On λ–DESIGNS WITH $\lambda = 2P$

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On λ-designs with $\lambda = 2P$

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Abstract: A λ-design is a family $B_1, B_2, ..., B_v$ of subsets of $X = \{1, 2, ..., v\}$ such that $|B_i \cap B_j| = \lambda$ for all $i \neq j$ and not all blocks are of the same size. Ryser's and Woodall's λ-design conjecture states that each λ-design can be obtained from a symmetric block design by complementing with respect to a fixed block. We prove some results on λ-designs in the case when λ is twice a prime number. In particular, we prove the λ-design conjecture for $\lambda = 10$, which was the smallest unsolved case.

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1. Introduction and statement of results

Definition 1.1. A λ-design D is a pair (X, L) such that

(i) $X = \{1, 2, ..., v\}$, $|L| = v$, and the elements of L are subsets of X (The elements of X are called *treatments* and the elements of L are *blocks*.)
(ii) For all $B_i, B_j \in L, i \neq j, |B_i \cap B_j| = \lambda$
(iii) For all $B_j \in L, |B_j| = k_j > \lambda$ and not all k_j are equal.

λ-designs were first defined by Ryser[Ry] and Woodall[Wo70]. The only known examples are obtained from symmetric block designs by the following complementation procedure. Let (X, S) be a symmetric (v, k, λ') design and let $C_0 \in S$ be fixed. Then $L = \{B \subset X : B = X \setminus C_0$ or $B = (C \setminus C_0) \cup (C_0 \setminus C)$ for some $C \in S\}$ is the block system of a $(k - \lambda')$-design. λ-designs obtained by this procedure are called *type*-1 designs. The *λ-design conjecture* [Ry],[Wo70] states that all λ-designs are type-1. The conjecture was proven by deBruijn and Erdős [BE] for $\lambda = 1$, by Ryser [Ry] for $\lambda = 2$, and, in a series of papers by Bridges and Kramer [Br70], [Kr69], [BK], for $3 \leq \lambda \leq 9$. Singhi and Shrikhande [SS76] showed the validity of the conjecture for λ =prime.

Throughout the paper, D denotes a λ-design on v points. Ryser and Woodall independently proved the following theorem.

Theorem 1.2. [Ry],[Wo70] If D is a λ-design then there exist integers r_1, r_2 such that $r_1 + r_2 = v + 1$ and every treatment occurs either in r_1 blocks or in r_2 blocks.

We say that a treatment i is in *class n* if its replication number is r_n and use the notation $i \in C_n$.

The present paper contains partial results of an attempt to prove the λ-design conjecture when λ is twice a prime. Our starting point is the following result of Singhi and Shrikhande.

Theorem 1.3. [SS84] Let D be any λ-design. Suppose that $r_1 > r_2$ and let $\rho = \frac{r_1 - 1}{r_2 - 1} = \frac{x}{y}$, $(x, y) = 1$. If $(\lambda, x - y) = 1$ then D is type-1.

It is easy to see (cf. [SS76]) that $x - y < \lambda$ and $y < \lambda$. Hence in the case $\lambda = 2P$, P is an odd prime, Theorem 1.3 leaves open the following three possibilities:
Case 1: x, y are odd, $y > 1$;
Case 2: x is odd, $y = 1$; and
Case 3: $x - y = P$.

We are able to handle Case 1 and a part of Case 2. More precisely, we prove the following theorem:

Theorem 1.4. Let D be a λ-design with $\lambda = 2P$, P an odd prime and let $\rho = \frac{x}{y}$, $(x, y) = 1$. If either x, y are odd, $y > 1$ or $x \geq P$, $y = 1$ then D is type-1.

The method of proof is a mixture of divisibility considerations (in the spirit of [SS76], [SS84]) and the analytic methods of [Se]. The main difficulty in proving the conjecture for $\lambda = 2P$ lies in Case 3 where all block sizes are divisible by P and the methods of Singhi and Shrikhande break down. However, using the analytic methods, we are able to handle the smallest unsolved value of P.

Theorem 1.5. The λ-design conjecture is true for $\lambda = 10$.

The organization of the paper is as follows. In Section 2, we give the relations among the parameters of λ-designs and list the results of [Se] used in this paper. In Section 3, we prove some results valid for all values of λ. Section 4 contains the proof of Theorem 1.4 while the proof of Theorem 1.5 is given in Section 5.

2. Previous results

The parameters of λ-designs are most often expressed as functions of v, λ, ρ. For our purposes, it will be more convenient to use $d = e_1 - r_2$ and write the other parameters as functions of d, λ, ρ. Let e_1, e_2 be the number of class 1 and class 2 treatments, respectively. Wlog, we may assume that a treatment i is in class 1 if and only if $i \leq e_1$. We denote by k'_m, k^*_m the number of class 1 and class 2 treatments in B_m, respectively. Clearly, $k_m = k'_m + k^*_m$. The following equations were essentially obtained in [Ry] and [Wo70] (cf. [SS76],[Kr74]).

(2.1)
$$k'_m = \lambda - \frac{k_m - 2\lambda}{\rho - 1},$$

(2.2)
$$k^*_m = \lambda + \frac{(k_m - 2\lambda)\rho}{\rho - 1},$$

(2.3)
$$e_1 = \lambda + \frac{\lambda + d}{\rho},$$

(2.4)
$$e_2 = \lambda\rho - d\rho - \rho + \lambda,$$

(2.5)
$$r_1 = \lambda\rho - d\rho - \rho + \lambda + d + 1,$$

(2.6)
$$r_2 = \lambda - d + \frac{\lambda + d}{\rho},$$

(2.7)
$$\frac{r_1^2 - r_1}{v - 1} = \frac{r_1\rho}{\rho + 1} = \lambda\rho - (d+1)\rho\frac{\rho - 1}{\rho + 1},$$

(2.8)
$$\frac{r_1(r_2 - 1)}{v - 1} = \frac{r_1}{\rho + 1} = \lambda - (d+1)\frac{\rho - 1}{\rho + 1},$$

$$(2.9)\qquad \frac{r_2^2 - r_2}{v-1} = \frac{r_2}{\rho+1} = \frac{\lambda}{\rho} - \frac{d}{\rho}\frac{\rho-1}{\rho+1},$$

and

$$(2.10)\qquad \frac{r_2\rho}{\rho+1} = \lambda - d\frac{\rho-1}{\rho+1}.$$

Let
(2.11)

$$R_{ij} = \sum_{\{m: i,j \in B_m\}} \frac{1}{k_m - \lambda},\quad R_{i\bar{j}} = \sum_{\{m: i \in B_m, j \notin B_m\}} \frac{1}{k_m - \lambda},\quad R_{\bar{i}\bar{j}} = \sum_{\{m: i,j \notin B_m\}} \frac{1}{k_m - \lambda}.$$

