ON THE UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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ON THE UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT*

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Introduction. We shall consider an inverse problem for electrically conductive material occupying a domain Ω in $\mathbf{R}^n$, $n \geq 2$. Let D be a subdomain of Ω and suppose that the conductivity coefficient of D is 2 and of $\Omega \setminus D$ is 1. We wish to determine the location of D by injecting current with density g across $\partial\Omega$ and measuring the voltage u on a portion Γ_0 of the boundary $\partial\Omega$. The voltage u satisfies the refraction problem:

$$\text{div}\,((1+\chi(D))\nabla u) = 0 \quad \text{in} \quad \Omega,$$
$$\frac{\partial u}{\partial N} = g \quad \text{on} \quad \partial\Omega \qquad (N = \text{ normal to } \partial\Omega)$$

where $\chi(D)$ is the indicator function of D, and u is normalized by

$$\int_{\partial_0\Omega} u = 0 \quad \text{where } \partial_0\Omega \text{ is a compact subset of } \partial\Omega.$$

Thus the problem is to determine D either by one measurement, i.e., by using one specific choice of g, or perhaps by a finite number of measurements $g_1, g_2, \ldots, g_N$.

If any number of measurements is allowed or, more precisely, if the Dirichlet–to–Neumann operator

$$u|_{\partial\Omega} \longrightarrow \frac{\partial u}{\partial N}\Big|_{\partial\Omega}$$

is known for all functions u defined on $\partial\Omega$, then D is uniquely determined; this was proved, for ∂D piecewise analytic, by Kohn and Vogelius [6], and, for ∂D Lipschitz, by Isakov [5]; see also [4].

The case of a single measurement g was studied by Friedman and Gustafsson [2] and Bellout and Friedman [1]. They proved, roughly, that the local dependence of the data $u|_{\Gamma_0}$ on the domain D is a function whose Frechet derivative does not vanish. Friedman and Vogelius [3] established a Lipschitz estimate for the location of extreme inhomogeneities in a smooth conductive medium using one measurement only.

In this paper we prove that if D is known to be a convex polyhedron (although its specific shape is not known) then the shape and location of D are determined by one measurement only.

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§1. Statement of main results. Let Ω be either a bounded domain in $\mathbf{R}^n$ $(n = 2,3)$ with Lipschitz and piecewise $C^{1,1}$ boundary $\partial\Omega$, or a half space in $\mathbf{R}^n$. Let D be a bounded subdomain of Ω with Lipschitz and piecewise C^2 boundary ∂D, such that $\overline{D} \subset \Omega$.

Let g be any function in $L^\infty(\partial\Omega)$ such that $g \not\equiv 0, g$ has compact support, $\int\limits_{\partial\Omega} g = 0$, and let S be a compact subset of $\partial\Omega$ containing the support of g. Consider the refraction problem

$$\text{div}\,((1+\chi(D))\nabla u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial N} = g \quad \text{on} \quad \partial\Omega \tag{1.2}$$

where N is the outward normal; u is normalized, say, by

$$\int\limits_S u = 0\,. \tag{1.3}$$

It is well known (see, for instance, [7]) that if Ω is bounded then this problem has a unique solution in $H^1(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha > 0$; further, setting

$$D^e = \Omega \setminus \overline{D},\ D^i = D.\ u^e = u\big|_{D^e},\ u^i = u\big|_{D^i},$$

we have

$$\Delta u^e = 0 \quad \text{in} \quad D^e,\ \Delta u^i = 0 \quad \text{in} \quad D^i, \tag{1.4}$$

$$u^e = u^i \quad \text{on the smooth part of} \quad \partial D, \tag{1.5}$$

$$\frac{\partial u^e}{\partial N} = 2\,\frac{\partial u^i}{\partial N} \quad \text{on the smooth part of} \quad \partial D, \tag{1.6}$$

where the "smooth part" refers to the portion of ∂D which is $C^{1,1}$. The same is true of D is a half space. Indeed, to prove existence we set $\Omega_\rho = \Omega \cap \{|x| < \rho\}$ and denote by u_n $(n = n_0, n_0+1, \dots)$ the solution of (1.1) in Ω_n satisfying (1.2) on $\partial\Omega_n$ ($g = 0$ on $\partial\Omega_n \setminus S$) and normalized by (1.3); n_0 is such that $S \subset \{|x| < n_0\}$. Clearly

$$\frac{1}{2}\int\limits_{\Omega_n} (1+\chi(D))|\nabla u_n|^2 \le \int\limits_S |g|\,|u_n|\,,$$

and using Poincare's inequality we get

$$\int\limits_{\Omega_{n_0}} (|u_n|^2 + |\nabla u_n|^2) \le C_{n_0} \qquad \text{if} \quad n \ge n_0.$$

Using this to estimate $\int\limits_S |g|\,|u_n|$ in the previous inequality, we find that

$$\int\limits_{\Omega_n} (1+\chi(D))|\nabla u_n|^2 \le C, \qquad C \text{ independent of } n.$$

It follows that a subsequence of u_n is convergent to a solution u as asserted above. Uniqueness follows by multiplying the difference of the equations (for solutions u_1 and u_2) by $u_1 - u_2$ and integrating over Ω_ρ, and then taking an appropriate sequence $\rho \to \infty$.

THEOREM 1.1. *Let D_1, D_2 be two convex polyhedrons such that $\overline{D}_j \subset \Omega$ and*

$$\text{diam } D_j < \text{dist } (D_j, \partial\Omega) \qquad (j = 1, 2), \tag{1.7}$$

and denote by u_j the solution of the refraction problem (1.1)–(1.3) for $D = D_j$. If

$$u_1 = u_2 \quad \text{on} \quad \Gamma_0 \tag{1.8}$$

where Γ_0 is a nonempty open subset of $\partial\Omega$, then $D_1 = D_2$.

