A NONLINEAR NONLOCAL WAVE EQUATION ARISING IN COMBUSTION THEORY

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IMA Preprint Series # 462

December 1988

A NONLINEAR NONLOCAL WAVE EQUATION ARISING IN COMBUSTION THEORY*

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Abstract. We consider the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\frac{\partial T}{\partial t} = \left(\gamma\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)e^T \quad , \quad \gamma > 1$$

which describes a model of shock–induced ignition of initially cold combustible gas. Under some assumptions on the initial data we prove that there exists a unique solution T, and $|\nabla T_t|$ blows up along a Lipschitz curve $t = \phi(x)$ with $|\phi'(x)| < 1$ a.e. We also study the behavior of the solution as $\gamma \to 1$.

Key words. Nonlocal wave equations, shock, blow–up of solutions, combustion

AMS(MOS) subject classifications. 35L05, 35630, 35L67, 35L70

§1. **Introduction; statement of the main results.** This paper deals with the nonlinear partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\frac{\partial T}{\partial t} = \left(\gamma\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)e^T \quad , \quad \gamma > 1\,. \tag{1.1}$$

This equation describes the behavior of combustible gas located between a moving piston and the shock wave that it generates. More precisely (cf. [4] [7]), assume that the gas is capable of undergoing an exothermic reaction of the Arrhenius type, and suppose that the shock wave created by the moving piston is strong enough to raise the temperature of the gas to ignition temperature. Then, assuming a very high activation energy, the state of the gas between the piston and the shock is described, to a first order approximation, by the set of first order partial differential equations

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \frac{\rho_0}{\sqrt{\overline{T}_0}}\,\frac{\partial v_1}{\partial \xi} = 0,\\
&\rho_0\frac{\partial v_1}{\partial t} + \frac{1}{\gamma\sqrt{\overline{T}_0}}\,\frac{\partial p_1}{\partial \xi} = 0,\\
&p_1 - \rho_0\overline{T}_1 - \overline{T}_0\rho_1 = 0,\\
&\rho_0\frac{\partial \overline{T}_1}{\partial t} - \frac{\gamma-1}{\gamma}\,\frac{\partial p_1}{\partial t} = \rho_0 e^{\overline{T}_1}\,.
\end{aligned} \tag{1.2}$$

*This paper is partially supported by National Science Foundation Grant DMS–86–12880, and CICYT Research Grant PB86–0112–C02–02.

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Here ξ is a new spatial variable (moving with the piston speed); $\overline{T}_0, \rho_0, p_0$ and v_0 are, respectively, the temperature, density, pressure and velocity of the gas in the absence of chemistry, and $\overline{T}_1, \rho_1, p_1, v_1$ are the corresponding first order corrections generated by the shock front taking the chemistry into account. Thus, if T, ρ, p and v are the actual temperature, density, pressure and velocity of the gas, then

$$\begin{aligned} T &= \overline{T}_0 + \epsilon \overline{T}_1 + \dots, & \rho &= \rho_0 + \epsilon \rho_1 + \dots, \\ p &= p_0 + \epsilon p_1 + \dots, & v &= v_0 + \epsilon v_1 + \dots, \end{aligned} \tag{1.3}$$

where $\epsilon \sim \overline{T}_0^2/\theta$ and θ is the activation energy; $\theta \gg 1$. Finally, γ is the ratio between the specific heats at constant pressure and constant volume. It is easy to verify that equation (1.1) for the function $\overline{T}_1$ can be derived from (1.2) by some manipulations, assuming appropriate smoothness which is expected to hold in the region between the piston and the shock. As long as the solution $\overline{T}_1$ of (1.1) remains bounded, the solution of (1.2) also remains bounded.

It is of interest in combustion theory to describe under what circumstances a combustible gas can be brought into ignition and what is the curve $t = \phi(x)$ along which ignition takes place. We refer to [4] [7] [8] [9] [11] and [13] for a description of several physical situations which give rise to ignition as well as to [15] for a general discussion on combustion theory.

Ignition is described mathematically by the fact that the temperature blows up in finite time. In [7] [8] (see also [4]) it is shown by formal asymptotic expansions that (1.2), or (1.1), may lead to ignition; the piston problem described above is considered in [7], whereas the Cauchy problem whereby the gas occupies the entire space is considered in [8]. Of course, near the blow–up curve $t = \phi(x)$ the approximation (1.3) is no longer valid. Thus the shape of the blow–up curve tells us only the approximate shape of the true ignition curve; a new scaling needs to be made in the vicinity of the blow–up curve.

In this paper we show under some restrictions on the initial data that the smooth solution of (1.1) exists for all $t < \phi(x)$ where $\phi(x)$ is a Lipschitz continuous function with $|\phi'(x)| < 1$ $a \cdot e$; the solution cannot be extended beyond this curve in the sense that some of its second derivatives become unbounded along $t = \phi(x)$.

It will always be assumed that $\gamma > 1$. It will be convenient to rewrite (1.1) in the form

$$\Box\, u = (\gamma - 1)\frac{\partial^2 e^T}{\partial t^2} \tag{1.4}$$

where

$$\frac{\partial T}{\partial t} - e^T = u \tag{1.5}$$

and $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$. Notice that (1.4) reduces to the free wave equation in the limit case $\gamma = 1$. Equation (1.4) with T satisfying (1.5) can be thought of as a wave equation with nonlinear nonlocal forcing terms.

