

THE SPACE CHARGE PROBLEM

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Abstract. We consider a space charge problem arising in modelling steady electric field in a neighborhood of an ionising conductor. Mathematically the problem reduces to a system consisting of one elliptic equation and one nonlinear hyperbolic equation. The existence of a weak solution is proved and its properties are studied, in part rigorously and in part heuristically and numerically.

Key words. space charge, ionising conductor, potential density, field lines, weak solution.

AMS(MOS) subject classifications. 35Q99, 78A35, 35J60, J5M05

§1 Introduction. In this paper we study a system of differential equations which arise in modelling of a steady electric field in the neighborhood of an ionising conductor. In contrast to an electro-static field where charge density is everywhere zero, the field in this situation has nonzero charge density which is known as the *space charge*. The resulting partial differential equations that describe this field consist of an elliptic and a hyperbolic equation coupled together in a nonlinear manner. The electric potential v and the charge density ρ satisfy:

$$(1.1) \quad \begin{aligned} \Delta v &= -\rho, \\ \nabla v \cdot \nabla \rho &= \rho^2 \end{aligned}$$

and thus v satisfies the third order nonlinear equation

$$(1.2) \quad \operatorname{div} (\Delta v \nabla v) = 0 .$$

Ionised fields arise in physical systems where there is a high DC electric field due to a conductor at a high potential. This field causes a local breakdown of the air molecules into positive and negative ions and a glowing corona forms around the conductor. The ions which have the opposite sign to that of the conductor are attracted onto its surface, whereas those of the same sign leave the neighbourhood of the corona and enter the medium surrounding the conductor. Experiments indicate that if the conductor is of positive polarity then the region occupied by the corona is very thin. Thus a macroscopic

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approximation for our system consists of studying the field derived from a single positive ion species with the thin corona acting as a source of these ions.

The fields arising from an ionised gas are important in the study of DC transmission lines and in the performance of electrostatic precipitators used in electricity power stations for the removal of dust from furnace gases. In the former problem the migration of the ions from the surface of the conductor to ground causes a current and consequent power loss. In addition, there may be radio interference if the resulting field is time dependent. A typical geometry for this system is illustrated in Figure 1.1 and comprises a section through a thin wire at high potential contained within an earthed conductor at zero potential.

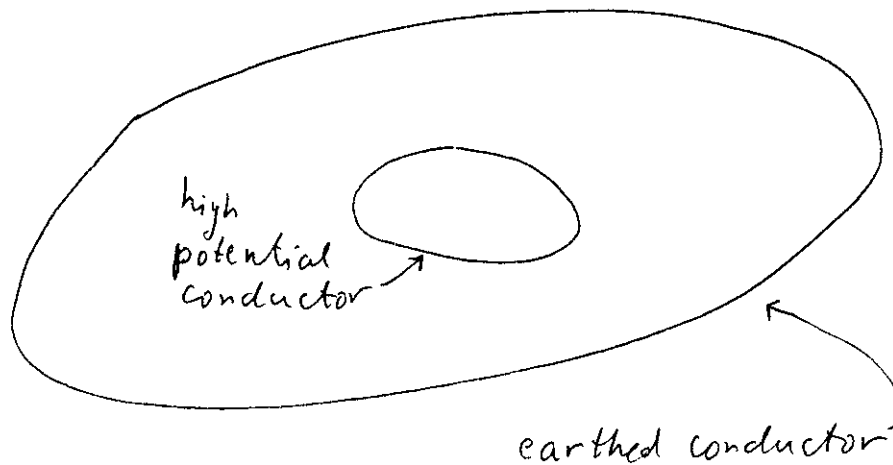


Figure 1.1

The resulting geometry is thus one-connected. In contrast, an electrostatic precipitator comprises, in general, an array of thin coronating electrodes at high potential between two parallel earthed plates. This geometry is illustrated in Figure 1.2 and it is evident from this figure that it is multiply connected. This geometry leads to discontinuities in the gradient of the space charge, the existence of which has been observed experimentally in a test rig at the Central Electricity Research Laboratory (CERL), Letherhead, England.

For the literature on the space charge problem see Felici [6], Atten [3], Khalifa [10], Janischewski and Gela [9], Abdel-Salam et al. [1], Okubo et al. [11], Sunaga and Sawada [15], Smith [14], Hutton [8], and Budd and Wheeler [4] [5]. In much of the earlier work many approximations were made concerning the distribution of the field lines. One in particular, known as Deutch's approximation (see Sarma and Janischewski [13]) assumes the field lines to be the same as the field lines obtained by finding a harmonic field with

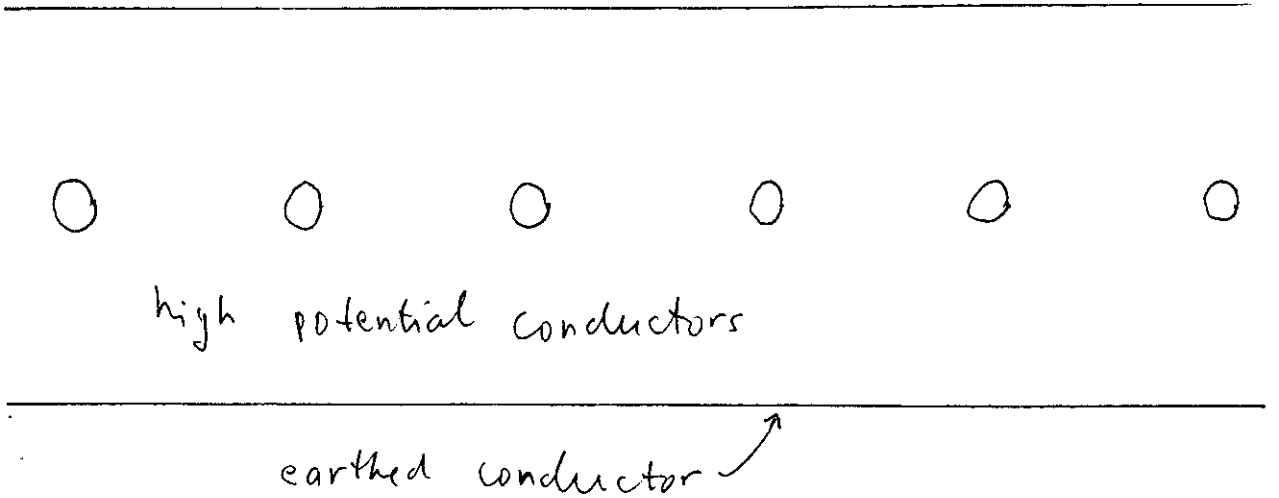


Figure 1.2

zero space charge. Also some special solutions have been constructed for simple geometries such as parallel plates and concentric cylinders; see reference in [4] [5]. Solutions for the full equations in the absence of boundaries have been constructed by Felici [6], Varley [17] and Smith [14]. More recent work has been done on the numerical solution of the space charge equations usually with finite element technique. Although this can be very successful (see Abdel Salam et al [2] and Hutton [8]), it is also expensive, requiring a special treatment at the shock boundaries and a careful mesh refinement close to the high tension conductor. A new numerical method for solving the space charge equations in a simply connected geometry which overcomes these difficulties was given by Budd and Wheeler [5]; it is based on a hodograph transformation.

In §2 we shall establish the existence of a weak solution of (1.1) in a bounded domain Ω in \mathbb{R}^2 with suitable boundary conditions. The boundary $\partial\Omega$ is a union of two sets Γ_i and Γ_e , and $v = M$ on Γ_i $v = 0$ on Γ_e ; ρ is prescribed only on Γ_i . Our method is based on first studying the approximating system

$$(1.3) \quad \begin{aligned} \Delta v &= -\rho, \\ \epsilon \Delta \rho + \nabla v \cdot \nabla \rho - \rho^2 &= 0 \quad (\epsilon > 0) \end{aligned} .$$

and then letting $\epsilon \rightarrow 0$. The solution (v, ρ) satisfies:

$$(1.4) \quad \begin{aligned} v &\in C^{1+\alpha}(\Omega) \quad \forall \alpha \in (0, 1), \\ \rho &\geq 0, \quad \rho \in L^\infty(\Omega) . \end{aligned}$$

In the special case of Ω 1-connected and M large, existence of a classical solution was established by Atten [3] by a different method.

Motivated by the physical model of the electrostatic precipitators, we shall concentrate, for the rest of the paper, on the geometry given in Figure 1.2 (some of the results however easily extend to general geometry). Thus in §3 we state the results of §2 for this geometry, and, due to periodicity, we may concentrate on one period of Ω ; see Figure 3 below.

In §4 we study the level lines of v and, in particular, prove that the open sets $\{v > \lambda\}$, $\{v < \lambda\}$ must be connected to the boundary. This implies that

$$(1.5) \quad v \text{ cannot take a strict local maximum at an interior point.}$$

In §5 we show that, for any $\delta > 0$, if M is large enough then

$$(1.6) \quad \begin{aligned} \rho > 0 & \quad \text{if} \quad a < x < \frac{a}{2} - \delta ; \\ \rho = 0 & \quad \text{if} \quad \frac{a}{2} + \delta < x < a . \end{aligned}$$

In §6 we show, under some conditions, that $\rho = 0$ in a neighborhood of the boundary $\gamma_1 = \Gamma_i \cap \{\rho = 0\}$.