Then R_{ij}, $R_{i\bar{j}}$, and $R_{\bar{i}\bar{j}}$ depend only on the class of i and j. We use the notation $R_{ij} = R(1)$ if $i = j$ and i is in class 1; $R_{ij} = R(1,1)$ if $i \neq j$ and they are both in class 1; etc.

$$(2.12)\qquad R(1) = \rho + 1,\quad R(2) = \frac{1}{\rho} + 1,$$

$$(2.13)\qquad R(1,1) = \rho,\quad R(1,2) = 1,\quad R(2,2) = \frac{1}{\rho},$$

$$(2.14)\qquad R(1,\bar{1}) = 1,\quad R(1,\bar{2}) = \rho,\quad R(\bar{1},2) = \frac{1}{\rho},\quad R(2,\bar{2}) = 1,$$

$$(2.15)\qquad R(\bar{1},\bar{1}) = \frac{1}{\rho} - \frac{1}{\lambda},\quad R(\bar{1},\bar{2}) = 1 - \frac{1}{\lambda},\quad R(\bar{2},\bar{2}) = \rho - \frac{1}{\lambda},$$

$$(2.16)\qquad R(\bar{1}) = 1 + \frac{1}{\rho} - \frac{1}{\lambda},\quad R(\bar{2}) = 1 + \rho - \frac{1}{\lambda},$$

and

$$(2.17)\qquad \sum_{m=1}^{v} \frac{1}{k_m - \lambda} = \rho + 2 + \frac{1}{\rho} - \frac{1}{\lambda}.$$

We need the following result of Kramer.

Theorem 2.1 [Kr74] (a) If $\rho \geq \lambda - 1$ then D is type-1.
(b) If $\rho = \frac{\lambda}{2}$ and $e_1 \geq \lambda + 2$ then D is type-1.

We shall also use the following results of [Se]. Let r_{ij} denote the number of blocks containing the treatments i and j. Moreover, let $x_1 = d\frac{\lambda\rho - \lambda}{\lambda\rho + \lambda - \rho}$ and $x_2 = (d+1)\frac{\lambda\rho - \lambda}{\lambda\rho + \lambda - 1}$.

Theorem 2.2. D is type-1 if and only if one of the following holds:
(i) $d \in \{0, -1\}$;
(ii) $\frac{r_1}{\rho+1}$ or $\frac{r_2}{\rho+1}$ is an integer; (let us note that (i) and (ii) were also announced in [Wo71])
(iii) $\frac{\lambda - x_1}{\rho}$ or x_2 is an integer;
(iv) there exists a treatment i such that r_{ij} depends only on the class of j.

Theorem 2.3. (a) Let $i \in C_1$. Then more than half of the numbers $\{r_{ij} : j > e_1\}$ are equal to $\lceil \frac{r_1}{\rho+1} \rceil$.
(b) Let $i \in C_2$. Then more than half of the numbers $\{r_{ij} : j > e_1\}$ are equal to $\lceil \frac{r_2}{\rho+1} \rceil$.

Lemma 2.4. $0 \leq \lceil \frac{r_2}{\rho+1} \rceil - \frac{r_2}{\rho+1} \leq \frac{\rho-1}{\rho+1}$.

We define

(2.18)
$$U_i = \sum_{\{m : i \in B_m\}} \left(k_m - \lambda - \frac{r_1}{\rho+1}\right), \quad i \in C_1,$$

(2.19)
$$V_i = \sum_{\{m : i \notin B_m\}} (k_m - 2\lambda + x_1), \quad i \in C_1,$$

(2.20)
$$U_i = \sum_{\{m : i \in B_m\}} \left(k_m - \lambda - \frac{r_2 \rho}{\rho+1}\right), \quad i \in C_2,$$

and

(2.21)
$$V_i = \sum_{\{m : i \notin B_m\}} (k_m - 2\lambda + x_2), \quad i \in C_2.$$

Theorem 2.5. $U_i \geq 0$ and $V_i \geq 0$ for all i. D is type-1 if and only if $U_i = 0$ or $V_i = 0$ for some i.

Also, the following equalities hold.
For all $i \in C_1$,

(2.22)
$$U_i + V_i = \frac{(\rho-1)^2(e_1-1)e_2}{(\rho+1)(\lambda\rho+\lambda-\rho)},$$

(2.23)
$$\sum_{j \leq e_1, j \neq i} r_{ij} = \frac{r_1 \rho}{\rho+1}(e_1 - 1) - \frac{U_i}{\rho-1},$$

(2.24)
$$\sum_{j > e_1} r_{ij} = \frac{r_1 e_2}{\rho+1} + \frac{U_i \rho}{\rho-1},$$

$$(2.25) \qquad U_i + V_i = \sum_{j \le e_1, j \ne i} \left((r_{ij} - \frac{r_1 \rho}{\rho+1})^2 + (r_{ij} - r_1 + \lambda - x_1)^2 \right) +$$

$$\sum_{j > e_1} \left((r_{ij} - \frac{r_1}{\rho+1})^2 + (r_{ij} - r_2 + \frac{\lambda - x_1}{\rho})^2 \right),$$

and

$$(2.26) \quad \sum_{j \le e_1, j \ne i} (r_{ij} - r_1 + \lambda - x_1)^2 + \sum_{j > e_1} (r_{ij} - r_2 + \frac{\lambda - x_1}{\rho})(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho} + 1) = -\frac{V_i}{\rho - 1}.$$

