beta/\sigma - 2) + a_2a_4/\sigma \\ &= a_2a_4/\sigma. \end{aligned}$$

Equating the coefficients of y in (3.16) and (3.17) gives

$$c_1\sigma + c_2(\beta - 1) = a_2b_6 + a_4b_2$$

or

$$\begin{aligned} c_1 &= \big(a_2b_6 + a_4b_2 + c_2(1-\beta)\big)/\sigma \\ &= \big(a_2a_4 + a_1a_4 + a_2a_4(1 - 1/\sigma) + a_2a_4(1-\beta)/\sigma\big)/\sigma \\ &= \big(a_2a_4(2 - \beta/\sigma) + a_1a_4\big)/\sigma \\ &= a_1a_4/\sigma; \end{aligned}$$

of xy in (3.19) and (3.20) gives

$$a_3\beta/\sigma = a_1c_2 + a_2c_1 + a_4c_7$$

or

(3.28)
$$\begin{aligned} 2a_3 &= a_1a_2a_4/\sigma + a_1a_2a_4/\sigma + a_1a_2a_4/\sigma \\ &= 3a_1a_2a_4/\sigma. \end{aligned}$$

Since $a_1a_2a_3 = 0$, one of a_1, a_2, and a_3 must be zero. If $a_3 = 0$ then we are done. If a_1 or a_2 is zero then (3.28) implies $a_3 = 0$.

If $\beta = 2$ then by the argument in the preceding paragraph we may assume, without loss of generality, that $\sigma \neq 1$. Thus by (3.25) $0 = a_2^2(1 - 1/\sigma)$ which implies $a_2 = 0$. Equating the coefficients of z^3 in (3.19) and (3.20) gives $0 = a_3c_6$. If $a_3 = 0$ then we are done. If $c_6 = 0$ then from (3.27) we have

$$2(c_6 - c_5) = (a_3^2 - a_2^2)/\sigma$$

or

$$c_5 = -a_3^2/2\sigma.$$

Equating the coefficients of y^2z in (3.19) and (3.20) gives

$$\begin{aligned} 0 &= a_2c_9 + a_3c_5 \\ &= -a_3^3/2\sigma \end{aligned}$$

which implies $a_3 = 0$.

If $\beta \neq 0$, $\beta \neq 2$, and $\beta \neq 2\sigma$ then we may rewrite (3.24) as

$$\begin{aligned} -c_6 &= a_3^2 \frac{1-\beta/\sigma}{\beta} \\ &= a_3^2(1/\beta - 1/\sigma). \end{aligned}$$

We may rewrite (3.25) as

$$c_5 = a_2^2 \frac{1-1/\sigma}{\beta - 2}.$$

Plugging into (3.27) gives

$$2\left[-a_3^2(1/\beta - 1/\sigma) - a_2^2(1-1/\sigma)/(\beta-2)\right] = (a_3^2 - a_2^2)/\sigma$$

or

$$a_3^2 \frac{\beta - 2\sigma}{\beta\sigma} = a_2^2 \frac{2\sigma - \beta}{\sigma(\beta-2)}. \tag{3.29}$$

Since $\beta \neq 2\sigma$, equation (3.26) implies that $a_2 = 0$ or $a_3 = 0$. Equation (3.29) implies that if one of a_2 and a_3 is zero then so is the other. Thus $a_3 = 0$. □

Theorem 3.5 provides us with a proof that the Lorenz system does not have a polynomial flow. Recall that theorem 3.4 ensures us that the group of p-symmetries of a p-f vector field is uncountable. Since the Lorenz system's group of p-symmetries contains only two elements, the Lorenz system does not have a polynomial flow. Thus we have proved all statements made in the main theorem.

REFERENCES

[A] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.

[BM] H. Bass and G. H. Meisters, *Polynomial flows in the plane*, Adv. in Math., 55 (1985), pp. 173–203.

[C] B. Coomes, *Polynomial flows, symmetry groups, and conditions sufficient for injectivity of maps*, Ph.D. thesis, University of Nebraska-Lincoln, 1988.

[GH] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[L] E. N. Lorenz, *Deterministic non-periodic flows*, J. Atmospheric Sci., 20 (1963), pp. 130–141.

[LE] S. Lie and F. Engel, *Theorie der Transformationsgruppen I, II, III*, Leipzig, 1888, 1890, 1893. Reprinted by Teubner, Leipzig, 1930.

[M1] G. H. Meisters, *Jacobian problems in differential equations and algebraic geometry*, Rocky Mountain J. Math., 12 (1982), pp. 679–705.

[M2] ————, *Polynomial flows on* $\mathbb{R}^n$, in *Proceedings of the Semester on Dynamical Systems*, Autumn 1986 at the Stefan Banach International Mathematics Center, ul. Mokotowska 25, Warszawa, Poland.

[MO] G. H. Meisters and C. Olech, *A poly-flow formulation of the Jacobian conjecture*, Bull. Acad. Polon. Sci. Sér. Sci. Math. (to appear).

[O] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[S] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer-Verlag, New York, 1982.

[TW] M. Tabor and J. Weiss, *Analytic structure of the Lorenz system*, Phys. Rev. A, 24 (1981), pp. 2157–2167.