For the simpler situation

$$\Box u = f(u) , \quad f(u) \sim |u|^p , \ p > 1$$

the occurrence of blow–up was established in [6] [10] [12]; analysis of the blow–up surface was carried out in [2] [3].

In this paper we study the initial value problem corresponding to (1.1); thus we prescribe the initial values

$$T(x,0) = T_0(x), \quad T_t(x,0) = T_1(x), \quad T_{tt}(x,0) = T_2(x); \tag{1.6}$$

the corresponding combustion problem is described in [8]. Since our techniques are local in nature, they apply in particular also to bounded domains and, therefore, can be used to describe the gas behavior in some regions of the gas between the expanding piston and the shock in the problem described above.

We shall impose the following conditions:

$$\begin{aligned} &T_0 \in C^3(\mathbf{R}), \quad T_1 \in C^2(\mathbf{R}), \quad T_2 \in C^1(\mathbf{R}), \quad T_0 \geq 0; \\ &T_0, T_0', T_0'', T_1, T_1' \ \text{ and } T_2 \text{ are uniformly bounded,} \end{aligned} \tag{1.7}$$

$$T_1 \geq e^{T_0} , \tag{1.8}$$

and there exist a $\theta \in (0,1)$ such that

$$\theta T_1 \geq |T_0'| , \tag{1.9}$$

$$\theta(T_2 - e^{T_0}T_1) \geq |T_1' - e^{T_0}T_0'| , \tag{1.10}$$

$$\begin{aligned} &\theta \ D_t(T_2 - e^{T_0}T_1) \pm D_t(T_1' - e^{T_0}T_0') \\ &\quad \geq (\gamma - 1)e^{T_0} \left\{\theta(T_2 - e^{T_0}T_1) \pm \left(T_1' - e^{T_0}T_0'\right)\right\} , \end{aligned} \tag{1.11}$$

where the various terms in (1.11) are computed using (1.6) and (in the case of D_tT_2) (1.1).

Example. Assume that $\rho_0 > 0$, $\overline{T}_0 > 0$. Then a family of explicit solutions of (1.2) is given by

$$\begin{aligned} &\rho_1 \equiv \text{const.}, \quad v_1 \equiv \text{const.}, \\ &T_1(t) = \log \frac{1}{1-\gamma t} , \\ &\rho_1(t) = \rho_0 \ \log \frac{1}{1-\gamma t} + \rho_1 \overline{T}_0 . \end{aligned}$$

Notice that $T(x,t) \equiv T_1(t)$ solves (1.1) and assumption (1.7)–(1.11) are fulfilled in this case.

REMARK 1.1. From (1.10) it follows that

$$T_2 - e^{T_0} T_1 \geq 0 \,. \tag{1.12}$$

We now state the main result of this paper.

THEOREM 1.1. *Assume that (1.7)–(1.11) hold. Then there exists a unique positive valued Lipschitz continuous function $\phi(x)$ and a unique pair of functions $u(x,t)$, $T(x,t)$ such that*

$$|\phi'(x)| \leq \theta \quad \text{a.e.,} \qquad \theta \text{ as in (1.9)–(1.11)}, \tag{1.13}$$

(1.14) *u and T satisfy (1.4), (1.5) in the domain* $Q_\phi = \{(x,t);\ x \in \mathbf{R}, \quad 0 < t < \phi(x)\}$,

and all the second derivatives of T and T_t are continuos in $Q_\phi \cup \{t=0\}$,

(1.15) T *satisfies (1.6),*

and, for any x_0 real and $\delta > 0$,

$$|T_{tt}| + |T_{tx}| \quad \text{is unbounded in } \{x_0 < x < x_0 + \delta, \quad \phi(x) - \delta < t < \phi(x). \tag{1.16}$$

In §2 we approximate the problem (1.4)–(1.6) by a sequence of truncated problems and prove some a priori estimates. In §3 we give the proof of Theorem 1.1, based on the results of §2. Finally, in §4, we consider the case $\gamma \to 1$ (almost compressible gas) and study the asymptotic behavior of the solution. Some formal asymptotic expansions for this case have been derived recently by Blythe and Crighton [1].

§2. Auxiliary estimates for approximating problems. Let $\tilde{u}(x,t)$ be a solution of the problem

$$\Box\, \tilde{u} = f(x,t), \tag{2.1}$$

$$\tilde{u}(x,0) = u_0(x), \quad \tilde{u}_t(x,0) = u_1(x) \tag{2.2}$$

in some region $Q_\zeta = \{(x,t);\ x \in \mathbf{R},\ 0 < t < \zeta(x)\}$ where $\zeta(x)$ is positive valued Lipschitz continuous function with $|\zeta'(x)| \leq 1$ a.e. Then the following well known representation formula holds:

$$\begin{aligned} \tilde{u}(x,t) = \frac{1}{2}\,[u_0(x+t) + u_0(x-t)] + \frac{1}{2} \int\limits_{x-t}^{x+t} u_1(s)\, ds \\ + \frac{1}{2} \iint\limits_{K^-(x,t)} f(\xi,\eta)\, d\xi\, d\eta \end{aligned} \tag{2.3}$$

for any $(x,t) \in Q_\zeta$, where

$$K^-(x,t) = \{(\xi,\eta); |x-\xi| < t-\eta,\ 0<\eta<t\}.$$

In the sequel we shall also use the notation:

$$\partial K^-(x,t) = \{(\xi,\eta);\quad |x-\xi| = t-\eta,\quad 0<\eta<t\},$$
$$K^+(x,t) = \{(\xi,\eta);\quad |x-\xi| < \eta-t,\quad t<\eta<\infty\}.$$