For all $i \in C_2$,

$$(2.27) \qquad U_i + V_i = \frac{(\rho-1)^2 e_1 (e_2 - 1)}{(\rho+1)(\lambda\rho + \lambda - 1)},$$

$$(2.28) \qquad \sum_{j \le e_1} r_{ij} = \frac{r_2 \rho}{\rho+1} e_1 - \frac{U_i}{\rho - 1},$$

$$(2.29) \qquad \sum_{j > e_1, j \ne i} r_{ij} = \frac{r_2 (e_2 - 1)}{\rho + 1} + \frac{U_i \rho}{\rho - 1},$$

$$(2.30) \qquad U_i + V_i = \sum_{j \le e_1} \left((r_{ij} - \frac{r_2 \rho}{\rho+1})^2 + (r_{ij} - r_1 + \lambda\rho - \rho x_2)^2 \right) +$$

$$\sum_{j > e_1, j \ne i} \left((r_{ij} - \frac{r_2}{\rho+1})^2 + (r_{ij} - r_2 + \lambda - x_2)^2 \right),$$

and

$$(2.31) \quad \sum_{j \le e_1} (r_{ij} - r_1 + \lambda\rho - \rho x_2)^2 + \sum_{j > e_1, j \ne i} (r_{ij} - r_2 + \lambda - x_2)(r_{ij} - r_2 + \lambda - x_2 + 1) = -\frac{V_i}{\rho - 1}.$$

3. Some results valid for all values of λ

By Theorem 2.2(i), the λ-design conjecture is equivalent to the fact that $d = e_1 - r_2 \in \{0, -1\}$. The main purpose of this section is to give a non-trivial lower bound for d (cf. Theorem 3.5). We shall use this result in the next two sections and it may well be a part of the final resolution of the λ-design conjecture.

Let $t_1 = \lceil \frac{r_1}{\rho+1} \rceil - \frac{r_1}{\rho+1}$, $t_2 = \lceil \frac{r_2}{\rho+1} \rceil - \frac{r_2}{\rho+1}$, $s_1 = \lceil \frac{\lambda - x_1}{\rho} \rceil - \frac{\lambda - x_1}{\rho}$, and $s_2 = \lceil \lambda - x_2 \rceil - (\lambda - x_2)$.

Lemma 3.1. $t_1 + t_2 = \frac{\rho - 1}{\rho + 1}$.

Proof: By (2.8) and (2.10), $r_2 - \frac{r_2}{\rho+1} - \frac{r_1}{\rho+1} = \frac{\rho-1}{\rho+1}$. Thus $t_1 + t_2 = \frac{\rho-1}{\rho+1}$ or $1 + \frac{\rho-1}{\rho+1}$; since, by Lemma 2.4, $t_2 \leq \frac{\rho-1}{\rho+1}$, $t_1 + t_2 = \frac{\rho-1}{\rho+1}$. ∎

Theorem 3.2. (a) Let $i \in C_1$. Then more than half of the numbers $\{r_{ij} : j > e_1\}$ are equal to $r_2 - \lceil \frac{\lambda - x_1}{\rho} \rceil$.
(b) Let $i \in C_2$. Then more than half of the numbers $\{r_{ij} : j > e_1\}$ are equal to $r_2 - \lceil \lambda - x_2 \rceil$.

Proof: (a) Let $i \in C_1$. We can suppose that $s_1 > 0$ and $V_i > 0$ otherwise, by Theorems 2.2(iii) and 2.5, the design is type-1 and all r_{ij} are equal to the stated value. From (2.26), we obtain

$$(3.1) \qquad \sum_{j>e_1} (r_{ij} - r_2 + \frac{\lambda - x_1}{\rho})(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho} + 1) < 0.$$

If $r_{ij} = r_2 - \lceil \frac{\lambda - x_1}{\rho} \rceil$ for some $j > e_1$ then $(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho})(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho} + 1)$ contributes $-s_1(1 - s_1)$ to the left-hand-side of (3.1); for all other $j > e_1$, $(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho})(r_{ij} - r_2 + \frac{\lambda - x_1}{\rho} + 1)$ contributes at least $\min\{s_1(1+s_1), (1-s_1)(2-s_1)\} > s_1(1-s_1)$. Hence, for more than $e_2/2$ treatments j of class 2, $r_{ij} = r_2 - \lceil \frac{\lambda - x_1}{\rho} \rceil$.
(b) This part follows similarly from (2.31). ∎

Corollary 3.3. (a) $s_1 + t_1 = \frac{\rho-1}{\rho+1} \frac{\lambda\rho + \lambda - \rho + d}{\lambda\rho + \lambda - \rho}$.
(b) $s_2 + t_2 = \frac{\rho-1}{\rho+1} \frac{\lambda\rho + \lambda + d}{\lambda\rho + \lambda - 1}$.

Proof: (a) By Theorems 2.3(a) and 3.2(a), for each $i \in C_1$ there exists $j \in C_2$ such that $r_{ij} = \lceil \frac{r_1}{\rho+1} \rceil = r_2 - \lceil \frac{\lambda - x_1}{\rho} \rceil$. Therefore $s_1 + t_1 = r_2 - \frac{r_1}{\rho+1} - \frac{\lambda - x_1}{\rho} = \frac{\rho-1}{\rho+1} \frac{\lambda\rho + \lambda - \rho + d}{\lambda\rho + \lambda - \rho}$.
(b) Similar to the proof of (a). ∎

Lemma 3.4. $-\lambda\rho - \lambda < d < \lambda + \frac{\lambda}{\rho} - 1$.

Proof: From the inequalities $e_1 > 0$ and $e_2 > 0$. ∎

Theorem 3.5. $d \geq -\frac{1}{2}\lambda\rho - \frac{1}{2}\lambda + \frac{\lambda(\rho-1)}{4\lambda-2}$.

Proof: By Lemma 3.1 and Corollary 3.3, $\frac{\rho-1}{\rho+1} \leq \frac{\rho-1}{\rho+1} \frac{\lambda\rho + \lambda - \rho + d}{\lambda\rho + \lambda - \rho} + \frac{\rho-1}{\rho+1} \frac{\lambda\rho + \lambda + d}{\lambda\rho + \lambda - 1}$. Solving this inequality for d, we obtain the stated bound. ∎

Proposition 3.6.

$$(3.2) \qquad \frac{\rho}{\rho - 1} \sum_{i \leq e_1} U_i + \frac{1}{\rho - 1} \sum_{i > e_1} U_i = e_1 e_2 \frac{\rho - 1}{\rho + 1}.$$

Proof: By equating the right-hand-sides of (2.24) and (2.28). ∎

Proposition 3.7. If $\rho = 3$ then the design is type-1.