It is anticipated that there is a free boundary $\Gamma : x = s(y)$ such that

$$\rho(x, y) > 0 \quad \text{if} \quad 0 < x < s(y) , \quad \rho(x, y) = 0 \quad \text{if} \quad s(y) < x < a ;$$

further, $\rho \in C^\alpha$ uniformly in $\{0 < x < s(y)\}$ but not across $\{x = s(y)\}$. In §7 we provide very partial results in this direction.

In §8 we introduce a numerical method for computing the field lines of ∇v and the density lines of ρ , and in particular the free boundary $\{x = s(y)\}$. This is an extension of the method of Budd and Wheeler [5] from simply connected domains to multiply connected domains. Some numerical results are given in §9 and they indicate some interesting properties of the free boundary.

§2. Existence for general domains. Let Ω be a bounded domain in \mathbb{R}^2 with $C^{2+\alpha_0}$ boundary $\partial\Omega = \Gamma_i \cup \Gamma_e$ ($0 < \alpha_0 < 1$), where $\Gamma_i = \bar{\Gamma}_i, \Gamma_e = \bar{\Gamma}_e$, $\Gamma_i \cap \Gamma_e = \emptyset$. Consider the problem:

$$(2.1) \quad \operatorname{div}(\Delta v \nabla v) = 0 \quad \text{in} \quad \Omega ,$$

$$(2.2) \quad \begin{cases} v = M & \text{on} \quad \Gamma_i , \\ v = 0 & \text{on} \quad \Gamma_e , \\ -\Delta v = g & \text{on} \quad \Gamma_i \end{cases}$$

where M is a given positive constant and g is a given function,

$$g \in C^{\alpha_0}, \quad g \geq 0.$$

We state a weak formulation:

$$(2.3) \quad \begin{cases} \int \nabla \phi \cdot \Delta v \nabla v = 0 & \forall \phi \in H_0^1(\Omega), \\ \Delta v \in L^\infty(\Omega) \\ v = M \quad \text{on } \Gamma_i, \quad v = 0 \quad \text{on } \Gamma_e \end{cases}$$

and

$$(2.4) \quad \Delta v + g = 0 \quad \text{on } \Gamma_i \quad \text{in some trace class sense.}$$

Notice that formully

$$(2.5) \quad \Delta v = -\rho,$$

$$(2.6) \quad \Delta v \cdot \nabla \rho - \rho^2 = 0.$$

This motivates us to introduce the ϵ -approximating system ($\epsilon > 0$)

$$(2.7) \quad \begin{cases} \Delta v = -\rho & \text{in } \Omega, \\ v = M & \text{on } \Gamma_i, \\ v = 0 & \text{on } \Gamma_e, \end{cases}$$

$$(2.8) \quad \begin{cases} \epsilon \Delta \rho + \nabla v \cdot \nabla \rho - \rho^2 = 0 & \text{in } \Omega, \\ \rho = g & \text{on } \Gamma_i, \\ \rho = 0 & \text{on } \Gamma_e. \end{cases}$$

LEMMA 2.1. *For any $\epsilon > 0$ there exists a solution $(v_\epsilon, \rho_\epsilon)$ of (2.7), (2.8) with $v_\epsilon \in C^{2+\alpha}(\overline{\Omega})$, $\rho_\epsilon \in C^{2+\alpha}(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Proof. Let

$$\mathcal{A} = \{\rho; \rho \in L^\infty(\Omega), \quad 0 \leq \rho \leq g_0\}, \quad g_0 = \max g.$$

Given $\rho \in \mathcal{A}$, denote by v the solution of (2.7) and by $\bar{\rho}$ the solution of

$$(2.9) \quad \begin{cases} \epsilon \Delta \bar{\rho} + \nabla v \cdot \nabla \bar{\rho} - \rho \bar{\rho} = 0 & \text{in } \Omega, \\ \bar{\rho} = g & \text{on } \Gamma_i, \\ \bar{\rho} = 0 & \text{on } \Gamma_e. \end{cases}$$

By elliptic estimates, for any $0 < \beta < 1$,

$$(2.10) \quad |v|_{C^{1,\beta}(\Omega)} \leq A$$

where A is a constant depending only on Ω, g_0, β . By the maximum principle

$$(2.11) \quad 0 < \bar{\rho} < g_0 \quad \text{in } \Omega .$$

Using (2.10) and elliptic estimates we also get

$$|\bar{\rho}|_{C^\alpha(\Omega)} \leq B \quad \text{for some } \alpha \in (0, 1)$$

where B is a constant depending only on Ω, g . It follows that the mapping $T : \rho \rightarrow \bar{\rho}$ maps \mathcal{A} into a compact subset. It is also easy to check that T is a continuous map. Hence, by Schauder's fixed point theorem, T has a fixed point, and Lemma 2.1 follows.

Multiplying the differential equation in (2.8) by ϕ , $\phi \in H_0^1(\Omega)$, and integrating over Ω , we get

$$(2.11) \quad -\epsilon \int_{\Omega} \phi \Delta \rho_\epsilon - \int_{\Omega} (\nabla v_\epsilon \cdot \nabla \rho_\epsilon) \phi + \int_{\Omega} \rho_\epsilon^2 \phi = 0 .$$

Since

$$\begin{aligned} - \int_{\Omega} (\nabla v_\epsilon \cdot \nabla \rho_\epsilon) \phi &= \int_{\Omega} \rho_\epsilon \nabla v_\epsilon \cdot \phi + \int_{\Omega} \rho_\epsilon \nabla v_\epsilon \cdot \nabla \phi \\ &= - \int_{\Omega} \rho_\epsilon^2 \phi + \int_{\Omega} (\Delta v_\epsilon) \nabla v_\epsilon \cdot \nabla \phi \quad \text{by (2.7),} \end{aligned}$$

we get from (2.11),

$$(2.12) \quad -\epsilon \int_{\Omega} \rho_\epsilon \Delta \phi + \int_{\Omega} (\Delta v_\epsilon) \nabla v_\epsilon \cdot \nabla \phi = 0 .$$

Recalling that $0 \leq \rho_\epsilon \leq g_0$, $|\Delta v_\epsilon| \leq 1$, we can extract a sequence $\epsilon = \epsilon_j \rightarrow 0$ such that

$$\begin{aligned} \rho_\epsilon &\rightarrow \rho \quad \text{weakly in } (L^\infty(\Omega))^* , \\ v_\epsilon &\rightarrow v \quad \text{in } C^{1+\alpha}(\bar{\Omega}) \quad \forall 0 < \alpha < 1 . \end{aligned}$$

Taking $\epsilon \rightarrow 0$ in (2.12) we obtain the first relation in (2.3). The other relations in (2.3) are obviously also valid.

On the other hand it is not clear whether condition (2.4) is satisfied. Nonetheless we shall henceforth work with the pair (v, ρ) just constructed and view it as a *weak solution* of the space charge problem. Our objective is to establish various physical properties for (v, ρ) . In some situations we can also prove (see §5) that ρ is continuous on Γ_i and satisfies (2.4) pointwise.

Remark 2.1. It will be shown in §4 that $0 \leq v \leq M$ in Ω , so that $\partial v / \partial \nu \geq 0$ on Γ_i where ν is the outward normal. If

$$\begin{aligned} \frac{\partial v}{\partial \nu} &> 0 \quad \text{on a component } \Gamma_{i,j} \text{ of } \Gamma_i, \\ g = \text{const} = c &> 0 \quad \text{on } \Gamma_{i,j}, \end{aligned}$$

then ρ is continuous on $\Gamma_{i,j}$. Indeed, denoting by d the distance function to $\Gamma_{i,j}$, we can compare ρ_ϵ in an η -neighborhood of $\Gamma_{i,j}$ with super- and subsolutions

$$w_+ = c + Ad, \quad w_- = c - Ad$$

where η is small enough and A is sufficiently large so that

$$\epsilon \Delta w_\pm + \nabla v \cdot \nabla w_\pm - (w_\pm)^2 \leq 0;$$

we deduce that

$$|\rho_\epsilon - c| \leq Ad$$

from which the assertion follows. The above extends also to the case $c = 0$.

Remark 2.2. Atten's proof of existence [3] is based on the formal relation

$$(2.13) \quad \nabla v \cdot \nabla \frac{1}{\rho} = -1,$$

or

$$(2.14) \quad \frac{1}{\rho(X)} - \frac{1}{\rho(Y)} = \int_{C_{XY}} \frac{ds}{|\nabla v|}$$

where C_{XY} is the line of force (i.e., the curve in the direction of ∇v) from X to Y . Thus, defining the mapping $\rho \rightarrow v$ by (2.7) and $v \rightarrow \bar{\rho}$ by

$$\frac{1}{\bar{\rho}(X)} - \frac{1}{\bar{\rho}(Y)} = \int_{C_{XY}} \frac{ds}{|\nabla v|}$$

and assuming that Γ_i and Γ_e are both connected and M is sufficiently large, he shows that $\nabla v \neq 0$ in Ω and the mapping $T : \rho \rightarrow \bar{\rho}$ maps the admissible class

$$\{\rho = g \quad \text{on } \Gamma_i, \quad |\rho(X) - \rho(Y)| \leq C |X - Y|\}$$

into itself and has a fixed point. One can actually show that this is a contraction and thus the solution is unique. (Atten claims a general uniqueness theorem, but his proof is not clear.) In the case to be considered in the rest of the paper, Γ_i is not connected.