Rewriting (1.1) in the form (1.4), (1.5), we deduce that

$$\square\, u = (\gamma-1)e^T[e^T(e^T+u) + \frac{\partial u}{\partial t} + (e^T+u)^2]. \tag{2.4}$$

We shall find it convenient to work with the system (2.4), (1.5). In this section we approximate this system by

$$\square\, u_n = (\gamma-1)e_n(T_n)[e_n(T_n)(e_n(T_n)+u_n) + \frac{\partial u_n}{\partial t} + f_n(e_n(T_n)+u_n)], \tag{2.5}$$

$$\frac{\partial T_n}{\partial t} - e_n(T_n) = u_n \tag{2.6}$$

where $e_n(s),\ f_n(s)$ are nonnegative C^∞ functions satisfying the following properties:

$$\begin{gathered} e_n'(s) \geq 0,\quad e_n(s) = e^s \quad \text{if} \quad -\infty < s \leq n-1,\quad e_n(s) = e^n - \frac{1}{2} \quad \text{if} \quad s > n, \\ e_n'(s) \leq e^s, e_n(s) \leq e^s \quad \text{for any } n \text{ and } s \end{gathered} \tag{2.7}$$

$$f_n'(s) \geq 0,\quad f_n(s) = s^2 \quad \text{if} \quad |s| \leq n-1,\quad f_n(s) = n^2 \quad \text{if} \quad |s| > n. \tag{2.8}$$

LEMMA 2.1. *The system (2.5), (2.6), (1.6) has a unique solution* $(u_n,\ T_n)$ *for all* $t>0$.

Proof. For simplicity we drop the index n. Using the transformation (cf. [14])

$$v = \frac{1}{2}\,(u_x+u_t),\quad w = \frac{1}{2}\,(u_x-u_t)$$

we can rewrite (2.5), (2.6), (1.6), in the form

$$\begin{aligned} v_t - v_x &= \frac{1}{2}\, F(T, I(v,w),\ v-w), \\ w_t + w_x &= -\frac{1}{2}\, F(T, I(v,w),\ v-w), \\ T_t - e(T) &= I(v,w) \end{aligned} \tag{2.9}$$

where

$$
(2.10)\qquad
\begin{aligned}
I(v,w) &= T_1 - e(T_0) + \int_0^t (v-w)(0,s)\,ds + \int_0^x (v+w)(\xi,t)\,d\xi \\
F(T,I(v,w)),v-w) &= (\gamma-1)e(T)[e(T)(e(T)+I(v,w) \\
&\quad + v - w + f(e(T)+I(v,w))]
\end{aligned}
$$

with $v(x,0) \equiv v_0\ (x)$, $w(x,0) \equiv w_0(x)$ determined from (2.6), (1.6). We can rewrite (2.9) in the integrated form:

$$
(2.11)\qquad
\begin{aligned}
v(x,t) &= v_0(x+t) + \frac{1}{2}\int_0^t F(T,I(v,w),\ v-w)(x+t-s,s)\,ds \\
w(x,t) &= w_0(x-t) - \frac{1}{2}\int_0^t F(T,I(v,w),\ v-w)(x-t+s,s)\,ds \\
T(x,t) &= T_0(x) + \int_0^t (e(T)+I(v,w))(x,s)\,ds.
\end{aligned}
$$

We can now apply the successive iteration technique as used for ODE's in order to show that the system (2.11) with F, I defined by (2.10) has a unique solution (cf. [14]); since e and f are bounded functions, both I and F have a linear growth in v, w and are bounded in T and, therefore, the solution of (2.10) exists for any t-interval.

LEMMA 2.2. *The solution (u_n, T_n) satisfies:*

$$
(2.12)\qquad u_n \geq 0,
$$

$$
(2.13)\qquad T_n \geq 0,
$$

$$
(2.14)\qquad \frac{\partial u_n}{\partial t} \geq 0\ .
$$

for $n \geq n_0$ and some n_0 large enough (depending on $\|T_0\|_\infty$)

Proof. For simplicity we drop the index n. By assumptions (here we use (1.7) and (1.12)) the inequalities (2.12)–(2.14) hold for $t = 0$. We shall suppose first that

$$
(2.15)\qquad u > 0\ ,\ T > 0,\ \ \frac{\partial u}{\partial t} > 0 \quad \text{at} \quad t = 0
$$

and prove that

$$(2.16) \qquad u > 0, \quad T > 0, \quad \frac{\partial u}{\partial t} > 0 \qquad \text{for all} \quad t > 0.$$

Suppose that (2.16) is not true. Then there exists a triangle Δ with vertices $A = (a,0), B = (b,0), P$ and sides $l_1 = \{(x,0),\ a \le x \le b\}, l_2 = \{(x,t); x - t = a\},\ \ l_3 = \{(x,t); x + t = b\}$ such that (2.16) holds in $\overline{\Delta} \setminus P$ but at least one of the inequalities becomes an equality at P.