The restriction (1.7) can be relaxed provided some geometric assumptions are made on D. In particular:

THEOREM 1.2. *If Ω is a half space then the assertion of Theorem 1.1 is valid even if the condition (1.7) is dropped.*

The uniqueness results of Theorems 1.1, 1.2 imply, by a compactness argument, the following stability property:

THEOREM 1.3. *Let D_j ($j = 1, 2, \dots$) and D_∞ be convex polyhedrons with closure in Ω and denote by u_j, u_∞ the solutions of (1.1)–(1.3) corresponding to D_j, D_∞ respectively. Assume further that the number of vertices of D_j is bounded by a constant independent of j. If*

$$|u_j - u_\infty|_{L^\infty(\Gamma_0)} \longrightarrow 0 \quad \text{as} \quad j \to \infty \tag{1.9}$$

where Γ_0 is a nonempty open subset of $\partial\Omega$, then

$$D_j \longrightarrow D_\infty \quad \text{as} \quad j \longrightarrow \infty. \tag{1.10}$$

The convergence in (1.10) is taken in the sense that there is a pairing of the vertices v_{jm} of D_j to the vertices $v_{\infty m}$ of D_∞ such that

$$v_{jm} \longrightarrow v_{\infty m} \quad \text{as} \quad j \longrightarrow \infty, \quad \forall\, m.$$

It would be interesting to replace (1.10) by a quantitative estimate If the D_j, D_∞ were uniformly $C^{1,1}$ smooth then a Lipschitz continuity estimate

$$d(D_j, D_\infty) \le C\, |u_j - u_\infty|_{L^\infty(\Gamma_0)} \quad (C \text{ constant}), \tag{1.11}$$

where d in the Hausdorff distance, has been derived by Bellout and Friedman [1]. However the results in [1] do not extend to the case where $\partial D_j, \partial D_\infty$ are not in $C^{1,1}$.

In §2 we establish some auxiliary results needed in the proofs of Theorems 1.1, 1.2. Theorems 1.1, 1.2 for $n = 2$ are proved in Section 3, and for $n = 3$ in Section 4; the proof for $n = 3$ can easily be extended to any $n \ge 3$. In §5 we extend Theorem 1.2 to the case where D is a circle. Finally in §6 we establish (1.11) in some (very special) cases.

§2. Auxiliary results.

Definition 2.1. Two bounded domains D_1, D_2 with closure in Ω are said to have *i–contact* (interior contact) if

$$\begin{aligned}(2.1)\qquad &\text{the sets } \Omega \setminus \overline{D}_j,\ \Omega \setminus (\overline{D}_1 \cup \overline{D}_2), D_1 \cap D_2 \text{ are connected,}\\ &\text{the set } \partial D_1 \cap \partial D_2, \text{ int } (\overline{D}_1 \cup \overline{D}_2) \text{ are disjoint, and}\\ &\text{there is a nonempty } C^2 \text{ hypersurface } \Gamma \text{ which}\\ &\text{belongs to the boundaries of both } \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ and } D_1 \cap D_2.\end{aligned}$$

REMARK 2.1. If D_1, D_2 are convex polyhedrons with int $(D_1 \cap D_2) \neq \emptyset$ such that their boundaries have nonempty interior and such that $\overline{D}_1, \overline{D}_2$ are contained in Ω, then they have i–contact.

LEMMA 2.1. *Suppose D_1 and D_2 have i–contact and $D_1 \neq D_2$. Denote by u_j the solution of (1.1)–(1.3) with $D = D_j$. Then (1.8) cannot hold.*

Proof. If (1.8) holds then u_1^e, u_2^e have the same Cauchy data on Γ_0 and, consequently, by uniqueness for the Cauchy problem for harmonic functions, $u_1^e \equiv u_2^e$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$. Using (1.5), (1.6) we conclude that $u_1^i = u_2^i, \partial u_1^i/\partial N = \partial u_2^i/\partial N$ on Γ and therefore, again by uniqueness for the Cauchy problem,

$$u_1^i = u_2^i \quad \text{in} \quad D_1 \cap D_2. \tag{2.2}$$

Since $D_1 \neq D_2$, we may assume that $D_2 \setminus \overline{D}_1$ is nonempty. The functions u_1^e, u_2^i are harmonic in $D_2 \setminus \overline{D}_1$. From (2.1) we have that $\partial(D_2 \setminus \overline{D}_1)$ consists only of points which belong to either $\partial(D_1 \cup D_2)$ or $\partial(D_1 \cap D_2)$. On $\partial(D_1 \cup D_2)$ we have $u_1^e = u_2^e = u_2^i$ (using (1.5)), and on $\partial(D_1 \setminus \overline{D}_2) \cap \partial(D_1 \cap D_2)$ we have $u_1^e = u_1^i = u_2^i$ (by (1.5) and (2.2)). Thus

$$u_1^e = u_2^i \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1)$$

and, by the maximum principle,

$$u_1^e = u_2^i \quad \text{in} \quad D_2 \setminus \overline{D}_1. \tag{2.3}$$

This implies that

$$\frac{\partial u_1^e}{\partial N} = \frac{\partial u_2^i}{\partial N} \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1). \tag{2.4}$$

Since $u_1^e = u_2^e$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$,

$$\frac{\partial u_1^e}{\partial N} = \frac{\partial u_2^e}{\partial N} \quad \text{on} \quad \partial D_2 \setminus \overline{D}_1.$$

Combining this with (2.4) we get

$$\frac{\partial u_2^e}{\partial N} = \frac{\partial u_2^i}{\partial N} \quad \text{on} \quad \partial D_2 \setminus \overline{D}_1$$

and, by comparison with (1.6),

$$\frac{\partial u_2^e}{\partial N} = 0 \quad \text{on} \quad \partial D_2 \setminus D_1. \tag{2.5}$$

Next, from (2.2),

$$\frac{\partial u_1^i}{\partial N} = \frac{\partial u_2^i}{\partial N} \quad \text{on} \quad \partial(D_1 \cap D_2).$$

Recalling (2.4), we deduce that

$$\frac{\partial u_1^e}{\partial N} = \frac{\partial u_1^i}{\partial N} \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1\ . \tag{2.6}$$

Since, by (1.6),

$$\frac{\partial u_1^e}{\partial N} = 2\frac{\partial u_1^i}{\partial N} \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1$$

we deduce that

$$\frac{\partial u_1^e}{\partial N} = 0 \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1.$$

From this relation and from (2.5) it follows that $\partial u_1^e/\partial N = 0$ on the boundary of $D_2 \setminus \overline{D}_1$. Hence $u_1^e \equiv \text{const.}$ in $D_2 \setminus \overline{D}_1$. This easily implies that also $u_1^e \equiv \text{const.}$ in $\Omega \setminus \overline{D}_1$ and, in particular,

$$g = \frac{\partial u_1^e}{\partial N} = 0 \quad \text{on} \quad \partial\Omega,$$

which is a contradiction.