§3. **Special geometry.** From now on we concentrate on the special geometry where

$$\Omega = \tilde{\Omega} \setminus \left(\bigcup_{j=-\infty}^{\infty} B_j \right),$$

$$\tilde{\Omega} = \{-\infty < x < \infty, -b < y < b\},$$

B_j is the ball with center $(ja, 0)$ and radius μ , $0 < \mu < \frac{1}{2}a, 0 < \mu < b$.

$$\Gamma_i = \bigcup_{j=-\infty}^{\infty} \partial B_j, \quad \Gamma_e = \{y = \pm b\}$$

and

$$\begin{aligned} \rho &= 1 \quad \text{on } \partial B_j, & j & \text{ even,} \\ &= 0 \quad \text{on } \partial B_j, & j & \text{ odd.} \end{aligned}$$

One can establish the existence of $(v_\epsilon, \rho_\epsilon)$ by applying the proof of Lemma 2.1 in truncated domains. The solution will be symmetric with respect to $\{x = ja\}$ ($j = 0, \pm 1, \pm 2, \dots$) and with respect to $\{y = 0\}$. Alternatively we can work in the bounded domain

$$\Omega_0 = \tilde{\Omega}_0 \setminus (\tilde{B}_1 \cup \tilde{B}_0)$$

where

$$\begin{aligned} \tilde{\Omega}_0 &= \{(x, y) ; 0 < x < a, 0 < y < b\}, \\ \tilde{B}_0 &= \{x^2 + y^2 < \mu^2\} \cap \tilde{\Omega}_0, \\ \tilde{B}_1 &= \{(x - a)^2 < \mu^2\} \cap \tilde{\Omega}_0 \end{aligned}$$

and set

$$\begin{aligned} \gamma_j &= \partial \tilde{B}_j \cap \partial \Omega_0, \quad \Gamma_i = \gamma_0 \cup \gamma_1, \\ S &= \{y = b\} \cap \partial \Omega_0, \\ \ell_0 &= \{(0, y) ; \mu < y < b\}, \\ \ell_1 &= \{(a, y) ; \mu < y < b\}, \\ \ell_2 &= \{(x, 0) ; \mu < x < a - \mu\}. \end{aligned}$$

Then

$$(3.1) \quad \begin{aligned} v_\epsilon &= M \quad \text{on } \gamma_0 \cup \gamma_1, \quad v_\epsilon = 0 \quad \text{on } S, \\ \frac{\partial v_\epsilon}{\partial \nu} &= 0 \quad \text{on the remaining parts of } \partial \Omega_0, \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} \rho_\epsilon &= 1 \quad \text{on } \gamma_0, \quad \rho_\epsilon = 0 \quad \text{on } \gamma_1, \\ \rho_\epsilon &= 0 \quad \text{on } S, \\ \frac{\partial \rho_\epsilon}{\partial \nu} &= 0 \quad \text{on the remaining parts of } \partial \Omega_0. \end{aligned}$$

It will be convenient to have in sight the following figure:

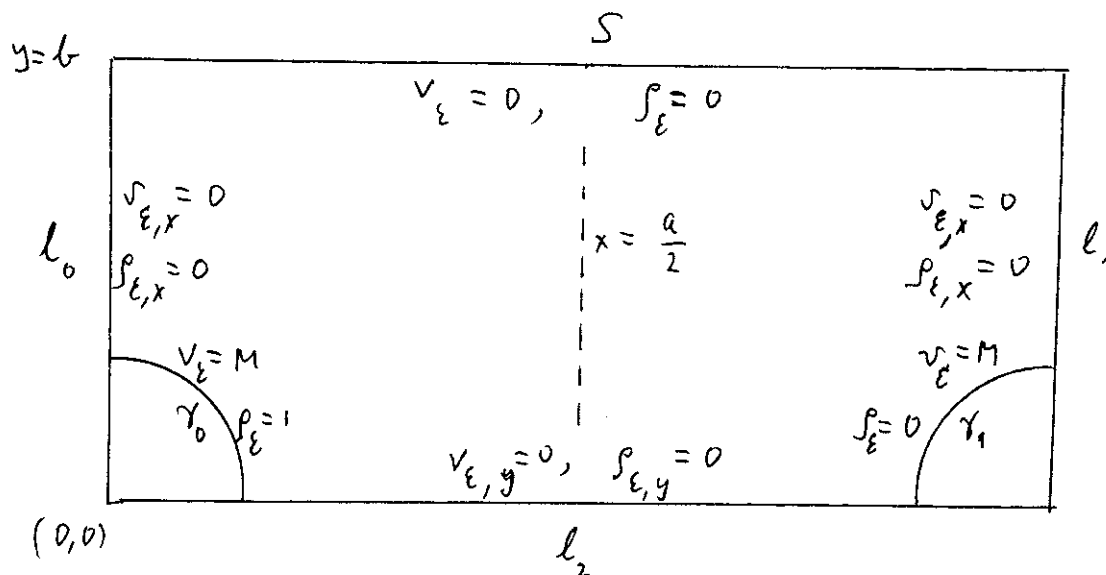


Figure 3.1

The solution $(v_\epsilon, \rho_\epsilon)$ in all of Ω is obtained from the solution in Ω_0 by extension based on symmetry and periodicity in the obvious way.

In the sequel we shall be concerned with $(v_\epsilon, \rho_\epsilon)$ and the limit (v, ρ) only in the fundamental domain Ω_0 . Recall that

$$(3.3) \quad v \in C^{1+\alpha}(\bar{\Omega}_0) \quad , \quad \rho \in L^\infty(\Omega_0),$$

$$(3.4) \quad 0 \leq \rho \leq 1.$$

§4. The level lines of v .

THEOREM 4.1. $0 < v \leq M$ in Ω_0 .

Proof. Since $-\Delta v = \rho \geq 0$, the maximum principle implies that $v > 0$ in Ω_0 . To prove that $v \leq M$, introduce the function

$$(4.1) \quad w_\epsilon = e^{\frac{v_\epsilon}{\epsilon}} \rho_\epsilon$$

It satisfies

$$(4.2) \quad \operatorname{div} (e^{\frac{-v_\epsilon}{\epsilon}} \nabla w_\epsilon) = 0.$$

Suppose the set $\{v > M\} \cap \Omega_0$ is nonempty. Then, for any compact subset K of $\{v > M\} \cap \Omega_0$,

$$v > M + 2c \quad \text{in } K, \quad \text{for some } c > 0;$$

hence

$$(4.3) \quad v_\epsilon \geq M + c \quad \text{in } K$$

if ϵ is small enough. Clearly

$$\begin{aligned} \frac{\partial w_\epsilon}{\partial x} &= 0 \quad \text{on } \ell_0 \cup \ell_1, & \frac{\partial w_\epsilon}{\partial y} &= 0 \quad \text{on } \ell_2, \\ w_\epsilon &= e^{\frac{M}{\epsilon}} \quad \text{on } \gamma_0, & w_\epsilon &= 0 \quad \text{on } S \cup \gamma_1. \end{aligned}$$

Hence, by applying the maximum principle to w_ϵ in Ω_0 , we get

$$w_\epsilon \leq e^{\frac{M}{\epsilon}} \quad \text{in } \Omega_0.$$

It follows that

$$\rho_\epsilon(X) \leq e^{\frac{M}{\epsilon}} e^{-\frac{v_\epsilon}{\epsilon}} \leq e^{-\frac{\epsilon}{\epsilon}} \quad \text{in } K,$$

where (4.3) was used. Taking $\epsilon \rightarrow 0$ we get $\rho = 0$ in K and thus also $\Delta v = 0$ in K . Since K is arbitrary,

$$\Delta v = 0 \quad \text{in } \{v > M\} \cap \Omega_0,$$

which is a contradiction to the maximum principle.

THEOREM 4.2. *For any $\lambda > 0$ the set $\{v > \lambda\}$ must be connected to the boundary Γ_i ; consequently it has at most two components, one connected to γ_0 and the other to γ_1 .*

Proof. Take a component K of $\{v > \lambda\}$. If K is not connected to Γ_i , then on $\partial K \cap \Omega_0$ we have $v = \lambda$, and

$$(4.4) \quad \frac{\partial w_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial K \cap \partial \Omega_0.$$

Choose a point $X_0 \in K$ such that

$$v(X_0) > \lambda = \max_{\partial K \cap \Omega_0} v.$$

Then, by continuity,

$$(4.5) \quad v_\epsilon(X) > \max_{\partial K \cap \Omega_0} v_\epsilon + c \quad \text{in } B_\delta(X_0) \equiv \{|X - X_0| < \delta\}$$

for some $c > 0$, $B_\delta(X_0) \subset K$ and all ϵ small enough.

Recalling (4.4) and applying the maximum principle to w_ϵ in K , we get

$$\rho_\epsilon(X) e^{\frac{v_\epsilon(X)}{\epsilon}} \leq \max_{\partial K \cap \Omega_0} \rho_\epsilon e^{\frac{v_\epsilon}{\epsilon}} \leq \exp\left\{\frac{1}{\epsilon} \max_{\partial K \cap \Omega_0} v_\epsilon\right\}.$$

Comparing with (4.5) we conclude that

$$\rho_\epsilon(X) \leq e^{-c/\epsilon} \quad \text{if } x \in B_\delta(X_0)$$

and therefore $\rho = 0$ in $B_\delta(X_0)$. It follows that $\Delta v = 0$ in $B_\delta(X_0)$. If we now choose X_0 to be a point where $v(X)$ takes its maximum in K , then we get a contradiction to the maximum principle.