Since the right–hand side of (2.5) is positive in Δ, from the representation formula (2.3) we deduce that $u(P) > 0$. From (2.6) it is also clear that $T(P) > 0$. Finally, from the representation formula (2.3) applied to u_t we have

$$u_t(P) = \frac{1}{2}\,(u_t(a,0) + u_t(b,0)) + \frac{1}{2}\int_a^b u_{tt}(\xi,0)\,d\xi$$
$$+\frac{1}{2}\iint_{K^-(x_0,t_0)} \Box\, u_t$$

where $P = (x_0, t_0)$, and the last term is equal to

$$\frac{1}{2\sqrt{2}}\int_{\partial K^-(x_0,t_0)} \Box\, u\ ds - \frac{1}{2}\int_a^b \Box\, u(\xi,0)\ d\xi\ .$$

Since $\Box\, u > 0$ on $\partial K^-(x_0,t_0) \setminus \{(x_0,t_0)\}$ whereas

$$-\frac{1}{2}\int_a^b \Box\, u(\xi,0)\ d\xi = -\frac{1}{2}\int_a^b [u_{tt}(\xi,0) - u_{xx}(\xi,0)]\ d\xi,$$

we get

$$u_t(P) > \frac{1}{2}\,(u_t(a,0) + u_t(b,0)) + \frac{1}{2}\,(u_x(b,0) - u_x(a,0)) \ge 0$$

by (1.10) if n is large enough, since $e_n(T_0) = e^{T_0}$ in that case. Thus (2.16) holds at P, which is a contradiction.

Having proved the lemma in case (2.15) holds, we note that the lemma is also valid if we modify the general initial data satisfying (2.12)–(2.14) by replacing T and its derivatives at $t = 0$ by $T + \epsilon e^{\alpha t}$ for some $\alpha > 0$ and any small $\epsilon > 0$. Since the corresponding solutions $T_{\epsilon n}, u_{\epsilon n}$ converge to T_n, u_n as $\epsilon \to 0$ (by using the representation (2.9), (2.10)), the proof of the lemma is complete

LEMMA 2.3. *The solution* (u_n, T_n) *satisfies:*

$$\frac{\partial}{\partial t} u_n \geq \left|\frac{\partial}{\partial x} u_n\right|, \tag{2.17}$$

$$\frac{\partial}{\partial t} T_n \geq \left|\frac{\partial}{\partial x} T_n\right|. \tag{2.18}$$

Proof. By Lemma 2.2 and (2.5), u_n is a supersolution, i.e. $\Box\, u_n \geq 0$. Hence (2.17) follows by (1.9) (with $\theta = 1$) and the representation formula (2.3), as in proof of Lemma 1.3 of [3]. Set

$$\widehat{T} = \frac{\partial T_n}{\partial t} \pm \frac{\partial T_n}{\partial x} .$$

Then, by (2.6), (2.17),

$$\frac{\partial \widehat{T}}{\partial t} - e'(T)\widehat{T} = \frac{\partial u_n}{\partial t} \pm \frac{\partial u_n}{\partial x} \geq 0 ,$$

and, by (1.9), $\widehat{T}(x,0) \geq 0$. Hence $\widehat{T} \geq 0$ for all $t > 0$, i.e., (2.18) holds.

§3. **Proof of Theorem 1.1.** We proceed by the method of Caffarelli and Friedman [3]. We choose a dense sequence $X_j = (x_j, t_j)$ in $\mathbf{R} \times (0, \infty)$. Since $u_n \geq 0$, there exists a subsequence of u_n (which is also labelled u_n) such that, for each j,

$$\lim_{n\to\infty} u_n(X_j) \quad \text{either exists and is finite, or is equal to } +\infty. \tag{3.1}$$

By Lemma 2.3,

$$\text{if} \quad \lim_{n\to\infty} u_n(X_j) < \infty \quad \text{then} \quad u_n(X) \leq C \quad \text{in } K^-(X_j), \tag{3.2}$$

where $C = \lim u_n(X_j)$, whereas

$$\text{if} \quad \lim_{n\to\infty} u_n(X_j) = \infty \quad \text{thus } u_n(X) \to \infty \text{ as } n \to \infty, \text{ uniformly in } K^+(X_j). \tag{3.3}$$

We claim that

$$\text{if (3.2) holds then } T_n \text{ and } \partial T_n/\partial t \text{ are uniformly bounded in any compact subset } Q \text{ of } K^-(X_j). \tag{3.4}$$

Indeed, suppose

$$T_n(\overline{x},\overline{t}) \to \infty \quad \text{for some } (\overline{x},\overline{t}) \in K^-(X_j).$$

Thus, in view of (2.18), there exists a triangle Δ with closure in $K^-(X_j)$, base

$$\sigma = \{(x,t_0); \quad a < x < b, \quad t_0 < \overline{t}\}$$

and vertices $(a,t_0), (b,t_0), P$, such that, as $n \to \infty$,

$$T_n(x,t) \to \infty \quad \text{uniformly in } \Delta.$$

Since further $u_n \geq 0, \quad \partial u_n/\partial t \geq 0$, we see from (2.5) that

$$\Box\, u_n \to \infty \quad \text{uniformly in } \Delta.$$

Using the representation formula (2.3) we then conclude that $u_n(P) \to \infty$, a contradiction to (3.2).

Having proved that T_n is uniformly bounded in Q, we now use (2.6) to deduce that also $\partial T_n/\partial t$ is uniformly bounded in Q.