Proof: By (2.8) and (2.10), $\rho = 3$ implies that either $\frac{r_1}{\rho+1}$ or $\frac{r_2}{\rho+1}$ is an integer. Hence, by Theorem 2.2(ii), the design is type-1. ∎

Lemma 3.8. (a) If $d > 0$ then there exists B_m with $k'_m > \lambda$.
(b) $\rho > \frac{\lambda}{2}$ and $e_1 \geq \lambda + 2$ is impossible.

Proof: (a) Let $i \in C_2$. By (2.6) and (2.12), $d > 0$ implies that $\sum_{\{m: i \in B_m\}} \frac{1}{k_m - \lambda} > \frac{r_2}{\lambda}$. Therefore, there exists m such that $\frac{1}{k_m - \lambda} > \frac{1}{\lambda}$ or, equivalently, $k_m < 2\lambda$, $k'_m > \lambda$.
(b) If $\rho \geq \lambda - 1$ then Theorem 2.1(a) implies that the design is type-1 and, from Theorem 2.2(i), $e_1 = \lambda + 1$. Suppose that $\rho < \lambda - 1$. By (2.3), $e_1 \geq \lambda + 2$ implies $d > 0$. By (a), there exists B_m with $k'_m > \lambda$. $k'_m > \lambda + 1$ is impossible because (2.1),(2.2) would imply $k^*_m < 0$. Hence $k'_m = \lambda + 1$ and $\frac{1}{k_m - \lambda} = \frac{1}{\lambda - \rho + 1}$. Since $\rho < \lambda - 1$, there exist $i, j \in C_2$ such that $i, j \in B_m$. Therefore $\frac{1}{\lambda - \rho + 1} \leq R(2,2) = \frac{1}{\rho}$, implying $\rho \leq \frac{\lambda+1}{2}$. Since $x - y < \lambda$, the only possibility is $\rho = \frac{\lambda+1}{2}$. By Theorem 1.3, the design is type-1, contradicting $d > 0$. ∎

4. The case $\lambda = 2P$

Throughout this section P always denotes an odd prime and $\lambda = 2P$.

Lemma 4.1. Suppose that $\rho = \frac{x}{y}$, x, y are odd, and $y > 1$. Then $P | (k_m - \lambda)$ if and only if $k_m = 2\lambda$.

Proof: By (2.1), $\frac{k_m - 2\lambda}{\rho - 1}$ is an integer, i.e. $k_m - 2\lambda = (x - y)l_m$ for some integer l_m. $P \not| (x - y)$, so $P | (k_m - \lambda)$ implies $P | l_m$. If $|l_m| \geq P$ then (2.1) or (2.2) gives negative value for k'_m or k^*_m; hence $l_m = 0$. ∎

Theorem 4.2. If $\rho = \frac{x}{y}$, x, y are odd, $y > 1$, and $P \not| x$ then $x = 2P - 1$, $y = P$, and the design is type-1.

Proof: Suppose that x, y satisfy the conditions of the theorem. Let us fix $i \in C_1$. By (2.16),

$$\text{(4.1)} \qquad \sum_{\{m: i \notin B_m\}} \frac{1}{k_m - \lambda} = 1 + \frac{y}{x} - \frac{1}{2P}.$$

Let a be the number of blocks B_m such that $P | (k_m - \lambda)$ and $i \notin B_m$. By Lemma 4.1, $k_m = 2\lambda$ for these blocks. Moreover, let $M = \{m : i \notin B_m \wedge P \not| (k_m - \lambda)\}$. By (4.1),

$$\text{(4.2)} \qquad \frac{a+1}{2P} = 1 + \frac{y}{x} - \sum_{\{m \in M\}} \frac{1}{k_m - \lambda}.$$

The denominator of the fraction on the right-hand-side of (4.2) is not divisible by P, so $P | (a+1)$. Now let $j \in C_1, j \neq i$, and let b be the number of blocks B_m such that $P | (k_m - \lambda)$ and $i, j \notin B_m$. Similarly to the previous argument, we obtain from (2.15) that $P | (b + 1)$. $b \geq 2P - 1$ is impossible because $\frac{y}{x} - \frac{1}{2P} < 1 - \frac{1}{2P}$. Hence $b = P - 1$ and $\frac{y}{x} - \frac{1}{2P} \geq \frac{P-1}{2P}$, so $\rho < 2$. $a > b$ since we can choose j such that $j \in B_m$ for some m with $k_m = 2\lambda$. By

(4.1), $a \geq 4P-1$ would imply $\frac{4P-1}{2P} \leq 1 + \frac{y}{x} - \frac{1}{2P}$, a contradiction, so the possible values for a are $a = 3P-1$ and $a = 2P-1$.

If $a = 3P-1$ then $\sum_{\{m \in M\}} \frac{1}{k_m - \lambda} = \frac{1}{\rho} - \frac{1}{2} < \frac{1}{2}$. By (2.14), $R(\bar{1},2) = \frac{1}{\rho}$, hence $j \in B_m$ for all $j \in C_2$ and $m \in M$. Similarly, $R(\bar{1},1) = 1$ implies that $j \notin B_m$ for all $j \in C_1$ and $m \in M$. Therefore, for all $m \in M$, $k'_m = 0$ and $k^*_m = e_2$. By (2.1), $k'_m = 0$ implies $k_m = \lambda\rho + \lambda$, so (2.4) gives $d = -1$. By Theorem 2.2(i), the design is type-1. From the integrality of e_2 we obtain that $y|2P$, so $y = P$. Moreover, $x|2P-1$ since e_1 is an integer and $x > y$ implies $x = 2P-1$.

Finally, we prove that $a = 2P-1$ is impossible. In this case, each $j \in C_1, j \neq i$ is contained in exactly P blocks with $k_m = 2\lambda$ and not containing i. Conversely, each such block contains exactly $2P$ treatments of class 1, hence $P(e_1 - 1) = (2P-1)2P$, i.e. $e_1 = 4P-1$. From (2.3), we obtain that $\rho = 1 + \frac{d+1}{2P-1}$ and, from (2.4), $e_2 = 4P - \frac{d(d+1)}{2P-1}$. $d \geq 0$ because $\rho > 1$; moreover, $d \neq 0$ since we assumed that $P \not| x$. Therefore $d(d+1) > 0$ and even so $e_2 \leq 4P-2$.