The above proof yields:

COROLLARY 4.3. *The function v cannot take a strict local maximum at any point in Ω_0 .*

THEOREM 4.4. *For any $\lambda \in (0, M)$ the set $\{v < \lambda\}$ is connected to S .*

Proof. If the assertion is not true then there is a component K of $\{v < \lambda\}$ with $\partial K \cap S = \emptyset$. Then $v = \lambda$ on $\partial K \cap \Omega_0$ and

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial K \cap \partial \Omega_0$$

($\partial K \cap \gamma_i = \emptyset$ since $v = M$ on γ_i). Since $-\Delta v \geq 0$, the maximum principle yields $v > \lambda$ in K , which is a contradiction.

Remark 4.1. For any $\lambda < M$ the set $\{v = \lambda\}$ cannot be the closure of an open set. Indeed, move any disc in $\{v < \lambda\}$ until it touches $\{v = \lambda\}$, say at X_0 . Then, by the maximum principle, $\partial v(X_0)/\partial \nu > 0$; at the same time, if $X_0 = \lim_i X_i$ where $X_i \in \text{int } \{v = \lambda\}$, then $\nabla v(X_0) = \lim_i \nabla v(X_i) = 0$, which is a contradiction.

§5. M large.

THEOREM 5.1. *Given any small $\delta > 0$, there exists an $M_0 > 0$ such that if $M > M_0$ then*

$$(5.1) \quad \rho > 0 \quad \text{in } \Omega_0 \cap \left\{x < \frac{a}{2} - \delta\right\},$$

$$(5.2) \quad \rho = 0 \quad \text{in } \Omega_0 \cap \left\{x > \frac{a}{2} + \delta\right\}.$$

Proof. The proof is by comparison. We first prove (5.1). Denote by \bar{v} the harmonic function in Ω_0 which satisfies the same boundary conditions as v with $M = 1$. Then

$$(5.3) \quad \frac{v_\epsilon}{M} \longrightarrow \bar{v} \quad \text{in } C^{1+\alpha}(\bar{\Omega}_0) \quad \text{as } M \longrightarrow \infty,$$

uniformly with respect to ϵ . By the maximum principle (using $\bar{v}_x(\frac{a}{2}, y) = 0$) we have

$$(5.4) \quad \bar{v}_x < 0, \quad \bar{v}_y < 0 \quad \text{in} \quad \{0 < x < \frac{a}{2}\} \cap \Omega_0$$

and thus, in particular,

$$(5.5) \quad |\nabla \bar{v}| \geq c > 0 \quad \text{in} \quad \{0 < x < \frac{a}{2} - \frac{\delta}{2}\} \cap \Omega_0 .$$

Consider a family of smooth curves σ_t for $0 \leq t \leq 1$, having the following properties:

- (i) As t varies from $t = 0$ to $t = 1$, σ_t traces a domain which contains $\Omega \cap \{0 < x < \frac{a}{2} - \delta\}$ and is contained in $\Omega \cap \{0 < x < \frac{a}{2} - \frac{\delta}{2}\}$; σ_1 coincides with γ_0 ;
- (ii) Each σ_t ($0 \leq t < 1$) consists of one vertical segment $\sigma_{t,1}$, one horizontal segment $\sigma_{t,2}$ and an arc $\sigma_{t,3}$ which connects them; along $\sigma_{t,3}$ x is monotone decreasing and y is monotone increasing (from the top-end-point of $\sigma_{t,1}$ to the right end-point of $\sigma_{t,2}$);
- (iii) $t \rightarrow \sigma_t$ is a smooth function, and its t -derivative is strictly positive at any point (t, σ_t) .

Define a function $w(x, y)$ by

$$w(\sigma_t) = t .$$

Then, by (5.4), (5.5)

$$\nabla \bar{v} \cdot \nabla w \geq c > 0 \quad \text{in} \quad \Omega_* \equiv \bigcup_{0 < t < 1} \{\sigma_t\}$$

and consequently,

$$(5.6) \quad \nabla v_\epsilon \cdot \nabla w \geq \frac{c}{2} M \quad \text{in} \quad \Omega_*$$

provided ϵ is small enough. It follows that

$$\epsilon \Delta w + \Delta v_\epsilon \cdot \nabla w - w^2 \geq 0 \quad \text{in} \quad \Omega_* .$$

Also, by (ii),

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \{x = 0\} \cap \partial \Omega_* \quad \text{and on} \\ \{y = 0\} \cap \partial \Omega_* ;$$

further, $w = 1 = \rho_\epsilon$ on γ_0 and $w = 0 \leq \rho_\epsilon$ on the remaining parts of $\partial \Omega_*$. We can therefore apply a comparison argument to deduce that

$$w \leq \rho_\epsilon \quad \text{in} \quad \Omega_* .$$

Taking $\epsilon \rightarrow 0$, the assertion (5.1) follows.

To prove (5.2) we proceed to construct a family of curve $\tilde{\sigma}_t$ in $\{\frac{a}{2} - \frac{\delta}{2} < x < a\}$ similar to the curves σ_t , with $\tilde{\sigma}_1 = \gamma_1$; here too $\tilde{\sigma}_t$ consists of a vertical segment $\tilde{\sigma}_{t,1}$, a horizontal segment $\tilde{\sigma}_{t,2}$ and an arc $\tilde{\sigma}_{t,3}$ connecting them; the coordinates x and y are increasing along $\tilde{\sigma}_{t,3}$. We define a function w by $w(\tilde{\sigma}_t) = t$ and try to find a supersolution in the form $\zeta = f(w)$ with $f' > 0$. Notice that now

$$(5.7) \quad \nabla \bar{v} \cdot \nabla w \leq -c < 0 .$$

Choose any $\theta \in (0, 1)$, θ close to 1, and any $\eta > 0, \eta$ small, and let

$$\begin{aligned} f(s) &= \frac{\eta}{\theta} s \quad \text{if } 0 < s < \theta , \\ &= \eta + \frac{\eta}{\theta} (s - \theta) + \lambda (s - \theta)^2 \quad \text{if } \theta < s < 1 . \end{aligned}$$

We determine λ by the condition $f(1) = 1$; then $\lambda > 0$. It is clear that $f \in C^1$ and

$$|\nabla f(w)| \leq C .$$

Hence

$$\nabla v_\epsilon \cdot \nabla f(w) \sim M(\nabla \bar{v} \cdot \nabla w) f'(w) \leq -c M \frac{\eta}{\theta}$$

by (5.7), and

$$\epsilon \Delta \zeta + \nabla v_\epsilon \cdot \nabla \zeta - \zeta^2 \leq \epsilon C - c M \frac{\eta}{\theta} - f^2(w) < 0$$

provided ϵ is sufficiently small.

By comparison we then get

$$\rho_\epsilon \leq f(w) \quad \text{if } M \geq M_0$$

provided ϵ is small enough depending on θ, η . Hence also

$$\rho \leq f(w) .$$

Taking $\eta \rightarrow 0$, $\theta \rightarrow 1$ we conclude that $\rho = 0$ in $\Omega_0 \cap \{\frac{a}{2} + \delta < x < a\}$.

Remark 5.1. From Remark 2.1 it follows that for M large ρ is continuous on γ_0 with $\rho = 1$ on γ_0 .

§6. General M.

LEMMA 6.1. Let D be an open set and $\rho \in C^\alpha(D)$, for some $\alpha \in (0, 1)$. If $X_0 \in D$, $\rho(X_0) \neq 0$, then $\nabla v(X_0) \neq 0$.

Proof. Let $-\sigma = \rho(X_0) > 0$. Then

$$\Delta v = \sigma + o(r^\alpha), \quad r = |X - X_0|.$$

If $\nabla v(x_0) = 0$ then, by [7; Lemma 5.2, p. 458], we must have

$$v = v(X_0) + \sigma r^2 + P_2 + R \quad (\sigma < 0)$$

where, after a rotation of the coordinates about $x = x_0$,

$$\begin{aligned} P_2 &= Ar^2 \cos 2\theta, \quad A \text{ constant,} \\ R &= o(r^{2+\alpha}), \quad \nabla R = o(r^{1+\alpha}). \end{aligned}$$

If $A = 0$ then the function v takes strict local maximum at $X = X_0$, a contradiction to Corollary 4.3. Hence $A \neq 0$ and then $\{v = v(X_0)\}$ is given locally by $-\sigma \sim A \cos 2\theta$, so that, for some small $\eta > 0$, the sets

$$\{v > v(X_0)\}, \quad \{v < v(X_0)\}$$

in $B_\eta(X_0)$ each consists of two regions. This contradicts the connectivity results of Theorems 4.2, 4.4 with $\lambda = v(X_0)$.

COROLLARY 5.2. Set $A_0 = (\mu, 0)$, and let D_0 be an Ω_0 -neighborhood A_0 . If $\rho \in C^\alpha(D_0)$ and $\rho = 1$ in $\partial D_0 \cap \gamma_0$, then $u(x, 0) \not\equiv M$ for $\mu < x < a - \mu$.