Next, as in [3], we define the curve $t = \phi(x)$ as the upper boundary of the closure of the union of the sets $K^-(X_j)$ with X_j as in (3.2). Then, since $\partial T_n/\partial t \geq e^{T_n}, T_n \geq 0$,

$$0 < \phi(x) < \infty \qquad \forall \quad x \in \mathbb{R};$$

furthermore, by (2.18)

$$|\phi'(x)| \leq 1 \quad \text{a.e.}$$

We may also assume that

$$\text{(3.5)} \qquad \left(u_n,\ T_n,\ \frac{\partial T_n}{\partial t}\right) \quad \text{is convergent } (L^\infty)^*\text{-weakly in every compact subset of } Q_\phi \cup \{t = 0\}.$$

Notice that in every such set

$$e_n(T_n) = e^{T_n}, f_n(e_n(T_n) + u_n) = (e^{T_n} + u_n)^2$$

if n is large enough.

We now use the representation formula (2.3) for u_n; the last integral on the right–hand side has the form

$$\iint\limits_{K^-(x,t)} \Psi_n d\xi d\tau + \frac{1}{\sqrt{2}}(\gamma - 1) \int\limits_{l(x,t)} e^{T_n} u_n\, ds$$

where $l(x,t)$ consists of the two non–horizontal edges of $\partial K^-(x,t)$, and $|\Psi_n| \leq C$. Consequently,

$$u_n(x,t) = \frac{1}{2\sqrt{2}}(\gamma - 1) \int_{l(x,t)} e^{T_n} u_n \, ds + H_n(x,t) \tag{3.6}$$

where $H_n(x,t)$ is uniformly Lipschitz continuous with Lipschitz constant independent of n. Similarly we can represent T_n by an integral formula involving u_n, solving

$$\frac{dT_n}{dt} + e^{T_n} = u_n.$$

Writing (3.6) at $(x+a,t)$ and taking the difference of the two equations, we get an integral representation for $u_n(x+a,t) - u_n(x,t)$. Similarly we get an integral representation for $T_n(x+a,t) - T_n(x,t)$. Applying Gronwall's inequality we deduce that

$$\begin{aligned} &\max_x\{|u_n(x+a,t) - u(x,t)| + |T_n(x+a,t) - T_n(x,t)|\} \\ &\leq C|a| \end{aligned}$$

where $0 \leq t < t_0$ and x varies in the set $\{\xi; (\xi,t) \in K^-(x_0,t_0)\}$. Similarly we can estimate differences of u_n and T_n with respect to the t variables. Hence

(3.7) u_n, T_n and $\dfrac{\partial T_n}{\partial t}$ are uniformly Lipschitz continuous in every compact subset of $Q_\phi \cup \{t=0\}$, with Lipschitz constant independent of n.

We now repeat the argument for $\dfrac{\partial u_n}{\partial t}$: Using the representation formula (2.3) for $\partial u_n/\partial t$ we get

$$\frac{\partial u_n}{\partial t}(x,t) = \frac{1}{2\sqrt{2}}(\gamma-1) \int_{l(x,t)} e^{T_n} \frac{\partial u_n}{\partial t} \, ds + \int_{l(x,t)} R_n \, ds + W_n$$

where R_n depends only on u_n and T_n, and R_n, W_n are uniformly Lipschitz continuous. Hence, taking finite differences as before and using Gronwall's inequality, we find that

(3.8) $\dfrac{\partial u_n}{\partial t}$ is uniformly Lipschitz continuous in every compact subset of $Q_\phi \cap \{t=0\}$, with Lipschitz constant independent of n.

The same can also be established for $\partial u_n/\partial x$ and T_n. Hence

$$\frac{\partial u_n}{\partial x} \longrightarrow \frac{\partial u}{\partial x}, \frac{\partial u_n}{\partial t} \longrightarrow \frac{\partial u}{\partial t}, \; T_n \longrightarrow T \tag{3.9}$$

uniformly in compact subsets of $Q_\phi \cup \{t=0\}$.

We can therefore go to the limit in the integral representation (2.3) for u_n and deduce that (u,T) forms a solution of (1.4), (1.5) with T_t, T in $C^2(Q_\phi \cup \{t=0\})$. As to (1.13), it is a consequence of our next result.

LEMMA 3.1. *The following inequalities hold in Q_ϕ:*

$$\theta \frac{\partial T}{\partial t} \geq \left|\frac{\partial T}{\partial x}\right| , \tag{3.10}$$

$$\theta \frac{\partial u}{\partial t} \geq \left|\frac{\partial u}{\partial x}\right| . \tag{3.11}$$

Proof. The inequalities (3.10), (3.11) hold at $t=0$ (by (1.9), (1.10)). By approximation of the initial data (as in the proof of Lemma 2.2) we may assume that, in fact

$$\theta \frac{\partial T}{\partial t} > \left|\frac{\partial T}{\partial x}\right| , \tag{3.12}$$

$$\theta \frac{\partial u}{\partial t} > \left|\frac{\partial u}{\partial x}\right| \tag{3.13}$$

at $t=0$. We proceed to prove these inequalities for all $t>0$; for these approximated data we also have the strict inequalities (2.16).

If (3.12), (3.13) are not valid for all $t>0$ thus there is a closed triangle $\triangle$ with vertices $A=(a,0)$, $B=(b,0), P$, as in the proof of Lemma 2.2, such that (3.12), (3.13) hold in $\triangle \setminus P$ but equality holds in either (3.12) or (3.13) at the point P.

