A QUASI–VARIATIONAL INEQUALITY ARISING IN ELASTOHYDRODYNAMICS

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A Quasi-Variational Inequality Arising in Elastohydrodynamics

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Abstract. In this paper we study a quasi-variational inequality arising in elastohydrodynamic lubrication. In the two dimensional case modeling a thin fluid film between an elastic ball and a plane, we prove the existence of a solution provided that the viscosity is assumed to be constant. We also establish in this case estimates for the support of the solution and prove uniqueness of the solution under some restrictions. In the case where the viscosity is not constant, we shall prove the existence and uniqueness under additional restrictions. Finally, for the one dimensional problem discribing a thin fluid between a rolling cylinder and a plane, we shall establish in addition to existence and uniqueness the fact that the free boundary consists of at most one point.

Key words: Variational inequality, Free boundary problem, A priori estimates, Fixed point.

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§1. The model

The lubrication of a ball rolling in the positive x direction gives rise to a variational inequality:

$$-\frac{\partial}{\partial x}(\frac{\rho h^3}{12\mu}\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\frac{\rho h^3}{12\mu}\frac{\partial u}{\partial y}) \geq -\frac{\partial}{\partial x}(\frac{\rho v h}{2}) \tag{1.1}$$

$$u \geq 0 \tag{1.2}$$

$$u\left[-\frac{\partial}{\partial x}(\frac{\rho h^3}{12\mu}\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\frac{\rho h^3}{12\mu}\frac{\partial u}{\partial y}) + \frac{\partial}{\partial x}(\frac{\rho v h}{2})\right] = 0 \tag{1.3}$$

where u is the pressure, v is the average surface speed ($v > 0$), ρ is the density of the liquid which shall be assume to be constant, $\mu = \mu(u)$ is the viscosity coefficient of the liquid, and h is the film thickness which takes the form:

$$h(x,y) = k + \frac{x^2+y^2}{2R} + \frac{2}{\pi E'}\int \frac{u(s,t)dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \tag{1.4}$$

where E' is the effective modulus, and k is a positive constant.

The variational inequality (1.1)–(1.3) (with h a given function) occurs in a simplified model of a lubrication problem (see [2]); the dependence of h on the pressure, as in (1.4), assumes that the ball is elastic; this is the case when the load is large. The system (1.1)–(1.4) forms an elastohydrodynamics lubrication model; for more details see [3], [6]. In this paper, we study the quasi-variational inequality (1.1)–(1.4) in a bounded, but large domain Ω. Since u is small on $\partial\Omega$, it seems natural to impose the boundary condition

$$u = 0 \quad \text{on } \partial\Omega \ . \tag{1.5}$$

If μ is constant, then setting

$$\delta = \frac{2}{\pi E'}, \quad \lambda = 6\mu v \tag{1.6}$$

we can rewrite (1.1)–(1.5) in the form

$$\begin{aligned} -\nabla(h^3\nabla u) &\geq -\lambda\frac{\partial h}{\partial x} \quad \text{for } (x,y)\in\Omega && (1.7)\\ u &\geq 0 \quad \text{for } (x,y)\in\Omega && (1.8)\\ u[-\nabla(h^3\nabla u) + \lambda\frac{\partial h}{\partial x}] &= 0 \quad \text{for } (x,y)\in\Omega && (1.9)\\ u &= 0 \quad \text{for } (x,y)\in\partial\Omega && (1.10)\end{aligned}$$

and

$$h(x,y) = k + \frac{x^2+y^2}{2R} + \delta\int_\Omega \frac{u(s,t)dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \ . \tag{1.11}$$

In §3 we prove the existence of a solution; the proof uses some estimates derived in §2. In §4 we take Ω to be a disc with large radius M, and obtain some estimates for the support of the solution, i.e., we prove that for some small $\epsilon > 0$,

$$u(x,y) > 0 \qquad \text{for } \; -M < x < -\epsilon M \; . \tag{1.12}$$

In §5 we prove the uniqueness of the solution provided δ is small. In §6 we study the problem (1.1)–(1.5) in the case where

$$\mu = \mu_0 e^{\alpha u}, \qquad \mu_0 > 0, \;\; \alpha > 0 \tag{1.13}$$

this case is of particular physical interest (see [1], [6]).

If, instead of a rolling ball we have a rolling cylinder, then the lubrication problem reduces to a one dimensional quasi-variationnal inequality:

$$\begin{aligned} -(h^3 \frac{u'}{\mu})' &\geq -6vh' \quad \text{ for } x \in [-M, M] && (1.14)\\ u &\geq 0 \quad \text{ for } x \in [-M, M] && (1.15)\\ u[-(h^3 \frac{u'}{\mu})' + 6vh'] &= 0 \quad \text{ for } x \in [-M, M] && (1.16)\\ u(\pm M) &= 0 && (1.17) \end{aligned}$$

and

$$h = k + \frac{x^2}{2R} + \frac{4}{\pi E'} \int_{-M}^{M} u(s) \log \frac{2M}{|x-s|} ds \; . \tag{1.18}$$