Proof. If v and ρ are extended by anti-reflection across $\{y = 0\}$ then we still have $\Delta v = -\rho$, $\rho \in C^\alpha$ in the domain $D = \bar{D}_0 \cup D_1$ where D_1 is the reflection of D_0 . By continuity we also have $\rho > 0$ on $\{y = 0\} \cap D$ if D_0 is small enough. Hence Lemma 6.1 implies that

$$\nabla v(x, 0) \neq 0 \quad \text{if } \mu < x < \mu + \delta$$

for some $\delta > 0$. Since, however, $v_y(x, 0) = 0$, we conclude that $v_x(x, 0) \neq 0$ and the assertion follows.

THEOREM 6.3. If ρ satisfies the condition imposed in Corollary 6.2 then there exists an Ω_0 -neighborhood N of γ_1 such that $\rho = 0$ in N .

Proof. In view of Corollary 6.2 there exists a point $(x_0, 0)$ with $\mu < x_0 < a - \mu$ such that $u(x_0, 0) = M'' < M$. Consider the set $D = \{u < M'\}$ in $\bar{\Omega}_0$, with $M' \in (M'', M)$.

Then D contains $(x_0, 0)$ and by Theorem 4.4 it is connected to S . Choose any smooth curve σ connecting $(x_0, 0)$ to S and contained in D , and denote by G the subdomain of Ω_0 lying to the right of σ . We apply the maximum principle to w_ϵ in G and conclude that

$$e^{\frac{v_\epsilon(X)}{\epsilon}} \rho_\epsilon(X) = w_\epsilon(X) \leq \max_\sigma e^{\frac{v_\epsilon}{\epsilon}} \leq e^{\frac{v_\epsilon}{\epsilon}} \leq e^{\frac{M'}{\epsilon}}.$$

But in an appropriate Ω_0 -neighborhood N of γ_0 we have

$$v_\epsilon(X) \geq M - \delta > M' + \delta$$

provided $\delta < (M - M')/2$ and ϵ is small enough. Consequently

$$\rho_\epsilon(X) \leq e^{-\delta/\epsilon} \longrightarrow 0 \quad \text{if } \epsilon \longrightarrow 0 \quad (X \in N),$$

and the assertion follows.

§7 Remarks on the free boundary. Physical experience suggests that the zones

$$(7.1) \quad \Sigma_0 = \{\rho > 0\} \quad \text{and} \quad \Sigma_1 = \{\rho = 0\}$$

are connected regions and are separated by a curve

$$(7.2) \quad \Gamma : \{x = s(y), \quad 0 \leq y \leq b\};$$

we refer to Γ as the free boundary. It is also anticipated that $\rho \in C^\alpha(\Sigma_0)$, and $\rho \in C^0(\bar{\Omega}_0)$, although ρ is not C^α across Γ .

From (2.13), (2.14) we see that Γ should be such that the lines of force of v go from γ_0 and from γ_1 into the domain Ω_0 until they reach the free boundary Γ , and the transversal derivative of v along Γ vanishes, whereas $\partial v / \partial s < 0$ along Γ as we go toward S .

In §9 we obtain some numerical description of the free boundary. In this section we shall assume that a C^1 curve (7.2) exists such that $\rho \in C^\alpha(\Sigma_-)$ where Σ_- is the subdomain of Ω_0 lying to the right of Γ . We also assume that

$$(7.3) \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{along } \Gamma, \quad \frac{\partial v}{\partial s} \neq 0 \quad \text{along } \Gamma$$

and that

$$(7.4) \quad \rho_\epsilon \longrightarrow \rho \quad \text{in } C^1(\bar{\Sigma}_-)$$

THEOREM 7.2. *Under the foregoing assumptions, $\rho = 0$ in Σ_- .*

Proof. Suppose $X_0 \in \Sigma_-$, $\rho(X_0) > 0$. Then, by Lemma 6.1, $\nabla v(X_0) \neq 0$. Because of the assumption (7.4), we can deduce from (1.3) that

$$\nabla v \cdot \nabla \left(\frac{1}{\rho} \right) = -1$$

and thus there is a line of force of v through X_0 in a neighborhood of X_0 ; denote it by γ . The functions v and ρ must both increase along one direction of γ , say γ^+ . But then $\rho > \rho(X_0) > 0$ along γ^+ and we can apply the previous step in order to continue the line of force beyond γ^+ . As we keep continuing γ^+ into a curve Γ^+ , we cannot have self-intersections. Indeed, otherwise we get a closed line of force which bounds a domain D_1 and

$$0 = \int_{\partial D} \frac{\partial v}{\partial \nu} = \int_D \Delta v = - \int_D \rho < 0 ,$$

a contradiction. Next, if Γ^+ meets $\{x = a\}$ then since $v_x(0, y) = 0$, the continuation of Γ is along the vertical direction. But either going upward or downward we arrive at a point on $\partial\Omega_0$ where $\rho = 0$, which is a contradiction (since ρ increases along Γ^+).

Similarly, if Γ^+ meets $\{y = 0\}$, its continuation must be in the negative direction of the x -axis.

The above considerations show that Γ^+ must meet Γ and then it must continue along with Γ upward. This again leads to a contradiction.

We have thus proved that $\rho(X_0) = 0$ for any $X_0 \in \Sigma_-$.

§8. A numerical method. We extend the numerical method of Budd and Wheeler [5] for the space charge equations on a simply connected domain to the present multiply connected geometry. Motivated by the discussion of the previous section we consider solving the system (1.1) in the two separate regions Σ_0 and Σ_1 defined by (7.1) and we treat Γ as an internal boundary upon which we specify appropriate boundary conditions which we derive in this section.

The numerical method described in Budd and Wheeler [5] may be regarded as a synthesis of the method of characteristics for a hyperbolic equation and a hodograph method for Laplace's equation. It is also closely related to techniques of orthogonal mesh generation described for example in Thompson et al. [16]. The advantage of this technique is that we may use it to consider a wide variety of different conductor geometries and boundary conditions. It is also appropriate for calculating the free boundary Γ , since this curve will be coincident with one of the boundaries of the mesh used in the solution procedure.

Consider the region described in Figure 8.1. In Σ_0 the following identity holds

$$(8.11) \quad \nabla \cdot (\Delta v \nabla v) = 0,$$

and ρ is assumed to be positive. In Σ_1 we may deduce from the results of §7 that $\rho \equiv 0$ and hence

$$(8.2) \quad \nabla \cdot (\nabla v) = 0.$$

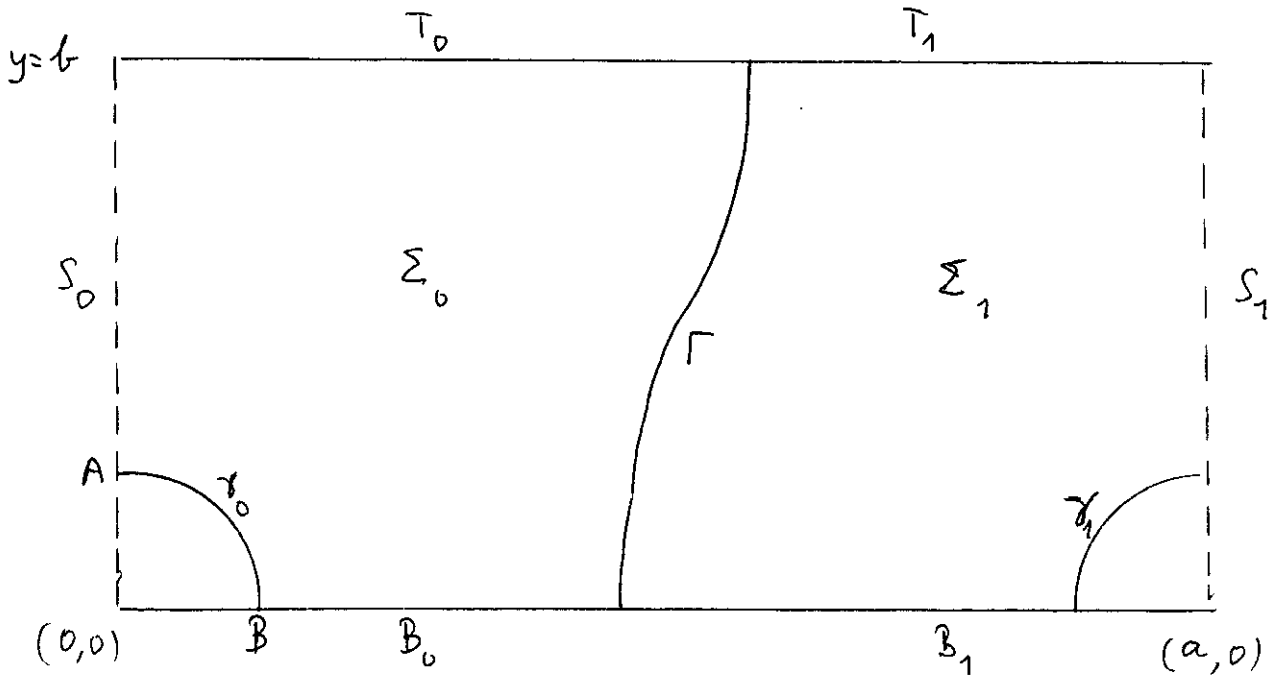


Figure 8.1

To proceed we observe that both of the differential identities (8.1), (8.2) may be integrated by introducing a vector potential $w_i(x, y)$ in each of the two regions Σ_i ($i = 0, 1$). By doing this we may express (8.1), (8.2) in the following form:

$$(8.3) \quad \begin{aligned} \rho v_x &= -a(w_0)w_{0,y} , \\ \rho v_y &= a(w_0)w_{0,x} \end{aligned}$$

for $(x, y) \in \Sigma_0$ and

$$(8.4) \quad \begin{aligned} v_x &= -b(w_1)w_{1,y} , \\ v_y &= b(w_1)w_{1,x} \end{aligned}$$

for $(x, y) \in \Sigma_1$, where the functions $a(w_0)$ and $b(w_1)$ are at the present stage undefined and arise from the degrees of freedom which have in defining $w_i(x, y)$ ($i = 0, 1$). The definition of these functions will become clear when we examine the conditions defining $w_i(x, y)$ on the boundaries γ_i ($i = 0, 1$). We make the important observation that the level curves of v and w_i intersect orthogonally. Indeed the level curves of $w_i(x, y)$ are everywhere parallel to the field lines of the system (1.1).