Introduce the pair of functions

$$\widehat{T} = \theta T_t \pm T_x, \quad \hat{u} = \theta u_t \pm u_x .$$

Then, as easily seen

$$\hat{u} = \widehat{T}_t - e^T \widehat{T} . \tag{3.14}$$

Applying $\Box$ to (3.14) and using (2.4) we get

$$\begin{aligned}\Box\, \hat{u} = &(\gamma-1)e^T[e^T T_t + u_t + (u+e^T)^2]\widehat{T} \\ &+ (\gamma-1)e^T[e^T\widehat{T}_t + \hat{u}_t + 2(u+e^T)\widehat{T}_t + e^T\widehat{T}\, T_t].\end{aligned}$$

Since $T_t > 0, u_t > 0,\ \widehat{T} > 0,\ \widehat{T}_t > 0,\ \widehat{T}_t = \hat{u} + e^T \widehat{T} > 0$ in $\triangle$, we obtain

$$\begin{aligned}\square\, \hat{u} &> (\gamma - 1)e^T(\hat{u}_t + 2T_t\hat{T}_t) \\ &= (\gamma - 1)\left[(e^T\hat{u})_t - e^T\hat{u}T_t + 2T_t\widehat{T}_t\right] \\ &> (\gamma - 1)(e^T\hat{u})_t.\end{aligned}$$

Using the representation formula (2.3) we get

$$\begin{aligned}\hat{u}(P) > \frac{1}{2}\,[\hat{u}(a,0) + \hat{u}(b,0)] &+ \frac{1}{2}\int_a^b \hat{u}_t(\xi,0)\, d\xi \\ &- \frac{1}{2}(\gamma-1)\int_a^b e^{T_0(\xi)}\hat{u}(\xi,0)\, d\xi,\end{aligned}$$

and since, by (1.11),

$$\hat{u}_t(\xi,0) > (\gamma-1)e^{T_0(\xi)}\hat{u}(\xi,0),$$

we get $\hat{u}(P) > 0$. From (3.14) we also obtain $\widehat{T}_t(P) > 0$, and thus the strict inequalities (3.12), (3.13) hold at P, a contradiction.

It remains to prove the assertion (1.16). If this is not true then, since $|\phi'(x)| \le \theta < 1$, for any $\delta > 0$ there exists a triangle $\triangle$ with base $\sigma = \{(x,t_0);\ a < x < b\}$, sides $\{x+t = x_0 + t_0\}$, $\{x - t = x_0 - t_0\}$, vertical height δ, and vertices $A = (a,t_0), B = (b,t_0), P = (x_0,t_0)$ such that σ lies in Q_ϕ,

$$t_0 > \phi(x_0),$$

and

$$|u_x| + |u_t| + |T| \le C_0 \qquad \text{on} \quad \sigma$$

By (3.9)

$$|u_{n,x}| + |u_n, t| + |T_n| \le C_0 + 1 \qquad \text{on} \quad \sigma$$

if n is sufficiently large.

Using the integral representation (2.11) we then find that

$$\left|\frac{\partial u_n}{\partial x}\right| + \left|\frac{\partial u_n}{\partial t}\right| + |T_n| \le C \quad \text{in } \triangle$$

provided δ is small enough depending only on C_0, where C is a constant depending only on C_0, δ. Thus $\triangle$ must lie in $\{t < \phi(x)\}$, which is a contradiction.

We have completed the proof of existence. To prove uniqueness we simply recall the argument of Lemma 2.1, noting that for any triangle with vertices $(a,s),(b,s),(x_0,t_0)$ where $t_0 > s$ and $x_0 - t_0 = a - s$, $\quad x_0 + t_0 = b + s$, if $(x_0,t_0) \in Q_\phi$ then $\triangle \subset Q_\phi$, by (1.13).

REMARK 3.1. Denote by $d(x_0, t_0)$ the distance from a point (x_0, t_0) in Q_ϕ to the blow-up curve $\{t = \phi(x)\}$. Then

$$T(x,t) \le \log \frac{1}{d(x,t)} \ . \tag{3.15}$$

Indeed, from (2.6), (2.12) we have

$$\frac{\partial T_n}{\partial t} \ge e_n(T_n)$$

and therefore

$$\int_{T_n(x,t)}^{T_n(x,t_1)} \frac{ds}{e_n(s)} \ge t_1 - t \ .$$

Taking $t_1 > \phi(x) > t$ and $n \to \infty$, we get

$$\int_{T(x,t)}^{\infty} \frac{ds}{e^s} \ge t_1 - t \ ,$$

or $T(x,t) \le \log 1/(t_1 - t)$. Taking finally $t_1 \downarrow \phi(x)$, (3.15) follows.

REMARK 3.2. The inequality

$$u(x,t) \le \frac{C}{d(x,t)} \qquad (C \quad \text{constant}) \tag{3.16}$$

holds in Q_ϕ. Indeed, applying (2.3) to u and using (1.4) we get

$$u(x_0, t_0) \le \frac{1}{2\sqrt{2}}(\gamma - 1) \int_{l_1 \cup l_2} e^T \frac{\partial T}{\partial t}\, ds + C_1$$

where l_1, l_2 are the two sides of $\partial K^-(x_0, t_0)$ with positive and negative slopes, respectively. Also, by Lemma 3.1,

$$\frac{\partial T}{\partial t} \le C(-1)^{i+1} \frac{dT}{ds} \quad \text{along } l_i.$$

Hence

$$u(x_0, t_0) \le (\gamma - 1) C e^{T(x_0, t_0)} + C_2 \ , \tag{3.18}$$

and (3.16) now follows from (3.15).