LEMMA 2.2. *For any domain D, the solution u of (1.1)–(1.3) cannot be harmonic in all of Ω; consequently u^e does not have harmonic continuation into all of Ω.*

Proof. Otherwise, the gradient of u is continuous across ∂D and, in particular,

$$\frac{\partial u^e}{\partial N} = \frac{\partial u^i}{\partial N}$$

on the smooth part of ∂D. Recalling (1.6) we deduce that $\partial u^i/\partial N = 0$ on the smooth part of ∂D. This implies that $u^i = \text{const.}$ in D and then also $u = \text{const.}$ in $\Omega \setminus D$. Consequently $g = \partial u/\partial N = 0$ on $\partial\Omega$, a contradiction.

LEMMA 2.3. *If solutions u_1, u_2 of (1.1)–(1.3) corresponding to two subdomains D_1, D_2 satisfy the relation (1.8), then $\overline{D}_1 \cap \overline{D}_2$ is nonempty.*

Proof. If $\overline{D}_1 \cap \overline{D}_2$ is empty then u_2^e is harmonic in a neighborhood of $\overline{D}_1$. Since $u_1^e = u_2^e$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$, it follows that u_1^e has harmonic continuation into all of Ω, a contradiction to Lemma 2.2.

Lemma 2.3 shows that in order to prove Theorems 1.1, 1.2 we may assume, without loss of generality, that $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$.

Notation. We denote by $B_\epsilon(a)$ a ball with center a and radius ϵ.

LEMMA 2.4. *Consider a solution u to the refraction problem (1.1)–(1.3) and suppose a_0 is a point of ∂D such that*

$$D \cap B_\epsilon(a_0) = P_1 \cap \cdots \cap P_k \cap B_\epsilon(a_0) \qquad (\epsilon > 0)$$

where P_j are half-spaces $\{x; (x - a_0) \cdot N_j > 0\}$, for some unit vectors N_j. If u^e has harmonic continuation from $\Omega \setminus \overline{D}$ into $B_\epsilon(a_0)$, then u^i has harmonic continuation from D into $B_{\epsilon_1}(a_0)$, for some $0 < \epsilon_1 < \epsilon$.

Proof. We may assume that ∂P_1 has a common part with $\partial D \cap B_\epsilon(a_0)$. Consider the Cauchy problem

$$\begin{aligned} &\Delta w = 0 \quad \text{in} \quad B_{\epsilon_1}(a_0), \\ &w = u^e, \frac{\partial w}{\partial N} = \frac{1}{2} \frac{\partial u^e}{\partial N} \quad \text{on} \quad \partial P_1 \cap B_{\epsilon_1}(a_0). \end{aligned} \tag{2.7}$$

Since u^e is analytic in $B_\epsilon(a_0)$, by the Cauchy–Kowalewski theorem there exists a unique analytic solution of (2.7) if ϵ_1 is small enough. In view of (1.4)–(1.6), u^i satisfies the same differential equation in $D \cap B_{\epsilon_1}(a_0)$ and the same Cauchy condition on $\partial P_1 \cap \overline{D}$. Hence, by uniqueness for the Cauchy problem, $w = u^i$ in $D \cap B_{\epsilon_1}(a_0)$, and the lemma follows.

§3. Proof of Theorems 1.1, 1.2 for n = 2. Denote by $x(\beta)$ the image under rotation $(r, \phi) \to (r, \phi + \beta)$ about the origin of a point $x = (r, \phi)$. Similarly we define the image $x(\beta)$ of x under rotation with respect to any center x_0.

LEMMA 3.1. *Suppose $a_0 \in \partial D$ and $D \cap B_\epsilon(a_0)$ is a nonempty convex set given by $P_1 \cap P_2 \cap B_\epsilon(a_0)$ where P_1, P_2 are two distinct half-planes. If u^e has harmonic continuation from D^e into $D^e \cup B_\epsilon(a_0)$ then there is a rotation $x \to x(2\pi/p)$ $(p = 2, 3, \ldots)$ about a_0 such that*

$$u^e(x) = u^e\left(x\left(\frac{2\pi}{p}\right)\right) \quad \text{for} \quad x \in B_\epsilon(a_0). \tag{3.1}$$

Proof. We may assume that $a_0 = 0$ and that

$$D \cap B_\epsilon(a_0) = \{0 < \phi < \alpha, \quad r < \epsilon\}$$

where $0 < \alpha < \pi$. According to Lemma 2.4 the function u^i has harmonic continuation into a disc $B_{\epsilon_1}(a_0)$. We thus have, in $B_{\epsilon_1}(a_0)$,

$$(3.2)\qquad \begin{aligned} u^e(x) &= \sum_{k=0}^{\infty}(a_k^e \cos k\phi + b_k^e \sin k\phi)r^k, \\ u^i(x) &= \sum_{k=0}^{\infty}(a_k^i \cos k\phi + b_k^i \sin k\phi)r^k \end{aligned}$$

where $b_0^e = b_0^i = 0$.