We assume that $\rho \in C^\alpha(\Sigma_0)$; then $\nabla v \neq 0$ in Σ_0 (by Lemma 6.1). We shall also assume that $\nabla v \neq 0$ in Σ_1 ; note however that $\nabla v = 0$ at $\Gamma \cap \{y = 0\}$, and we assume this point to

be a saddle-point in the set of level curves of the functions $v(x, y)$. The non-vanishing of ∇v in the two domains permits us to invert the two relations (8.3), (8.4) and to express the variables (x, y) in the two regions Σ_i ($i = 1, 2$) in terms of v and w_i ($i = 1, 2$). It follows after some manipulation that if $(x, y) \equiv (x(v, w), y(v, w))$ then

$$(8.5) \quad \begin{aligned} \rho x_w &= a(w_0) y_v, \\ -\rho y_w &= a(w_0) x_v \end{aligned}$$

for $(x, y) \in \Sigma_0$ and

$$(8.6) \quad \begin{aligned} x_w &= b(w_1) y_v, \\ -y_w &= b(w_1) x_v \end{aligned}$$

for $(x, y) \in \Sigma_1$.

Further from the governing equation (1.1) in the region Σ_0 the function $\rho(x, y)$ satisfies the following ordinary differential equation problem

$$(8.7) \quad \begin{aligned} \frac{d(\rho^{-1})}{dv} &= - \left\{ \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 \right\} \quad \text{on } w = \text{constant}, \\ \rho^{-1}(M, w) &= 1 \quad ; \end{aligned}$$

this statement is equivalent to (2.14).

We now see that equation (1.1) may be expressed in our transformed coordinates as a pair of elliptic equations (8.5), (8.6) coupled to the simple ordinary differential equation problem (8.7). It is in this form that we shall numerically solve the space-charge equations.

To complete our derivation we must consider the boundary conditions satisfied by (x, y) . From Figure 8.1 we see that Σ_0 is bounded by the known curves B_0, γ_0, S_0 and T_0 , and by the unknown free boundary Γ . The region Σ_1 is similarly bounded. We first consider Σ_0 .

On the top surface T_0 the following two conditions are satisfied

$$(8.7a) \quad v = 0 \quad \text{and} \quad y(0, w) = b,$$

and hence from (8.5)

$$(8.7b) \quad x_v(0, w) = 0.$$

We now observe from the symmetry of the geometry of the domain that the line segments S_0 and B_0 are both parallel to field lines of the problem (1.1) and hence w_0 is constant along each. Without loss of generality we may take $w_0 = 1$ on S_0 and $w_0 = 0$ on B_0 . It therefore follows that on S_0

$$(8.8) \quad x(v, 1) = 0 \quad y_{w_0}(v, 1) = 0,$$

and on B_0

$$(8.9) \quad y(v, 0) = 0 \quad x_{w_0}(v, 0) = 0.$$

We now consider the curve γ_0 , on which $v = M$, and define it parameterically as follows:

$$(8.10) \quad \begin{aligned} x(M, w_0) &\equiv g_1(w_0), \\ y(M, w_0) &\equiv h_1(w_0) \end{aligned}$$

with $0 \leq w_0 \leq 1$, where $w_0 = 1$ at A and $w_0 = 0$ at B .

This parametric definition allows us freedom to calculate solutions for different curves γ_0 by simply changing the forms of g_1 and h_1 in the computer program.

The formulae given in (8.10) define w_0 upon the curve γ_0 in the sense that a field line which intersects the curve γ_0 at a point (x^*, y^*) such that $(x^*, y^*) = (g_1(w^*), h_1(w^*))$ is precisely the curve $(x(v, w^*), y(v, w^*))$ where $0 \leq v \leq M$. This in turn defines the form of the function $a(w_0)$ which follows the relation below given by Budd and Wheeler [5]

$$(8.11) \quad a(w_0) = |\nabla v| ((\partial g_1 / \partial w)^2 + (\partial h_1 / \partial w)^2)^{1/2},$$

where all of these quantities are evaluated upon the curve γ_0 . Here we have assumed that the value of $\rho(\psi_0)$ is explicitly given on γ_0 .

It now remains to consider the boundary Γ . From the discussion given in §7 it is apparent that Γ is a field line for the system (1.1). Further from figure 8.1 it is evident that the curve Γ_* (defined to be composed of the two curves B_0 and Γ) is the limit of the curve $\{(x, y) : w_0(x, y) = c\}$ as $c \rightarrow 0^+$. From the continuity of v across Γ it follows that

$$(8.12) \quad \begin{aligned} \lim_{w_0 \rightarrow 0} x(v, w_0) &= \lim_{w_1 \rightarrow 0} x(v, w_1), \\ \lim_{w_0 \rightarrow 0} y(v, w_0) &= \lim_{w_1 \rightarrow 0} y(v, w_1) \end{aligned}$$

for v such that (x, y) is in Γ .

Further from the continuity of ∇v and equation (8.5) we deduce that

$$(8.13) \quad \begin{aligned} \lim_{w_0 \rightarrow 0} \rho(v, w_0) x_{w_0}(v, w_0) / a(w_0) &= \lim_{w_1 \rightarrow 0} x_{w_1}(v, w_1) / b(w_1) = y_v(v, 0), \\ \lim_{w_0 \rightarrow 0} \rho(v, w_0) y_{w_0}(v, w_0) / a(w_0) &= \lim_{w_1 \rightarrow 0} y_{w_1}(v, w_1) / b(w_1) = x_v(v, 0) \end{aligned}$$

for v such that (x, y) belongs to Γ .

Equations (8.12) and (8.13) serve as a jump condition across Γ .

This completes the boundary conditions for the region Σ_0 . A similar set of boundary conditions apply for the region Σ_1 namely

$$(8.14) \quad \begin{aligned} y(0, w_1) &= b, & x_v(0, w_1) &= 0, \\ y(v, 0) &= 0, & x_{w_1}(v, 0) &= 0, \\ x(v, 1) &= a, & y_{w_1}(v, 1) &= 0, \end{aligned}$$

and

$$(8.15) \quad \begin{aligned} x(M, w_1) &= g_2(w_1), \\ y(M, w_1) &= h_2(w_1), \end{aligned}$$

with $0 \leq w_1 \leq 1$, $0 \leq v \leq M$.

To solve the problem (1.1) numerically we now solve the elliptic problems (8.5), (8.6) together with the ordinary differential equation (8.7). This calculation is made on the two regions defined by $\{(v, w_0) : 0 \leq v \leq M, 0 \leq w_0 \leq 1\}$ and $\{(v, w_1) : 0 \leq v \leq M, 0 \leq w_1 \leq 1\}$ with the boundary conditions (8.8)–(8.15). Here we solve for the unknowns $x(v, w_i), y(v, w_i), a(w_0), b(w_1)$ and $\rho(v, w_0)$.

The numerical scheme uses a finite difference formulation to solve the elliptic system (8.5), (8.6) with a simple trapezium method to integrate the ordinary differential equation (8.7). This is allied with an iterative technique to calculate the functions $a(w_0)$ and $b(w_0)$. We take care in our calculations as $w_i \rightarrow 0$ especially when we are close to the saddle-point. We omit a detailed description of the numerical techniques used, and refer the reader to Budd and Wheeler [15] for further details.