§4. **The case $\gamma \to 1$.** When $\gamma = 1$ the system (1.4)–(1.6) becomes

$$\Box\, u = 0, \tag{4.1}$$

(4.2)
$$\frac{\partial T}{\partial t} - e^T = u$$

with initial conditions

(4.3)
$$\begin{cases} u(x,0) = T_1 - e^{T_0} \equiv u_0(x), \\ u_t(x,0) = T_0'' - e^{T_0} T_1 \equiv u_1(x) \end{cases}$$

and

(4.4)
$$T(x,0) = T_0 \ .$$

Denote by $\overline{u}$ the solution of (4.1), (4.3) and denote by $\overline{T}$ the solution of (4.2), (4.4) with $u = \overline{u}$. Then

$$\overline{u}(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi \ ,$$

and $\overline{T}$ satisfies the integral equation

(4.5)
$$e^{-T_0(x)} - e^{-\overline{T}(x,t)} = \int_0^t \left[1 + e^{-\overline{T}(x,s)}\overline{u}(x,s)\right]\, ds \ .$$

The function $\overline{u}$ is smooth for all $t > 0$, but the function $\overline{T}$ blows up in finite time $t = \overline{\phi}(x)$, where $\overline{\phi}$ is defined by

(4.6)
$$\overline{e}^{T_0(x)} = \int_0^{\overline{\phi}(x)} \left[1 + e^{-\overline{T}(x,s)}\overline{u}(x,s)\right]\, ds \ ;$$

notice that

(4.7)
$$-\overline{\phi}'(x) = e^{-T_0(x)} T_0'(x) \ .$$

In addition to (1.7)–(1.11) we shall assume that

(4.8)
$$e^{-T_0}|T_0'| \leq 1 \ ;$$

therefore

(4.9)
$$|\overline{\phi}'(x)| \leq 1.$$

Denote by $(u_\gamma, T_\gamma, \phi_\gamma)$ the solution (u, T, ϕ) of (1.4)–(1.6) corresponding to γ, and recall that

$$(4.10) \qquad |\phi'_\gamma(x)| \le \theta \quad \text{a.e.}$$

For any $\eta > 0,\ R > 0$, set

$$\Omega_{\eta,R} = \{(x,t);\ |x| < R,\ 0 \le t \le \overline{\phi} - \eta\}, \Omega_\eta = \Omega_{\eta,\infty}\ .$$

THEOREM 4.1.

$$(4.11) \qquad \liminf_{\gamma \to 1} \phi_\gamma(x) \ge \overline{\phi}(x) \qquad \forall \quad x \in \mathbf{R}\ ,$$

and

$$(4.12) \qquad \nabla u_\gamma \longrightarrow \nabla \overline{u}\ ,\ T_\gamma \longrightarrow \overline{T} \qquad \text{uniformly in } \Omega_{\eta,R}$$

for any $\eta > 0, R > 0$.

Proof. Set

$$C_\eta = \max_{\Omega_{\eta/2}}\{|\nabla \overline{u}| + |\overline{T}|\}\ .$$

For any (x_0, t_0) in $\Omega_{\eta/2}$, introduce a triangle $\triangle$ with sides lying on $x \pm t = x_0 \pm t_0$ and on $t = 0; (x_0, t_0)$ is a vertex of $\triangle$. By (4.9), $\triangle$ is contained in $\Omega_{\eta/2}$ and, therefore,

$$(4.13) \qquad |\nabla \overline{u}| + |\overline{T}| \le C_\eta \qquad \text{in } \triangle$$

Denote by $\triangle_\delta$ any triangle with sides on $x \pm t = \text{const.}, t = 0$, and height $2\delta; \triangle_\delta \subset \triangle$. Using (2.11) and Gronwall's inequality we deduce that

$$(4.14) \qquad |\nabla u_\gamma| + |T_\gamma| \le C \qquad \text{in } \triangle_\delta$$

provided δ is small enough; C is a constant which is independent of γ, but depends on δ and on

$$(4.15) \qquad \max\{|\nabla u_\gamma| + |T_\gamma|\} \qquad \text{on } \triangle_\delta \cap \{t = 0\}.$$

Using (4.14) and the representation (2.3) for u_γ we can further derive the bound

$$(4.16) \qquad |\nabla^2 u_\gamma| + |\nabla T_\gamma| \le C'$$

where C' is a constant dependent only on C and δ. It follows that, as $\gamma \longrightarrow 1$, $u_\gamma \longrightarrow \overline{u}$ and $T_\gamma \longrightarrow \overline{T}$ in $C^1(\Delta_\delta)$. Consequently,