From the conditions (1.5), (1.6) at $\phi = 0$ we get

$$(3.3)\qquad a_k^e = a_k^i \ , \ b_k^e = 2b_k^i$$

and from the same conditions at $\phi = \alpha$ we get, after using (3.3),

$$(b_k^e - b_k^i)\sin k\,\alpha = 0, \ (a_k^e - 2a_k^i)\sin k\,\alpha = 0$$

or

$$(3.4)\qquad b_k^e \sin k\,\alpha = a_k^e \sin k\,\alpha = 0.$$

If α/π is irrational then $\sin k\alpha \neq 0$ and consequently $b_k^e = a_k^e = 0$ for $k = 1, 2, \ldots$; hence $u^e \equiv$ const. which is a contradiction. We conclude that α/π is a rational number, say

$$\frac{\alpha}{\pi} = \frac{q}{p} \quad \text{when} \quad p = 2, 3, \ldots, q \geq 1, \quad (p, q) = 1.$$

For all nonzero coefficients a_k^e, b_k^e we must have $\sin(k\pi q/p) = 0$, i.e., kq/p is a positive integer; since $(p, q) = 1$, it follows that $k = mp$ where $m = 0, 1, 2, \ldots$. Recalling the representation (3.2) and noting that

$$\cos(mp(\phi + \frac{2\pi}{p})) = \cos(mp\phi), \quad \sin(mp(\phi + \frac{2\pi}{p})) = \sin(mp\phi),$$

the assertion (3.1) follows.

LEMMA 3.2. *Let D and u^ϵ satisfy the conditions of Lemma 3.1 and assume that u^ϵ has harmonic continuation into an open set S containing $B_\epsilon(a_0)$, and that the images S_j of S under the rotation $\phi \to \phi + 2\pi j/p$ (about a_0) satisfy:*

$$S_j \cap S_k \quad \text{are connected for all } j,k \; .$$

Then u^ϵ has harmonic continuation into the open set $\bigcup_{j=1}^{p} S_j$.

Proof. Define $u_j^\epsilon(r,\phi) = u^\epsilon(r, \phi - (j-1)2\pi/p)$. The function $u_j^\epsilon(r,\phi)$ is harmonic in S_j. In view of Lemma 3.1 the harmonic function u_j^ϵ and u_k^ϵ agree on $S_j \cap S_k \cap B_\epsilon(a_0)$. Since the intersection $S_j \cap S_k$ is connected and contains $B_\epsilon(a_0)$, u_j^ϵ and u_k^ϵ agree on $S_j \cap S_k$. It follows that the function defined by $u = u_j^\epsilon$ on S_j, for $1 \le j \le p$, is a harmonic continuation of u^ϵ into $S_1 \cup \cdots \cup S_p$.

Proof of Theorem 1.1. If $D_1 \neq D_2$ then we may assume that the origin 0 is a vertex of the convex hull of $D_1 \cup D_2$ and that it belongs to $\overline{D}_1$ but not to $\overline{D}_2$. As in Lemma 2.1, $u_1^\epsilon = u_2^\epsilon$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$. Since u_2^ϵ is harmonic in a neighborhood of $0, u_1^\epsilon$ has harmonic continuation from $\Omega \setminus D_1$ into $B_\epsilon(0)$, for some $\epsilon > 0$. From the condition (1.7) it follows that there are two different half–discs of radius $d >$ diam D_1, centered at 0, which belong to $\Omega \setminus D_1$. Let $\tilde{S}$ denote the union of these two half–discs and $B_\epsilon(0)$. By Lemma 3.2 u_1^ϵ has harmonic continuation into $S_1 \cup \cdots \cup S_p = B_d(0)$. Since $\overline{D}_1 \subset B_d(0), u_1^\epsilon$ has harmonic continuation into all of Ω, which is a contradiction to Lemma 2.2

Proof of Theorem 1.2. We assume that $D_1 \neq D_2$ and take $\Omega = \{x_2 < 0\}$. Let $a(1) = (a_1(1), a_2(1))$ be a vertex in the convex hull D_* of $D_1 \cup D_2$ with the largest distance to $\partial\Omega$. If u_1^ϵ (or u_2^ϵ) has harmonic continuation into a neighborhood of $a(1)$ then using the fact that u_j^ϵ is harmonic in the half–plane $\{x_2 < a_2(1)\}$ we can construct two half–discs in $\Omega \setminus D_*$ with radius larger than the diameter of D_* and center at $a(1)$ and show, as in the previous proof, that u_1^ϵ (or u_2^ϵ) has harmonic continuation into Ω, which is a contradiction. Thus in the sequel we may assume that

(3.5) u_1^ϵ and u_2^ϵ do not have harmonic continuation from $\Omega \setminus D_*$ into a neighborhood of a(1).

It follows that $a(1)$ is a vertex of both D_1 and D_2. Consider the vertices $a(0) = (a_1(0), a_2(0))$, $a(2) = (a_1(2), a_2(2))$ of D_* adjacent to $a(1)$. If $a(0)$ is a vertex of both D_1 and D_2, then the convex polygons D_1 and D_2 have i–contact (with $\Gamma = \overline{a(0)a(1)}$), which is a contradiction to Lemma 2.1. Therefore

(3.6) $a(0)$ is not a common vertex of D_1, D_2, and the same holds for $a(2)$.

We may assume that $a_1(0) \le a_1(2)$. Consider the case

$$a_1(1) < a_1(2). \tag{3.7}$$

Since $a(2)$ is not a common vertex, the functions $u_1^\epsilon, u_2^\epsilon$ have harmonic continuation into an ϵ–neighborhood of $a(2)$. We can therefore apply Lemma 3.1 to u_1^ϵ, considering rotations with respect to $a(2)$. Denote by $\tilde{S}$ the set $B_\epsilon(a(2)) \cup (\Omega \setminus \overline{D}_*)$. If the related rotation number is $p = 2$ then the sets $S_1 = \tilde{S}$ and S_2 satisfy:

$$S_1 \cup S_2 \supset \mathbf{R}^2 \setminus \overline{D}_3$$

where D_3 is the union of D_1 and its image under rotation $x \longrightarrow x(\pi)$ about $a(2)$. We conclude that u_1^ϵ has harmonic continuation into $\mathbf{R}^2 \setminus \overline{D}_3$.

We now use $a(0)$ as a center of rotation for u_1^ϵ, taking

$$\tilde{S} = B_\epsilon(a(0)) \cup (\mathbf{R}^2 \setminus \overline{D}_3). \tag{3.8}$$

It is easy to check that $S_1 \cup \cdots \cup S_p$ contains $a(1)$, no matter what p is. Hence u_1^ϵ has harmonic continuation to a neighborhood of $a(1)$, which is a contradiction to (3.5).