Having calculated x, y and ρ as functions of v and w_i ($i = 1, 2$) it is a simple matter to then plot the level lines of v as there are precisely the curves

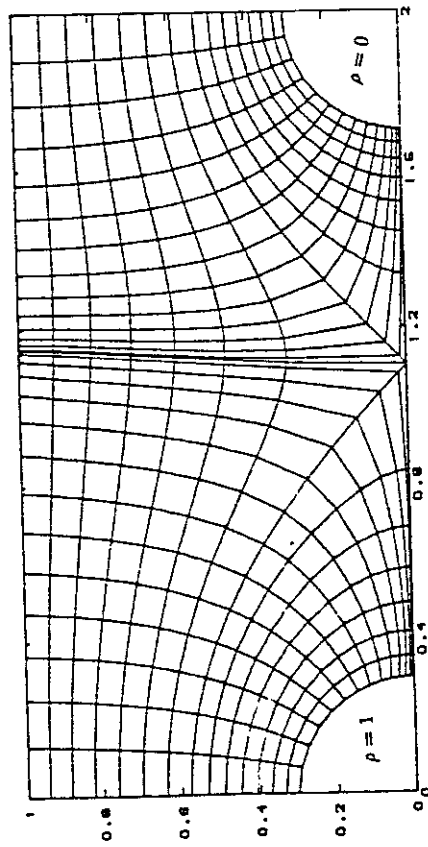
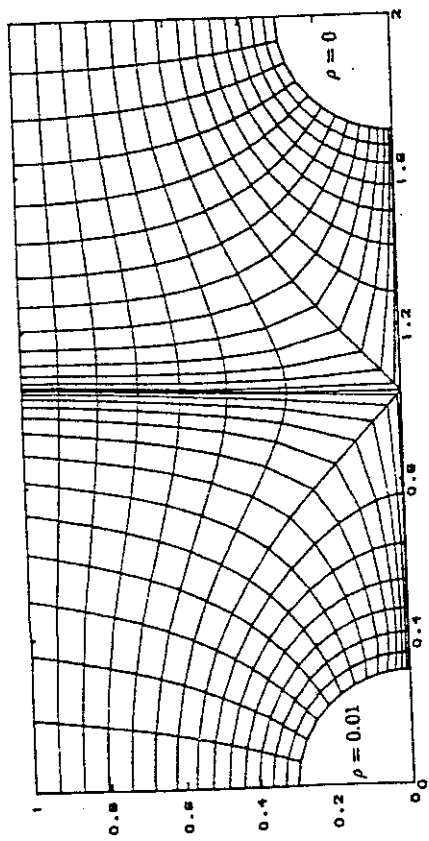
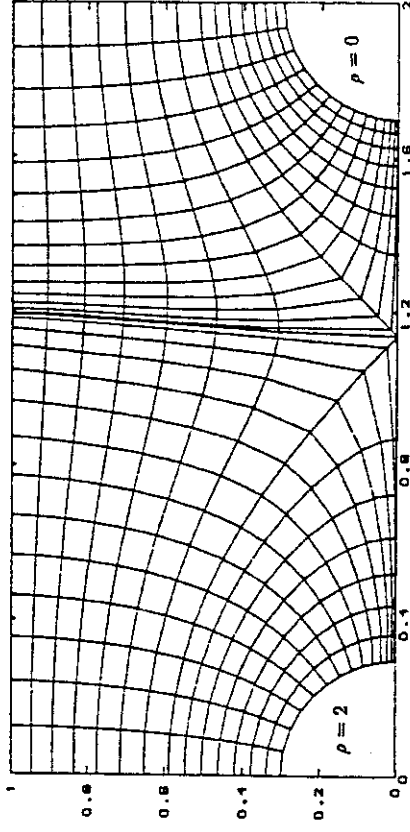
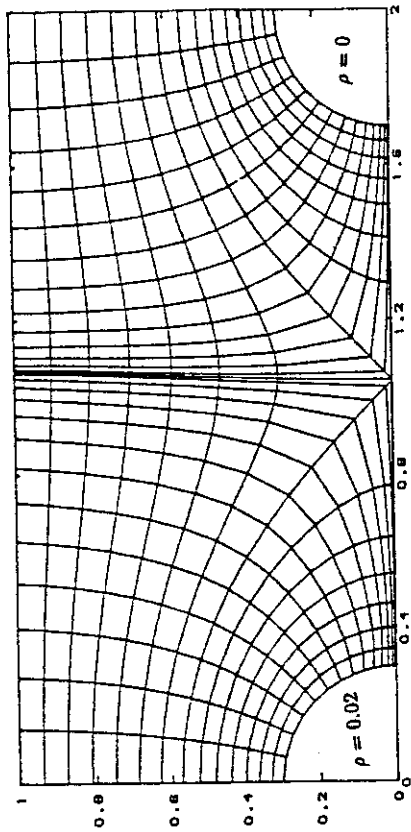
$$\{(x(v, w_i), y(v, w_i)) : v = \text{constant}\}.$$

An equally simple calculation then allows us to determine the field lines of the solution.

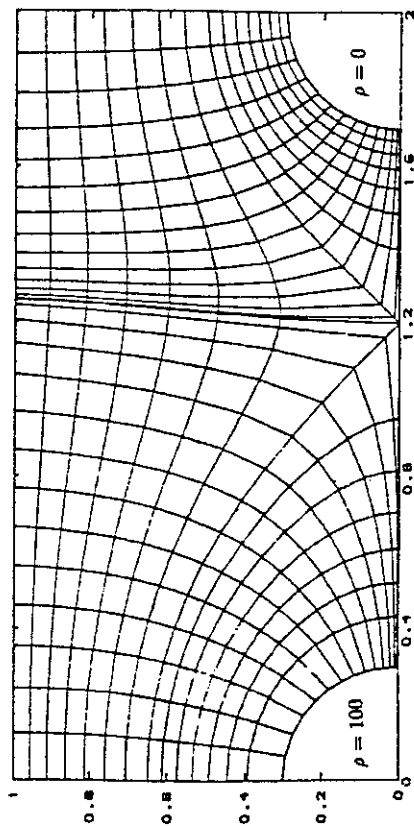
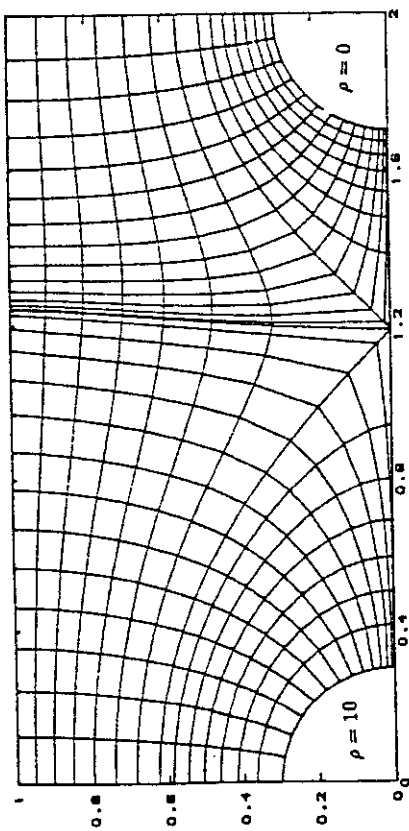
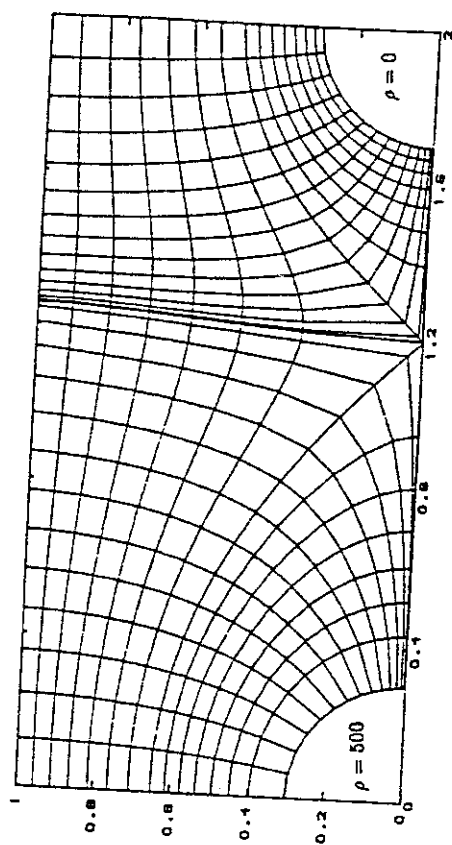
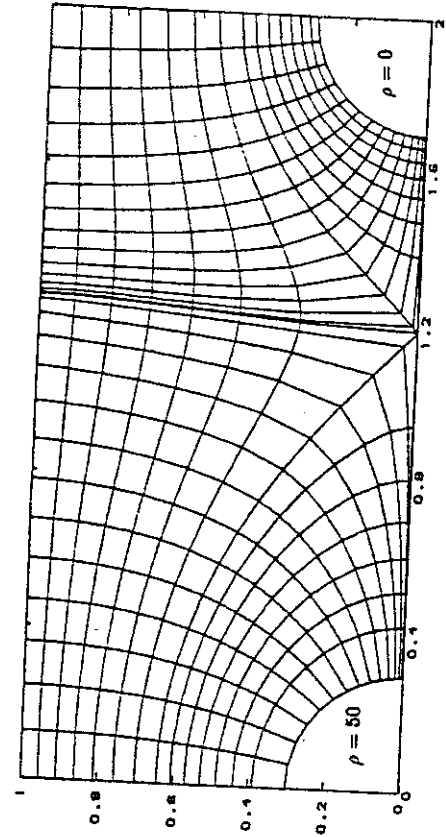
§9. Numerical results.