Let us denote the number of blocks with $k_m = 2\lambda$ by c. For each pair $h, j \in C_1$, there are exactly $P-1$ such blocks B_m with $h, j \notin B_m$. On the other hand, for fixed B_m there are $\binom{e_1 - 2P}{2}$ pairs $h, j \in C_1$ with $h, j \notin B_m$. Thus $\binom{e_1}{2}(P-1) = c\binom{e_1-2P}{2}$ and, using $e_1 = 4P-1$, we obtain $c = 4P-1$. Therefore there are $2P$ blocks with $k_m = 2\lambda$ and containing the fixed $i \in C_1$. Each of these $2P$ blocks contains $2P$ treatments of class 2. Conversely, each treatment of class 2 is contained in $0, P$, or $2P$ of these blocks (since the denominator of $R(1,2)$ is not divisible by P). Since $e_2 \leq 4P-2$, there are two treatments $h, j \in C_2$ contained in all of these $2P$ blocks, contradicting $R(2,2) = \frac{1}{\rho} < 1$. ∎

Lemma 4.3. Suppose that $\rho = \frac{x}{y}$, x, y are odd, $y > 1$, and $P|x$. Then $P = x$.

Proof: $x \geq 5P$ is impossible since $x - y < \lambda$ and $y < \lambda$. If $x = 3P$ then $P < y < 2P$, and, by (2.1) and (2.2), the only possible values for k'_m are $2P$ and $2P - y$. Moreover, since e_1 is an integer, $3P|2P + d$. $d > 0$ contradicts Lemma 3.8(a). If $d \leq -2P$ then $e_1 \leq 2P$ and all blocks not containing a fixed $i \in C_1$ are of the same size. Therefore r_{ij} depends only on the class of j, contradicting Theorem 2.2(iv). ∎

Theorem 4.4. If $\rho = \frac{P}{y}$, y is odd, and $y > 1$ then $y|2P-1$ and the design is type-1.

Proof: We distinguish the following two cases: (i) $y \leq \frac{P-1}{2}$ and (ii) $y \geq \frac{P+1}{2}$.
(i) Let $i \in C_1$ be fixed. For $j \in C_1, j \neq i$, let us denote the number of blocks with $k_m = 2\lambda$ and $i, j \notin B_m$ by a_j. By (2.15), $\frac{2y-1-a_i}{2P}$ is a non-negative fraction with denominator not divisible by P, so it must be equal to 0. Hence $a_j = 2y-1$ for all $j \in C_1$, all blocks B_m with $i, j \notin B_m$ are of size 2λ, and r_{ij} is constant for all j in class 1. Similarly, for $j \in C_2$, let b_j be the number of blocks with $k_m = 2\lambda$, $i \notin B_m$, and $j \in B_m$. From $R(\bar{1},2) = \frac{y}{P}$ we obtain that $P|(2y - b_j)$; thus all blocks containing a fixed treatment $j \in C_2$ and not containing i are of size 2λ, and r_{ij} is constant for all j in class 2. Therefore, by Theorem 2.2(iv), the design is type-1. From the integrality of e_1 and e_2 we get $d = 0$ and $y|2P-1$.
(ii) As above, we can conclude that $P|(2y - b_j)$ for all $j \in C_2$. In particular, $b_j > 0$, hence there exists a block of size 2λ not containing a treatment of class 1. This implies $e_1 \geq \lambda + 1$ and $d > -2P$. On the other hand, Lemma 3.4 gives $d < 2P + 2y - 1 < 4P$. The integrality

of e_1 implies $P|d$. Thus the possible values for d are $3P, 2P, P, 0$ and $-P$. We examine the integrality condition given by e_2. If $d = 3P$ then $y|(-P-1)$. Since $y \geq \frac{P+1}{2}$, the only possibility is $y = \frac{P+1}{2}$ implying $e_2 = 0$, a contradiction. If $d = 2P$ then $y|(-1)$, a contradiction. If $d = P$ then $y|P-1$, a contradiction. If $d = 0$ then, by Theorem 2.2(i), the design is type-1. Finally, if $d = -P$ then $y|3P-1$. Since $y \geq \frac{P+1}{2}$, the only possibilities are $y = \frac{3P-1}{5}$ and $y = \frac{3P-1}{4}$. We prove that these cases are impossible.

If $d = -P$ and $y = \frac{3P-1}{5}$ then $r_1 = 6P+1$ and

$$\frac{r_1}{\rho+1} = \frac{(6P+1)(3P-1)}{8P-1} = \frac{9}{4}P - \frac{1}{32}(3 + \frac{35}{8P-1}). \tag{4.3}$$

Thus $t_1 \geq \frac{11}{32}$. On the other hand, Corollary 3.3(a) gives $t_1 \leq \frac{(2P+1)(13P-6)}{(8P-1)(16P-7)}$, a contradiction. If $d = -P$ and $y = \frac{3P-1}{4}$ then $r_1 = 5P+1$ and

$$\frac{r_1}{\rho+1} = \frac{(5P+1)(3P-1)}{7P-1} = \frac{15}{7}P + \frac{1}{7}\frac{P-7}{7P-1}. \tag{4.4}$$

Thus $t_1 \geq \frac{1}{7} - \frac{1}{7}\frac{P-7}{7P-1} = \frac{6(P+1)}{7(7P-1)}$. From Corollary 3.3(a) we obtain $t_1 \leq \frac{(P+1)(11P-5)}{(7P-1)(14P-6)}$ which leads to a contradiction. ∎

Theorem 4.5. If $\rho = P$ then the design is type-1.

Proof: By Theorem 2.1(b), it is enough to prove that $e_1 \leq 2P+1$ is impossible. So let us suppose that $e_1 \leq 2P+1$ and let us fix $i, j \in C_1, i \neq j$. If $i, j \notin B_m$ for some block B_m and $P|(k_m - \lambda)$ then, similarly to the proof of Lemma 4.1, we can deduce that either $k'_m = P$ and $k_m - \lambda = P(P+1)$ or $k'_m = 0$ and $k_m - \lambda = 2P^2$. Let a be the number of blocks B_m with $k'_m = 0$ and $i, j \notin B_m$. Moreover, let b be the number of blocks B_m with $k'_m = P$ and $i, j \notin B_m$. By (2.15),

$$\frac{1}{\rho} - \frac{1}{\lambda} - \frac{a}{2P^2} - \frac{b}{P(P+1)} = \frac{\frac{P(P+1)}{2} - a\frac{P+1}{2} - bP}{P^2(P+1)} \tag{4.5}$$

is a non-negative fraction with denominator not divisible by P; hence $P^2|(\frac{P(P+1)}{2} - a\frac{P+1}{2} - bP)$. The only possibility is that

$$\frac{P(P+1)}{2} - a\frac{P+1}{2} - bP = 0; \tag{4.6}$$

in particular, the size of all blocks which do not contain i and j is divisible by P. (4.6) implies that $P|a$ so $a = 0$ or $a = P$.