Consider next the case where the rotation number at $a(2)$ is $p = 3$. In this case u_1^ϵ has harmonic continuation into $S_1 \cup S_2 \cup S_3$. The boundary of this union contains the images $a(1;2)$, $a(1;3)$ of the point $a(1)$ under the rotation. Since u_1^ϵ has no harmonic continuation into a neighborhood of $a(1)$, also

(3.9) $\quad u_1^\epsilon$ has no harmonic continuation
into a neighborhood of $a(1;2)$, or $a(1;3)$.

We now resort again to rotation about $a(0)$. If $p = 2$ then taking $\tilde{S}$ as in (3.8) we easily check that $S_1 \cup S_2$ contains either $a(1;2)$ or $a(1;3)$ and therefore u_1^ϵ has harmonic continuation into a neighborhood of one of these points, a contradiction to (3.9). If $p \ge 3$ then the set $S_1 \cup S_2 \cup S_3$ contains a sector of the plane with vertex at $a(0)$ and opening larger than $2\pi/3$. Thus u_1^ϵ has harmonic continuation to all of $\mathbf{R}^2$, a contradiction.

It remains to consider the case where the rotation number at $a(2)$ is $p \ge 4$. In this case u_1^ϵ has harmonic continuation to all of $\mathbf{R}^2$ which is obtained by rotation about $a(2)$ of a sector Σ of the plane with vertex at $a(2)$ and openning $\frac{\pi}{2} + \epsilon_1$ ($\epsilon_1 > 0$) which is entirely contained in Ω.

We have thus shown that the case (3.7) gives a contradiction. If (3.7) is not true, then $a_1(0) \le a_1(2) \le a_1(0)$ and both equalities cannot hold simultaneously. Hence $a_1(0) < a_1(1)$ and we can proceed as above, interchanging the roles of $a(0)$ and $a(2)$.

§4. Proofs of Theorem 1.1, 1.2 in case $n = 3$. For a convex polyhedron D and an edge γ in a direction σ, we shall use the representation

$$s\sigma + x' \quad \text{where} \quad x' \perp \sigma\ ,\ s \in \mathbb{R}^1$$

for any $x \in \mathbb{R}^3$.

LEMMA 4.1. *Assume that the origin 0 belongs to γ and denote by $u = (u^e, u^i)$ the solution of (1.1)–(1.3). If u^e has harmonic continuation into $B_\epsilon(0)$, for some $\epsilon > 0$, then there is a rotation $x'(2\pi/p)$, $\quad p = 2, 3, \ldots$, such that*

$$u^e(x) = u^e(s\sigma + x'(2\pi/p)) \quad \textit{for} \quad x \in B_\epsilon(0). \tag{4.1}$$

Proof. From Lemma 2.4 it follows that the function u^i has harmonic continuation into a ball $B_{\epsilon_1}(0), 0 < \epsilon_1 < \epsilon$. Introduce polar coordinates (r, ϕ) in the plane $E \equiv \{(x', 0)\}$. We may assume that $E \cap D$ is bounded by $\phi = 0$ and $\phi = \phi_1$ for some $0 < \phi_1 < \pi$. Consider the functions

$$u_1 = u^e - u^i \quad \text{and} \quad u_2 = u^e - 2u^i.$$

From the relations (1.5), (1.6) we have

$$u_1 = \frac{\partial u_2}{\partial \phi} = 0 \qquad \text{for} \qquad \phi = 0, \phi_1\ . \tag{4.2}$$

Since u_1, u_2 are harmonic in $B_{\epsilon_1(0)}$, the refraction formulas

$$u_1(x', s) = -u_1(x'*, s),\ u_2(x', s) = u_2(x'*, s) \tag{4.3}$$

hold for all points x', x^* symmetric with respect to the lines $\phi = 0, \phi = \phi_1$. This implies, in particular, the relations (4.2) for all $\phi = k\phi_1,\ k = 1, 2, \ldots$.

Let ϕ_0 be the smallest nonnegative $\tilde{\phi}_1$ such that (4.2) holds for $\phi = \tilde{\phi}_1$. If $\phi_0 = 0$ then both u_1 and u_2 do not depend on ϕ. If $\phi_0 > 0$ then $\phi_0 = \pi/p$ for some $p = 2, 3, \ldots$ and, from the relation (4.3), we conclude that

$$u_j(x', s) = u_j(x'\left(\frac{2\pi}{p}\right), s).$$

Since $u^e = 2u_1 - u_2$, the proof is complete.

LEMMA 4.2. *Let D, u be as in Lemma 4.1 and let B be a ball with a center at some point of γ such that $B \supset \overline{D}$. Let B_0 be a hemiball of B (i.e., $B_0 \subset B$, radius B_0 = radius B) and let $B_0 \subset S_* \subset B$ be such that*

S_* contains all sufficiently small rotations of B_0 about γ.

Assume that u^e has harmonic continuation into $\overline{S}_* \setminus \gamma_0$ where γ_0 is the straight line containing γ. Then u^e has harmonic continuation into a neighborhood of $\overline{D}$.

Proof. Let p be the rotation number from Lemma 4.1. We may assume that S_* contains $B \cap \{0 < \phi < \frac{2\pi}{p} + \epsilon'\}$ for some $\epsilon' > 0$. The function u^e thus has harmonic continuation into the set

$$S^* = (\{0 < \phi < \frac{2\pi}{p} + \epsilon'\} \cap) S_* \cup B_\epsilon(0).$$

Denote by S_j^* the image of S^* under the rotation $\phi \to \phi + 2(j-1)\pi/p$. Then

$$B \setminus \gamma_0 \subseteq S_1^* \cup \cdots \cup S_p^*$$

Define functions $u_j(r, \phi, s)$ as $u^e(r, \phi - 2(j-1)\pi/p, s)$. Then u_j is harmonic in S_j^*. In view of Lemma 4.1, $u_j = u_k$ on $S_j^* \cap S_k^* \cap B_{\epsilon_1}(0)$ and therefore u_j and u_k agree on $S_j^* \cap S_k^*$. Hence the function w defined by $w = u_j$ on S_j^* is a harmonic continuation of u^e into $B \setminus \gamma_0$. Since $w \in H^1(B)$, the singularity of w on $B \cap \gamma_0$ is removable (as can be seen by writing

$$\int \nabla w \cdot \nabla(\zeta_j \psi) = 0 \,, \quad \text{for any} \quad \psi \in C_0^\infty(B),$$

where $\zeta_j \in C_0^1(B)$, $\zeta_j = 0$ in a neighborhood of γ_0, $\zeta_j \longrightarrow 1$ as $j \longrightarrow \infty$, and taking $j \longrightarrow \infty$.).