$$|\nabla u_\gamma| + |T_\gamma| \le C_\eta + 1 \tag{4.17}$$

in Δ_δ if $1 < \gamma < \gamma_0$ and $\gamma_0 - 1$ is sufficiently small. Covering a small neighborhood of $\Delta \cap \{t \le \delta\}$ by a finite number of triangles Δ_δ we deduce that (4.17) holds in some neighborhood of $\Delta \cap \{t \le \delta\}$. We can now repeat the process (cf. [5] for a related argument), starting at $t = \delta$ and using (4.17) instead of the bound in (4.15). We deduce that (4.17) holds in some neighborhood of $\Delta \cap \{t \le 2\delta\}$ if $\gamma - 1$ is small enough, say $\gamma < \gamma_1$. This process can be repeated with the same δ in each step, provided δ is originally small enough (depending on η); in each subsequent step the initial values satisfy the same inequality (4.17) and $\gamma - 1$ is to be further decreased. We conclude that (4.16) holds in $\Delta \cap \{t \le t_0\}$ and

$$u_\gamma \longrightarrow \overline{u}, \quad \nabla u_\gamma \longrightarrow \nabla \overline{u}, \quad T_\gamma \longrightarrow \overline{T} \qquad \text{as} \qquad \gamma \longrightarrow 1\,,$$

uniformly in each $\Delta \cap \{t \le t_0\}$, and thus also uniformly in $\Omega_{\eta,R}$ for any $\eta > 0, R > 0$. This implies also the assertion (4.11)

REMARK 4.1. If we do not make the assumption (4.9) then the proof of (4.12) is still valid in every triangle Δ with sides on $x \pm t = x_0 + t_0, t = 0$ such that Δ lies in $\{t < \overline{\phi}(x)\}$.

REMARK 4.2. If the assertion

$$\lim_{\gamma \to 1} \phi_\gamma\,(x) = \overline{\phi}(x) \tag{4.18}$$

is true then from (4.10) we deduce that $|\overline{\phi}'(x)| \le \theta$. But this inequality cannot hold, in general, for data satisfying only (4.8). Consequently (4.18) is not true in general. Technically the difficulty in establishing (4.18) arises as follows: Suppose

$$\limsup_{\gamma \to 1} \phi_\gamma(x_0) > \overline{\phi}(x_0)\ . \tag{4.19}$$

Then for the corresponding sequence (u_γ, T_γ) we have, by Remarks 3.1, 3.2,

$$u_\gamma \le C,\ T_\gamma \le C$$

in a triangle Δ with sides on $x \pm t = \text{const.}, t = 0$, which contains $(x_0, \overline{\phi}(x_0))$ in its interior. If the estimate

$$|\nabla u_\gamma| \le C_1 \tag{4.20}$$

can be established in Δ, then we can use (3.5), as before, to show that

$$u_\gamma \longrightarrow \overline{u}\ ,\ T_\gamma \longrightarrow \overline{T}$$

uniformly in $\Delta \cap \{t < \overline{\phi}(x) - \eta\}$, for any $\eta > 0$. It follows that $\overline{T} \le C$ in $\Delta \cap \{t < \overline{\phi}(x)\}$, which is a contradiction. Thus the proof of (4.18) depends on establishing the estimate (4.20).

REFERENCES

[1] P.A. BLYTHE AND D.G. CRIGHTON, *Shock generated ignition: the induction zone*, to appear.

[2] L.A. CAFFARELLI AND A. FRIEDMAN, *The blow–up boundary for nonlinear wave equations*, Trans. Amer. Math. Soc., 297 (1986), 223–241.

[3] L.A. CAFFARELLI AND A. FRIEDMAN, *Differentiability of the blow–up curve for one dimensional nonlinear wave equations*, Arch. Rat. Mech. Anal., 91 (1985), 83–98.

[4] J.F. CLARKE AND R.S. CANT, *Nonsteady gasdynamic effects in the induction domain behind a strong shock wave*, in *Dynamic of Flames and Reactive Systems*, J.R. Bower et al ed., Progress in Astronautics and Aeronautics, vol. 98 (1984), 142–163.

[5] A. FRIEDMAN AND L. OSWALD, *The blow–up time for higher order semilinear parabolic equations with small leading coefficients*, J. Diff. Eqs., to appear.

[6] R. GLASSEY, *Finite time blow–up for solutions of nonlinear wave equations*, Math. Zeit., 177 (1981), 323–340.

[7] T.L. JACKSON AND A.K. KAPILA, *Shock induced thermal runaway*, SIAM J. Appl. Math., 45 (1985), 130–137.

[8] T.L. JACKSON AND A.K. KAPILA, *Dynamics of hot–spot evolution in a reactive, compressible flow*, in *Computational Fluid Dynamics and Reactive Gas Flows*, B. Engquist et al ed., IMA volume 12, Springer Verlag, New York, 1988, pp. 123–151.

[9] T.L. JACKSON, A.K. KAPILA AND D.S. STEWART, *Evolution of a reaction center in an explosive material*, to appear.

[10] F. JOHN, *Blow–up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., 28 (1979), 235–268.

[11] D.R. KASSOY, A.K. KAPILA AND D.S. STEWART, *A unified formulation for diffusive and nondiffusive thermal explosion theory*, to appear.

[12] T. KATO, *Blow–up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math., 32 (1980), 501–505.

[13] A. LIÑÁN AND F.A. WILLIAMS, *Theory of ignition of a reactive solid by a constant energy flux*, Combustion Science and Technology, 3 (1971), 91–94.

[14] M.C. REED, *Singularities in non–linear waves of Klein–Gordon type*, in *Nonlinear Partial Differential Equations and applications*, Springer–Lecture Notes, no. 648, 1977.

[15] F.A. WILLIAMS, *Combustion Theory*, Addison–Wesley, 1985.