If $a = 0$ then, for all $h \in C_1$, the blocks not containing i and h must be of size $P(P+3)$. In particular, r_{ih} is constant for all $h \in C_1, h \neq i$. Moreover, all blocks not containing i must satisfy $k'_m = P$ or $k'_m = e_1 - 1$. In the latter case,

$$k_m - \lambda = P + 1 - d + \frac{d}{P}. \tag{4.7}$$

For $h \in C_2$, let us denote by b_h the number of blocks satisfying $h \in B_m$, $i \notin B_m$, and $k'_m = P$. By (2.14), the denominator of $\frac{1}{P} - \frac{b_h}{P(P+1)} = \frac{P+1-b_h}{P(P+1)}$ is not divisible by P; hence $b_h = 1$ or $b_h = P+1$. If $b_h = P+1$ for all $h \in C_2$ then r_{ih} is constant for all $h \in C_2$ and Theorem 2.2(iv) leads to a contradiction with $e_1 \le \lambda + 1$. If there exists $h \in C_2$ with $b_h = 1$ then, by (4.7),

$$\frac{1}{P} - \frac{1}{P(P+1)} = c\frac{1}{P+1-d+\frac{d}{P}} \tag{4.8}$$

for some integer c. Solving this equation for c, we obtain $c = 1 + \frac{d}{P}\frac{1-P}{1+P}$. $(\frac{P-1}{2}, \frac{P+1}{2}) = 1$ implies $\frac{P(P+1)}{2} | d$. $d < 0$ since $e_1 \le \lambda + 1$ and $d \le -P^2 - P$ contradicts Theorem 3.5. Hence the only possibility is $d = -\frac{P^2+P}{2}$ implying $\frac{r_2}{\rho+1} = \frac{P+3}{2}$. From Theorem 2.2(ii), we obtain a contradiction anyway.

If $a = P$ then, for all $h \in C_1$, the only blocks not containing i and h are the same P blocks of size $2P^2 + 2P$. Hence there are $P + 1 - d + \frac{d}{P}$ blocks containing all class 1 treatments except i. The reciprocal sum of the $(k_m - \lambda)$'s for these blocks is 1. Repeating this argument with all class 1 treatments in the place of i, we obtain that each class 1 treatment is contained in exactly $(e_1 - 1)(P + 1 - d + \frac{d}{P})$ blocks satisfying $k'_m = e_1 - 1$ and the reciprocal sum of the $(k_m - \lambda)$'s for these blocks is $e_1 - 1$. By (2.12), $e_1 - 1 \le P + 1$ which is equivalent to $d \le -P^2$. Theorem 3.5 gives $d > -P^2 - P$. Since, from the integrality of e_1, $P|d$, the only possibility is $d = -P^2$. This implies $\frac{r_1}{\rho+1} = P^2 + 1$, contradicting Theorem 2.2(ii). ∎

Theorem 4.6. $\frac{\lambda}{2} < \rho < \lambda - 1$ is impossible.

Proof: $\frac{\lambda}{2} < \rho < \lambda - 1$ is impossible in type-1 designs. Hence, by Theorem 1.3, we have to prove that the assumption ρ is an odd integer, $P + 2 \le \rho \le 2P - 3$ leads to a contradiction. Also, by Lemma 3.8(b), we can suppose that $e_1 \le 2P + 1$.

Three possible values of $k_m - \lambda$ are divisible by P: (i) $k_m - \lambda = 2P$ which is equivalent to $k'_m = 2P$; (ii) $k_m - \lambda = P(\rho + 1) \iff k'_m = P$; and (iii) $k_m - \lambda = 2P\rho \iff k'_m = 0$. However, among blocks not containing two fixed treatments i, j of class 1, only cases (ii) and (iii) are possible.

Claim: There exists a block of size $2P\rho + 2P$.

Proof of the Claim: Suppose that there is no block of size $2P\rho + 2P$ and let a be the number of blocks not containing $i, j \in C_1$ and of size $P(\rho + 1) + 2P$. By (2.15),

$$\frac{a}{P(\rho+1)} + \frac{1}{2P} = \frac{a + \frac{\rho+1}{2}}{P(\rho+1)} \tag{4.9}$$

is a fraction $\le \frac{1}{\rho}$ and its denominator is not divisible by P. This implies $a = P - \frac{\rho+1}{2}$. Moreover, there exist some blocks not containg i and j such that the reciprocal sum of the $(k_m - \lambda)$'s for these blocks is $\frac{1}{\rho(\rho+1)}$. Since the smallest possible value of $\frac{1}{k_m - \lambda}$ is $\frac{1}{2P\rho}$ and $\frac{1}{\rho(\rho+1)} < 2\frac{1}{2P\rho}$, there must be a unique block satisfying $\frac{1}{k_m - \lambda} = \frac{1}{\rho(\rho+1)}$. Since, by (2.1), $(\rho - 1)|(k_m - 2\lambda)$, we obtain $(\rho - 1)|(\rho(\rho + 1) - 2P)$ which is equivalent to $(\rho - 1)|(2P - 2)$. However, this fact contradicts $P + 2 \le \rho \le 2P - 3$. ∎

Let b be the number of blocks of size $2P\rho+2P$. The Claim implies that $b>0$. On the other hand, by (2.15), $\frac{1}{\rho}-\frac{1}{\lambda}\geq\frac{b}{2P\rho}$ implying $b\leq 2P-\rho$. Let us fix a block B_m of size $2P\rho+2P$. Then B_m contains $2P\rho+2P$ treatments of class 2. If all of these treatments were contained in at least one other block of size $2P\rho+2P$ then at least $\frac{2P\rho+2P}{2P-\rho-1}>2P$ of them would be contained in a block B_n contradicting $|B_m\cap B_n|=\lambda$. Therefore we can fix $h\in C_2$ such that B_m is the only block of size $2P\rho+2P$ containing h. Let c be the number of blocks of size $P(\rho+1)+2P$ and containing h. Moreover, for all $j\in C_2$, let c_j be the number of blocks B_n with $h,j\in B_n$ and $k_n=P(\rho+1)+2P$. For $j\in C_1$, let c_j be the number of blocks B_n with $h\in B_n$, $j\notin B_n$, and $k_n=P(\rho+1)+2P$. Now be distinguish three cases: 1) $e_1<2P$, 2) $e_1=2P$, and 3) $e_1=2P+1$.
Case 1: $e_1<2P$. In this case, there are no blocks of size $4P$. If $j\in C_2$ and $j\notin B_m$ then $c_j=0$. Indeed, $\frac{c_j}{P(\rho+1)}$ must be a fraction $\leq\frac{1}{\rho}$ and denominator not divisible by P implying $c_j=0$ or $c_j=P$. As in the proof of the Claim, $c_j=P$ leads to contradiction. On the other hand, if $j\in B_m$ then