Proof of Theorem 1.1. We shall assume that $D_1 \neq D_2$ and derive a contradiction. As before, $u_1 = u_2$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$. We may assume that the origin belongs to an edge γ of the convex hull of $D_1 \cup D_2$ and that the interior of γ does not belong to $\overline{D}_1$.

As above the function u_1^e has harmonic continuation from $\Omega \setminus \overline{D}_1$ into a ball $B_\epsilon(0), 0 \in int\gamma$. From the condition (1.7) it follows that there are two different hemiballs of radius d, $d >$ diam D_1, centered at 0, that do not intersect D_1 and belong to Ω. Let $\tilde{S}$ be the union of these balls and $B_\epsilon(0)$. Repeating the proof of Theorem 1.1 for $n = 2$ but using Lemma 4.2 instead of Lemma 3.2, we get a contradiction.

Proof of Theorem 1.2. We again assume that $D_1 \neq D_2$ and proceed to derive a contradiction. We take $\Omega = \{x_3 < 0\}$. Since the function g has compact support, the functions u_1^e, u_2^e have harmonic extensions to $\mathbf{R}^3 \setminus (\overline{D}_* \cup \overline{D}_{**} \cup S_0)$ where D_* is the convex hull of $D_1 \cup D_2, D_{**}$ in its symmetric image under the map $(x_1, x_2, x_3) \longrightarrow (x_1, x_2, -x_3)$, and S_0 is a bounded set in $\{x_3 = 0\}$ containing the support of g.

Let a be a vertex of D_* with the largest distance to $\partial\Omega$. Then there is a face Γ of ∂D_* whose exterior normal has negative x_3-coordinate. Since, by Lemma 2.1, D_1 and D_2 do not have i-contact there must be an edge γ of ∂D_* with $a \in \overline{\gamma}, \gamma \subset \overline{\Gamma}$ such that γ is not contained in either $\overline{D}_1$ or $\overline{D}_2$; for definiteness we take $\gamma \not\subset \overline{D}_1$, and choose a point $b \in \gamma, b \notin \overline{D}_1$.

The plane containing Γ divides $\mathbf{R}^3$ into two half–spaces and, according to the choice of Γ, one of them, call it P, does not intersect $\overline{D}_* \cap \overline{D}_{**}$. Since D_*, D_{**} are convex polyhedrons, any small rotation of P about γ also does not intersect $\overline{D}_* \cup \overline{D}_{**}$.

We can now apply the proof of Lemma 4.2 to u_1^e and conclude that u_1^e has harmonic continuation into $\mathbf{R}^3 \setminus \tilde{S}$ where $\tilde{S}$ consists of S_0 and its image S_1 under the appropriate rotation. But since ∇u_1^e is in L^2_{loc}, the singularities of u_1^e on $\tilde{S}$ are removable (as in the proof of Lemma 4.2). Thus u_1^e is harmonic in $\mathbf{R}^3$, which is a contradiction.

§5. **The case where D is a circle.** In this section we assume that Ω is a half–space in $\mathbf{R}^2$, say $\{x_2 < 0\}$. We shall prove:

THEOREM 5.1. *Let B_1, B_2 be two open circles with $\overline{B}_j \subset \Omega$ and denote by u_j the solution of (1.1)–(1.3) with $D = B_j$. If (1.8) holds where Γ_0 is a nonempty open subset of $\partial\Omega$, then $B_1 = B_2$.*

For any ball $B = B(a, r)$ with center a and radius r, let $x^*(B)$ denote the inversion of a point x with respect to B, and set $K^*(B) = \{x^*(B); x \in K\}$. Also let $\sigma(A)$ denote the image of the set A under the mapping $(x_1, x_2) \longrightarrow (x_1, -x_2)$.

LEMMA 5.2. *Let S be any compact set on $x_2 = 0$ containing the support of g. Denote by u a solution of (1.1)–(1.3) when D is a circle $B = B(a, r)$ with $\overline{B} \subset \Omega$. Then u^e has harmonic continuation from $\Omega \setminus \overline{B}$ into*

$$\Omega \setminus (S(1) \cup \cdots \cup S(k) \cup \ldots, \{b\}, \{a\})$$

where $S(1)$ is $S^*(B), S(k)$ is $(\sigma(S(k-1)))^*(B)$, and b is some point of B; further, for any k, the first derivatives of the continuation are bounded in some neighborhood of $S(1) \cup \cdots \cup S(k)$.

Proof. Since $\partial u/\partial x_2 = 0$ on $\partial\Omega \setminus S, u^e$ has harmonic continuation onto $\mathbf{R}^2 \setminus (S \cup \overline{B} \cup \sigma(\overline{B}))$. The inversion $u_*^i(x) = u^i(x^*(B))$ yields a function u_*^i harmonic in $\mathbf{R}^2 \setminus \overline{B}$ and from the refraction conditions (1.5), (1.6),

$$u_*^i = u^e, -2\frac{\partial u_*^i}{\partial r} = \frac{\partial u^e}{\partial r} \quad \text{on } \partial B.$$

Let

$$u_1 = u_*^i - u^e, \quad u_2 = 2u_*^i + u^e.$$

Then u_1, u_2 are harmonic in $\mathbf{R}^2 \setminus (S \cup \overline{B} \cup \sigma(\overline{B}))$ and

$$u_1 = 0, \; \frac{\partial u_2}{\partial r} = 0 \quad \text{on } \partial B.$$

By inversion (with respect to B) the functions u_1, u_2 and therefore also u_*^i, u^e have harmonic continuation onto

$$\Omega \setminus (S(1) \cup E_1 \cup \{a\}), E_1 = (\sigma \overline{B})^*(B); \tag{5.1}$$

Moreover, the first derivatives of the continuation of u^e are bounded in a neighborhood of $S(1) = S^*(B)$.