We give below the description of the field lines, their orthogonal trajectories and the level lines of ρ in case $a = 1, b = 1, \mu = 0.3$ and $\rho_0 = 1/M$; for $\rho_0 = 0$ the free boundary is $\{x = 1\}$. Our code applies to any a, b, μ and any ρ_0 . The graphs suggest for any $\rho_0 > 0$, Γ lies to the right of $\{x = 1\}$. Since the existence of Γ has not been rigorously proved, we may reformulate this conjecture in the following rigorous way:

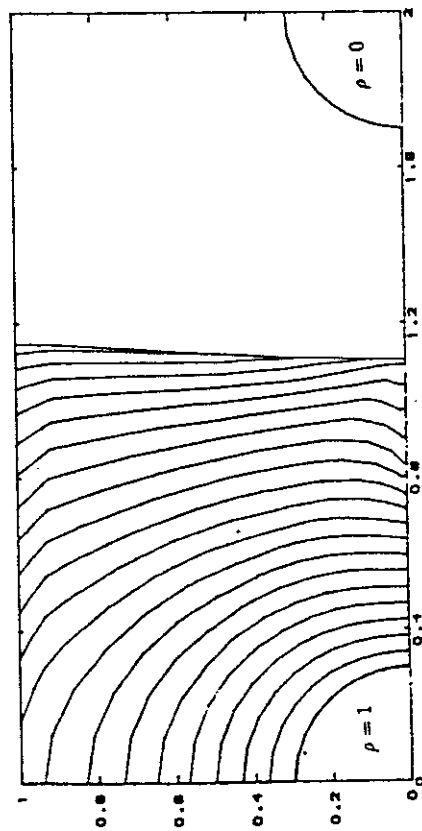
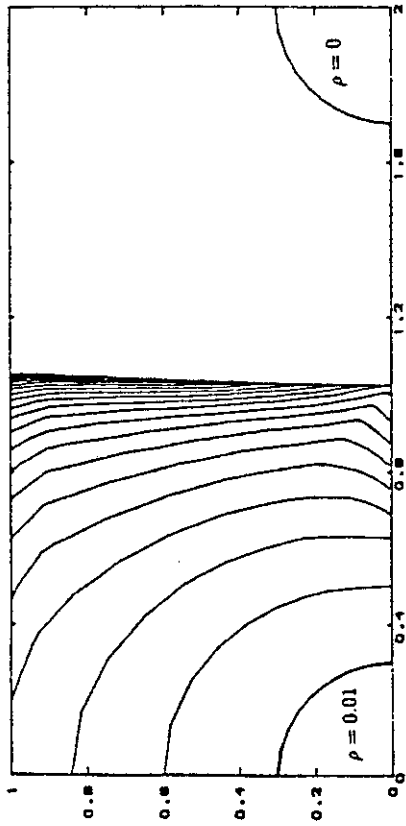
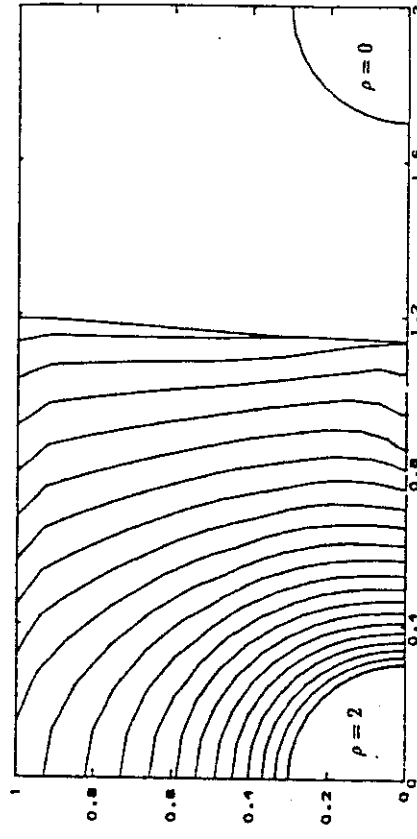
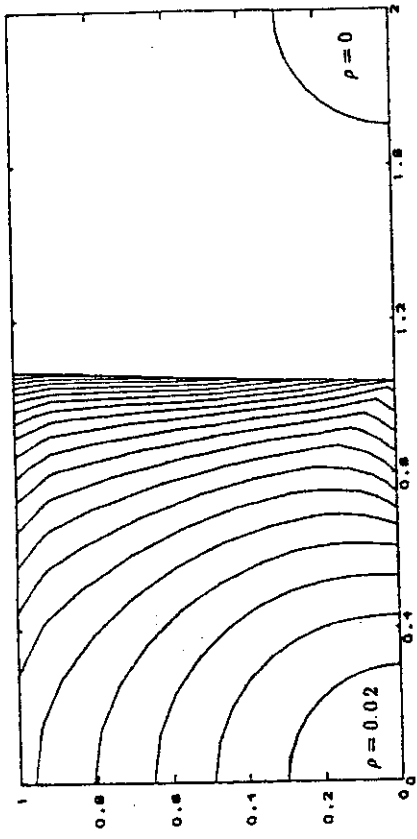
$$\rho(x, y) > 0 \quad \text{if} \quad 0 < x < \frac{a}{2}$$



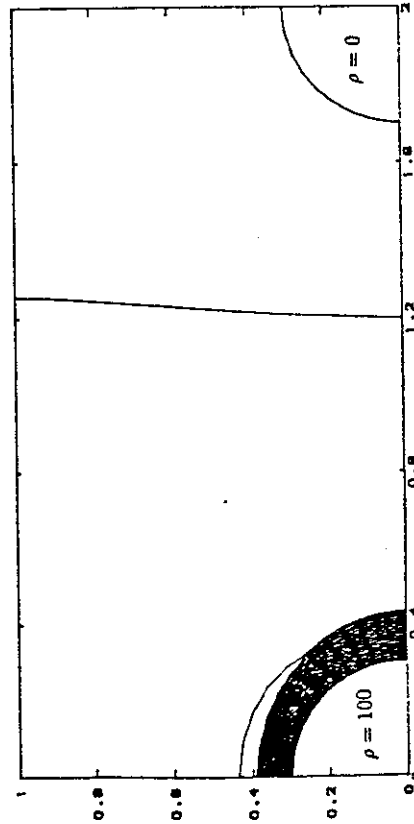
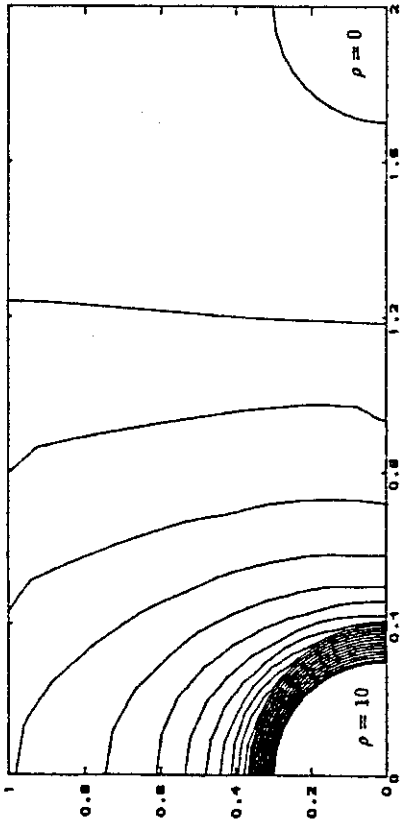
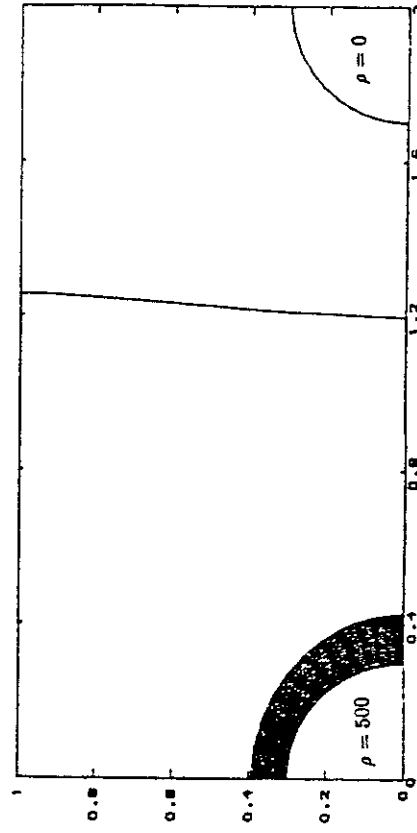
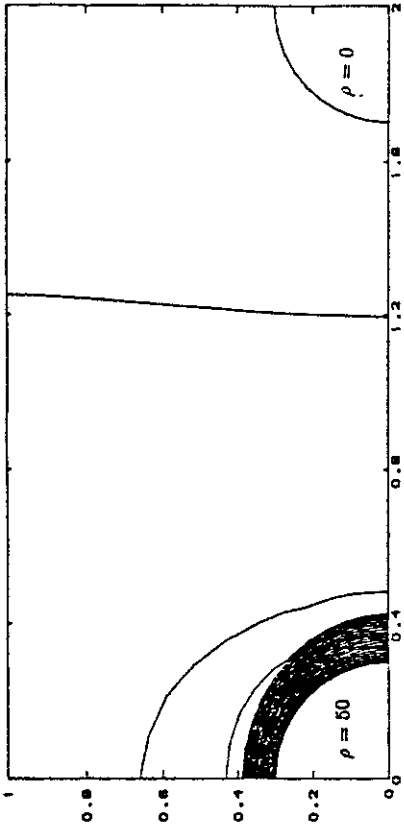
Field Lines



Field Lines



ρ Level Lines



ρ Level Lines

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