$$\frac{c_j}{P(\rho+1)}+\frac{1}{2P\rho}=\frac{c_j\rho+\frac{\rho+1}{2}}{P\rho(\rho+1)} \tag{4.10}$$

is a fraction $\leq\frac{1}{\rho}$ and denominator not divisible by P. Therefore $c_j=c^*$ where c^* is the unique positive solution of the system

$$P|(c^*\rho+\frac{\rho+1}{2}),\quad c^*<P. \tag{4.11}$$

There are $2P\rho+2P-1$ treatments with $c_j=c^*$. On the other hand, each of the c blocks of size $P(\rho+1)+2P$ contains $P(\rho+2)-1$ class 2 treatments beside h so

$$c^*(2P\rho+2P-1)=c(P\rho+2P-1). \tag{4.12}$$

This implies $c-c^*=lP$ for some positive integer l so (4.12) is equivalent to $c^*\rho=l(P\rho+2P-1)$, contradicting $c^*<P$.
Case 2: $e_1=2P$. Since none of the class 1 treatments are in B_m, $c_j=c^*$ for all $j\in C_1$. On the other hand, each block of size $P(\rho+1)+2P$ leaves out e_1-P treatments of class 1 so $e_1c^*=c(e_1-P)$, $c=2c^*$. Now let c^{**} be the number of blocks of size $4P$ and containing h. Since $e_1=2P$, these blocks contain all treatments of class 1. This implies $c^{**}\leq 1$ since otherwise the intersection of two such blocks would contain $\geq 1+2P$ treatments. Because of $c=2c^*$ and $R(1,2)=1$, $\frac{c^*}{P(\rho+1)}+\frac{c^{**}}{2P}$ is a fraction with denominator not divisible by P implying

$$P|(c^*+c^{**}\frac{\rho+1}{2}). \tag{4.13}$$

Thus $c^{**}=0$ is impossible, i.e. $c^{**}=1$. Adding (4.11) and (4.13), we obtain $P|(\rho+1)(c^*+1)$, $c^*=P-1$. So (4.13) gives $P|(P-1+\frac{\rho+1}{2})$ which is a contradiction.
Case 3: $e_1=2P+1$. Then $d=-2P+\rho$ and $r_2=4P-\rho+1$. Moreover, $\frac{e_2}{2}<2P\rho+2P$ so more than half of the class 2 treatments are contained in B_m. By Theorem 2.3(b), there

exists $i \in B_m$ such that $r_{ih} = \lceil \frac{r_2}{\rho+1} \rceil = \lceil \frac{4P+2}{\rho+1} \rceil - 1$. Since $\rho \geq P+2$, the possible values for r_{ih} are 2 and 3. $k_m^* \geq 2P$ for all blocks otherwise the intersection with B_m could not be of size λ. This implies that $r_{ih} = 2$ is impossible since $\frac{1}{2P\rho} + \frac{1}{2P} = \frac{\rho+1}{2P\rho} < \frac{1}{\rho}$. If $r_{ih} = 3$ then there must be at least one block besides B_m containing i and h and of size divisible by P. If the size of this block is $4P$ then the third block B_n satisfies

$$\frac{1}{k_n - \lambda} = \frac{1}{\rho} - \frac{1}{2P\rho} - \frac{1}{2P} = \frac{2P - \rho - 1}{2P\rho}. \tag{4.14}$$

Thus $P|(k_n - \lambda)$; however, both the assumptions $k_n - \lambda = 2P$ and $k_n - \lambda = P(\rho+1)$ lead to a contradiction easily. If the size of the second block is $P(\rho+1) + 2P$ then we obtain

$$\frac{1}{k_n - \lambda} = \frac{1}{\rho} - \frac{1}{2P\rho} - \frac{1}{P(\rho+1)} > \frac{1}{2P}, \tag{4.15}$$

again a contradiction. ∎

Proposition 4.7. (a) If $x - y = P$ then $y < P$.
(b) If $x - y = P$, $y < P$ then $d < 0$.

Proof: $x - y = P$ implies that each k_m is a multiple of P.
(a) Suppose that $y > P$. Then the only possible values for k'_m are $2P$ and $2P - y$, hence $\frac{1}{k_m - \lambda} = \frac{1}{2P}$ or $\frac{1}{3P}$. Let $i \in C_2$ and denote by a and b the number of blocks containing i and of size $4P$ and $5P$, respectively. By (2.12), $1 + \frac{y}{P+y} = \frac{3a+2b}{6P}$ implying that $(P+y)|6$. However, this is in contradiction with $y > P \geq 3$.
(b) Suppose that $d \geq 0$. From the integrality of e_1, we obtain $(P+y)|(2P+d)$; moreover, by Lemma 3.4, $d < 2P + \frac{2Py}{P+y} - 1 < 2P + 2y$. Thus the possible values for d are $d = P + 3y$ and $d = 2y$.

If $d = P + 3y$ then $r_2 = P$. The greatest possible value of $\frac{1}{k_m - \lambda}$ is $\frac{1}{P}$; hence, for any treatment i in class 2, $\sum_{i \in B_m} \frac{1}{k_m - \lambda} \leq P\frac{1}{P}$ contradicting (2.12). If $d = 2y$ then $r_2 = 2P$ and $e_1 = 2P + 2y$. Therefore, $k_m = k_n = 3P$, i.e. $k'_m = k'_n = 2P + y$ implies that the intersection of B_m and B_n consists of $2P$ treatments of class 1. In particular, any treatment i of class 2 is contained in at most one block of size $3P$. Thus $\sum_{i \in B_m} \frac{1}{k_m - \lambda} \leq \frac{1}{P} + (2P - 1)\frac{1}{2P}$ contradicting (2.12). ∎

5. The case $\lambda = 10$

By Theorem 1.4 and Propositions 3.7 and 4.7, the only possible values for ρ in a non-type-1 λ-design with $\lambda = 10$ are $\frac{9}{4}, \frac{8}{3}, \frac{7}{2}$, and 6. Moreover, in each case, d must be negative. In this section we shall eliminate these possibilities.