We now repeat the previous argument with u^e being the harmonic continuation of the original u^e into the region (5.1). This leads to a harmonic continuation of u^e into the set

$$\Omega \setminus (S(1) \cup S(2) \cup E_2 \cup \{a\})$$

where $E_2 = (\sigma(E_1))^*(B)$.

Proceeding step -by–step we continue u^e harmonically into

$$\Omega \setminus (S(1) \cup \cdots \cup S(k) \cup E_k \cup \{a\})$$

where $E_k = (\sigma(E_{k-1}))^*(B)$. Clearly E_k converge to a single point b in B as $k \longrightarrow +\infty$, and the proof is complete.

Proof of Theorem 5.1. Suppose $B_1 \neq B_2$. Let $S_1(k), S_2(k), b_1, b_2, a_1, a_2$ be the sets from Lemma 5.2 related to B_1, B_2 respectively. We may assume S is symmetric with respect to the x_1–coordinate of the center of B_2 and that the distance from $S_2(1)$ to S is not greater than the distance from $S_1(1)$ to S.

Since $B_1 \neq B_2$ the intersection S_0 of $S_2(1)$ with $S_1(1) \cup \cdots \cup S_1(k) \cup \{b_1\} \cup \{a_1\}$ has no more than one accumulation point a_∞. As above $u_1^e = u_2^e$ on $\Omega \setminus (\overline{B}_1 \cup \overline{B}_2)$ and therefore from the Lemma 5.2 we conclude that u_2^e has a bounded harmonic continuation into the set $V \setminus S_0$ where V is some neighborhood of $S_2(1)$. Since u_2^e is bounded near any isolated point of S_0 this point is a removable singularity and u_2^e has a harmonic continuation into $V \setminus \{a_\infty\}$; similarly a_∞ is also removable singularity. It follows that u_2^e has a harmonic continuation into a neighborhood of $S_2(1)$.

The set S is the image of $S_2(1)$ under the inversion with respect to ∂B_2. Since u_2^e has harmonic continuation into a neighborhood of $S_2(1)$, it has also harmonic continuation into a neighborhood of S. This is a contradiction since $(\partial u_1^e / \partial x_2)(x_1, 0) = g$ and g is not analytic in a neighborhood of S.

§6. A Lipschitz stability estimate. The results of Bellout and Friedman [1] can be used to derive a Lipschitz stability estimate of the form (1.11) for a family of balls $B(h)$ as $h \to 0$, with $D_j = B(h_j)$; it is assumed here that $B(h)$ is monotone in h (For non–monotone family $B(h)$, results are given in [1] only in case $n = 2$.). We shall subsequently deal only with the case where the domains $D(h)$ are convex polyhedrons. We shall further assume that $n = 2$, that $\Omega = \{(x,y); y < 0\}$ and that $D(h)$ obtained from D ($\overline{D} \subset \Omega$) by the mapping

$$(x,y) \longrightarrow (x, y+h).$$

Denote by $u(x,y;h)$ the solution of the refraction problem (1.1)–(1.3) corresponding to $D(h)$ (we assume that $\overline{D(h)} \subset \Omega$) and set

$$D_1 = D_2, \quad D_2 = D(h).$$

We shall also denote by u_j the solutions of (1.1)–(1.3) corresponding to D_j.

THEOREM 6.1. *Assume that one edge of D is parallel to the y–axis. Then the estimate*

$$\text{dist}\,(D_1, D_2) \le C\ |u_1 - u_2|_{L^\infty(\Gamma_0)} \tag{6.1}$$

holds, where C is a positive constant independent of h; Γ_0 is any nonempty open subset of $\partial\Omega$.

REMARK 1.1. A similar result can be established in case $D(h)$ is obtained from D by a translation $(x,y) \longrightarrow (x+h, y)$, and one edge of D is parallel to the x–axis.

REMARK 6.2. The complimentary estimate to (6.1), namely,

$$|u_1 - u_2|_{L^\infty(\partial\Omega)} \le C'\ \text{dist}(D_1, D_2),$$

follows by standard elliptic estimates applied to the function w_h defined below.

Proof. If the assertion of the theorem is not true then we have

$$\frac{u(x,0;h) - u(x,0)}{h} \longrightarrow 0 \quad \text{uniformly on } \Gamma_0.$$

The function

$$w_h(x,y) = \frac{u(x,y-h;h) - u(x,y)}{h}$$

satisfies (1.1) and, as easily seen,

$$\frac{\partial w_h}{\partial N} \qquad \text{is bounded on } \partial D.$$

It follows that w_h has a subsequence which is convergent (weakly in $H^1(\Omega)$) to a solution w of (1.1). On Γ_0,

$$\begin{aligned} w_h(x,0) = \{[u(x,0;h) - u_y(x,0;h)h] - u(x,0+) + o(h)\}/h \\ \longrightarrow -g(x)\ , \quad \text{by (6.2)}, \\ \frac{\partial}{\partial y}\, w(x,0) = -u_{yy}(x,0) + o(1) \longrightarrow -u_{yy}(x,0). \end{aligned}$$