Proposition 5.1. $\rho = \frac{9}{4}$ is impossible.

Proof: By Theorem 3.5, $d \geq -15$. From the integrality of e_1 and e_2, we obtain $9|(10 + d)$ and $4|(9 - d)$. However, this congruence system has no solution in the range $-15 \leq d \leq -1$. ∎

Proposition 5.2. $\rho = \frac{8}{3}$ is impossible.

Proof: Theorem 3.5 gives $d \geq -17$. The integrality of e_1 and e_2 imply $8|(10+d)$ and $3|(9-d)$. However, this congruence system has no solution in the range $-17 \leq d \leq -1$. ∎

Proposition 5.3. $\rho = \frac{7}{2}$ is impossible.

Proof: In this case, Theorem 3.5 gives $d \geq -21$. The congruences are $7|(10+d)$ and $2|(9-d)$ with solutions $d = -3$ and $d = -17$ in the range $-21 \leq d \leq -1$. If $d = -3$ then $r_2 = 15$ and $t_2 = \frac{6}{5}$, contradicting Lemma 3.1. If $d = -17$ then $r_2 = 25$ and $t_2 = \frac{4}{5}$. However, Corollary 3.3(b) implies $t_2 \leq \frac{35}{99}$; we obtained a contradiction. ∎

Theorem 5.4. $\rho = 6$ is impossible.

Proof: Theorem 3.5 gives $d \geq -33$ and the integrality of e_1 implies $6|(10+d)$. Thus the possible values for d are $-4, -10, -16, -22$, and -28. If $d = -4$ then $r_2 = 15$ and $t_2 = \frac{6}{7}$, contradicting Lemma 3.1. If $d = -22$ then $\frac{r_1}{\rho+1} = 25$ and if $d = -28$ then $\frac{r_2}{\rho+1} = 5$, contradicting Theorem 2.2(ii). The cases $d = -10$ and $d = -16$ require more careful considerations.

If $d = -10$ then $e_1 = 10$, $e_2 = 124$, $r_1 = 115$, and $r_2 = 20$. The possible values of $\frac{1}{k_m - \lambda}$ are $\frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \cdots, \frac{1}{60}$; however, blocks not containing two fixed treatments of class 1 must satisfy $\frac{1}{k_m - \lambda} \leq \frac{1}{20}$. Therefore, $R(\bar{1}, \bar{1}) = \frac{1}{15}$ implies that there are at least two blocks not containing i and j for any $i, j \in C_1$; hence $r_{ij} \geq 98$ for all $i, j \in C_1$. By (2.23) and (2.24), for fixed $i \in C_1$,

$$\sum_{j \leq e_1, j \neq i} r_{ij} = 9 \cdot (98 + \frac{4}{7}) - \frac{U_i}{5} = 9 \cdot 98 + \frac{36}{7} - \frac{U_i}{5} \tag{5.1}$$

and

$$\sum_{j > e_1} r_{ij} = 124 \cdot (16 + \frac{3}{7}) + \frac{6U_i}{5} = 124 \cdot 17 - 71 + \frac{1}{7} + \frac{6U_i}{5}. \tag{5.2}$$

Therefore $\frac{U_i}{5} \leq \frac{36}{7}$ and $\sum_{j > e_1} r_{ij} \leq 124 \cdot 17 - 40$. By (2.25),

(5.3)

$$U_i + V_i = \sum_{j \leq e_1, j \neq i} ((r_{ij} - 98 - \frac{4}{7})^2 + (r_{ij} - 97 - \frac{3}{16})^2) + \sum_{j > e_1} ((r_{ij} - 16 - \frac{3}{7})^2 + (r_{ij} - 17 - \frac{1}{32})^2) \geq$$

$$9 \cdot (\frac{4}{7})^2 + 9 \cdot (\frac{13}{16})^2 + 40 \cdot (\frac{3}{7})^2 + 40 \cdot (\frac{33}{32})^2 + 84 \cdot (\frac{4}{7})^2 + 84 \cdot (\frac{1}{32})^2 = 86 + \frac{31}{112}.$$

However, (2.22) implies that $U_i + V_i = 62 + \frac{31}{112}$, we obtain a contradiction.

If $d = -16$ then $e_1 = 9$, $e_2 = 160$, $r_1 = 145$, and $r_2 = 25$. By (2.28) and (2.29),

$$\sum_{j \leq e_1} r_{ij} = 9 \cdot (21 + \frac{3}{7}) - \frac{U_i}{5} = 9 \cdot 21 + 4 - \frac{1}{7} - \frac{U_i}{5} \tag{5.4}$$

and

$$(5.5) \qquad \sum_{j>e_1, j\neq i} r_{ij} = 159 \cdot (3 + \frac{4}{7}) + \frac{6U_i}{5} = 159 \cdot 4 - 69 + 6 \cdot (\frac{1}{7} + \frac{U_i}{5})$$

hold for all $i \in C_2$. In particular, $\frac{1}{7} + \frac{U_i}{5}$ is an integer. By Theorem 2.5 and Proposition 3.6, there exists $i \in C_2$ satisfying $\frac{U_i}{5} \leq e_1 \frac{\rho-1}{\rho+1} = \frac{45}{7}$, or, because of the integrality condition, $\frac{1}{7} + \frac{U_i}{5} \leq 6$. For this particular i, (5.5) gives $\sum_{j>e_1, j\neq i} r_{ij} \leq 159 \cdot 4 - 33$. By (2.30),

$$(5.6) \qquad U_i + V_i > \sum_{j>e_1, j\neq i} ((r_{ij} - 3 - \frac{4}{7})^2 + (r_{ij} - 4 - \frac{3}{23})^2) \geq$$

$$33 \cdot (\frac{4}{7})^2 + 33 \cdot (\frac{26}{23})^2 + 126 \cdot (\frac{3}{7})^2 + 126 \cdot (\frac{3}{23})^2 > 78.$$

However, (2.27) implies that $U_i + V_i = 74 + \frac{11}{161}$, we obtain a contradiction. ∎

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