Since $w_h(x,y)$ is a solution of (1.1) with Cauchy data converging to $(-u_y, -u_{yy})$ on Γ_0, w is harmonic in $\Omega \setminus D$ having the Cauchy data $(-u_y, -u_{yy})$ on Γ_0. But $-u^e_y$ is also harmonic in $\Omega \setminus D$ and it has the same Cauchy data on Γ_0; consequently

$$w = -u^e_y \qquad \text{in} \qquad \Omega \setminus D \,. \tag{6.3}$$

The refraction conditions for w and u are:

$$\begin{aligned} w^i &= w^e, \qquad w^i_N = \frac{1}{2}\, w^e_N \,, \\ u^i &= u^e, \qquad u^i_N = \frac{1}{2}\, u^e_N \qquad \text{on} \quad \partial D \end{aligned} \tag{6.4}$$

where $w^e = w\big|_{\Omega\setminus D}$, $\quad w^i = w\big|_D$. Let us write

$$\frac{\partial}{\partial N} = N_x \frac{\partial}{\partial x} + N_y \frac{\partial}{\partial y}$$

and introduce the tangential derivative

$$\frac{\partial}{\partial N_\perp} = -N_y \frac{\partial}{\partial x} + N_x \frac{\partial}{\partial y} \qquad \text{along} \quad \partial D.$$

We shall use (6.4) in order to obtain expressions for w^i and w^i_N in terms of derivatives of u^i on ∂D. First we derive from (6.4), (6.3) the relations

$$\begin{aligned} -N_y\, w^i_x + N_x\, w^i_y &= N_y\, u^e_{xy} - N_x\, u^e_{yy} \,, \\ N_x w^i_x + N_y\, w^i_y &= -\frac{1}{2}\, N_x\, u^e_{xy} - \frac{1}{2}\, N_y\, u^e_{yy} \,, \\ -N_y\, u^i_x + N_x\, u^i_y &= -N_y\, u^e_x + N_x\, u^e_y \,, \\ N_x\, u^i_x + N_y\, u^i_y &= \frac{1}{2}\, N_x u^e_x + \frac{1}{2}\, N_y\, u^e_y \,. \end{aligned} \tag{6.5}$$

Using (6.3) and expressing u^e_y from the third and fourth equations in (6.5), we get

$$\begin{aligned} w^i &= (N_y\, N_x - 2\, N_x\, N_y) u^i_x - (N_x^2 + 2\, N_y^2) u^i_y \\ &= -N_x\, N_y\, u^i_x - (1 + N_y^2) u^i_y \qquad \text{on} \quad \partial D. \end{aligned} \tag{6.6}$$

Next, the second equation in (6.4) can be written in the form

$$w^i_N = -\frac{1}{2}\, N_x\, u^e_{xy} - \frac{1}{2}\, N_y\, u^e_{yy} \qquad \text{on} \quad \partial D. \tag{6.7}$$

Applying $\partial/\partial N_\perp$ to the third and fourth equations in (6.5), we obtain

$$\begin{aligned} &N_y^2 u^i_{xx} - N_x\ N_y\ u^i_{xy} - N_y N_x u^i_{xy} + N_x^2\ u^i_{yy} \\ =&N_y^2\ u^e_{xx} - 2N_x N_y u^e_{xx} + N_x^2\ u^e_{yy} \end{aligned}$$

and

$$\begin{aligned} &- N_x N_y\ u^i_{xx} + N_x^2 u^i_{xy} - N_y^2 u^i_{xy} + N_x N_y u^i_{yy} \\ &= \frac{1}{2}\ N_x N_y u^e_{xx} + \frac{1}{2} N_x^2 u^e_{xy} - \frac{1}{2} N_y^2 u^e_{xy} + N_x N_y\ u^e_{yy} \end{aligned}$$

or

$$\begin{aligned} &-2N_y N_x u^i_{xy} + (N_x^2 - N_y^2) u^i_{yy} \\ &\quad = -2N_y N_x u^e_{xy} + (N_x^2 - N_y^2) u^e_{yy} \end{aligned} \tag{6.8}$$

$$\begin{aligned} &2N_x N_y u^i_{yy} + (N_x^2 - N_y^2) u^i_{xy} \\ &\quad = N_x N_y u^e_{yy} + \frac{1}{2}\ (N_x^2 - N_y^2) u^e_{xy}\ . \end{aligned} \tag{6.9}$$

From these equations we can solve for u^e_{xy} and u^e_{yy}:

$$u^e_{xy} = 2(N_x^2 + N_y^2) u^i_{xy} + 2N_x N_y (N_x^2 - N_y^2) u^i_{yy}\ , \tag{6.10}$$

$$u^e_{yy} = 2N_x N_y (N_x^2 - N_y^2) u^i_{xy} + (1 + 4N_x^2 N_y^2) u^i_{yy}\ . \tag{6.11}$$

Substituting these expressions into (6.7) we easily get

$$w^i_N = -N_x^3\ u^i_{xy} - (\frac{1}{2} + N_x^2) N_y\ u^i_{yy} \qquad \text{on} \quad \partial D. \tag{6.12}$$

Let σ be an edge of D parallel to the y–axis. Then $N_x = 1$, $N_y = 0$ on σ and (6.6), (6.12) yields:

$$w^i = -u^i_y, \quad w^i_N = -u^i_{xy} = -\frac{\partial}{\partial N} u^i_y \qquad \text{on} \quad \sigma.$$

It follows that

$$w^i \equiv -u^i_y \qquad \text{in} \qquad D. \tag{6.13}$$

Now let τ be any edge of D which is not parallel to the y–axis; then $N_y \neq 0$ along τ. Using (6.13) we obtain from (6.6), (6.12),

$$-N_x u^i_x + N_y\ u^i_y = 0\ , \tag{6.14}$$

$$N_x(1-N_x^2)u^i_{xy}+\left(\frac{1}{2}-N_x^2\right)N_y\ u^i_{yy}=0 \tag{6.15}$$

along τ. Applying $\partial/\partial N_\perp$ to (6.14) results in

$$(N_x^2-N_y^2)u^i_{xy}+2N_x\ N_y\ u^i_{yy}=0. \tag{6.16}$$

Consider (6.15), (6.16) as a linear system for the unknown variables u^i_{xy}, u^i_{yy} along τ. Since the determinant of the coefficients is equal to $-1/2$, it follows that

$$u^i_{xy}=u^i_{yy}=0 \qquad \text{along } \tau\ .$$

This implies that $\nabla w^i = 0$ along τ and therefore $w^i \equiv \text{const}.$ in D. It follows that also $w^e = \text{const}.$ and, in particular, $g = u^e_y = -w^e = \text{const}.$ on $\partial\Omega$, a contradiction.

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