Numerical work for this case can be found in [1]. In §7 we study this problem with μ given as in (1.13), i.e.,

$$\begin{aligned} -(h^3 \frac{u'}{e^{\alpha u}})' &\geq -\lambda h' \quad \text{ for } x \in [-M, M] && (1.19)\\ u &\geq 0 \quad \text{ for } x \in [-M, M] && (1.20)\\ u[-(h^3 \frac{u'}{e^{\alpha u}})' + \lambda h'] &= 0 \quad \text{ for } x \in [-M, M] && (1.21)\\ u(\pm M) &= 0 && (1.22) \end{aligned}$$

and

$$h = k + \frac{x^2}{2R} + \delta \int_{-M}^{M} u(s) \log \frac{2M}{|x-s|} ds \; . \tag{1.23}$$

Assuming α to be small, we prove the existence of the solution. It is also proved that the solution is unique and that the free boundary consists of at most one point provided δ is small.

§2. A priori estimates

Later on we shall need some estimates for the solution of the the quasi variational inequality

$$-\nabla(h^3\nabla u) \geq -\lambda N\frac{\partial h}{\partial x} \quad \text{for } (x,y)\in\Omega \tag{2.1}$$

$$u \geq 0 \quad \text{for } (x,y)\in\Omega \tag{2.2}$$

$$u[-\nabla(h^3\nabla u)+\lambda N\frac{\partial h}{\partial x}] = 0 \quad \text{for } (x,y)\in\Omega \tag{2.3}$$

$$u = 0 \quad \text{for } (x,y)\in\partial\Omega \tag{2.4}$$

and

$$h = k+\frac{N^2(x^2+y^2)}{2R}+N\delta\int_\Omega\frac{u(s,t)dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \tag{2.5}$$

where λ, k, δ, N are positive constants, and Ω is a smooth domain in R^2.

LEMMA 2.1. Assume that (u,h) is a solution of (2.1)-(2.5) with $u\in W_0^{1,2}(\Omega)$, $h\in W^{1,2}(\Omega)$. Then

$$\int_\Omega h^3|\nabla u|^2dxdy \leq \frac{\lambda^2|\Omega|}{k}N^2 \ . \tag{2.6}$$

Proof: Integrating (2.3) over Ω, we get

$$\begin{aligned}\int_\Omega h^3|\nabla u|^2dxdy &= -\lambda N\int_\Omega u\frac{\partial h}{\partial x} \\ &= \lambda N\int_\Omega h\frac{\partial u}{\partial x} \\ &\leq \lambda N\left(\int_\Omega h^2(\frac{\partial u}{\partial x})^2dxdy\right)^{\frac{1}{2}}\left(\int_\Omega dxdy\right)^{\frac{1}{2}} \\ &\leq \frac{\lambda N|\Omega|^{\frac{1}{2}}}{k^{\frac{1}{2}}}\left(\int_\Omega h^3|\nabla u|^2dxdy\right)^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

and hence (2.6) follows. □

Extend u by 0 outside Ω. Then $u \in W_0^{1,2}(R^2)$ and, by changing of variables,

$$h(x,y) = k + \frac{N^2(x^2+y^2)}{2R} + N\delta \int_{R^2} \frac{1}{\sqrt{s^2+t^2}} u(x-s, y-s) ds dt \ . \quad (2.8)$$

Thus

$$\nabla h(x,y) = \frac{N^2}{R}(x,y) + N\delta \int_{R^2} \frac{1}{\sqrt{s^2+t^2}} \nabla u(x-s, y-t) ds dt \ . \quad (2.9)$$

LEMMA 2.2. Under the assumption of Lemma 2.1, we have

$$\|\nabla h\|_{L^p(\Omega)} \le CN^2 \quad (2.10)$$

where $2 < p < \infty$, and C is a constant depending on Ω , p, R and λ.

Proof: Applying the Young's inequality (see, for example, Lemma 7.12 in [5]) to (2.9), we get

$$\|\nabla h\|_{L^p(\Omega)} \le \frac{C(\Omega)N^2}{R} + C(\Omega, p)N\|\nabla u\|_{L^2(\Omega)} \ . \quad (2.11)$$

Since, by (2.6),

$$k^3 \int_\Omega |\nabla u|^2 dx dy \le \frac{\lambda^2|\Omega|}{k} N^2 \quad (2.12)$$

and therefore,

$$\|\nabla u\|_{L^2(\Omega)} \le \left(\frac{\lambda^2|\Omega|}{k^4} N^2\right)^{\frac{1}{2}} = \frac{\lambda|\Omega|}{k^2} N \ . \quad (2.13)$$

Substituting this into (2.11), (2.10) follows. □

Setting

$$f = \frac{3\nabla h}{h} \nabla u - \frac{\lambda N}{h^3} \frac{\partial h}{\partial x} \ , \quad (2.14)$$

we can rewrite (2.1)–(2.4) as

$$-\Delta u \ge f \quad \text{for } (x,y) \in \Omega \quad (2.15)$$

$$u \ge 0 \quad \text{for } (x,y) \in \Omega \quad (2.16)$$

$$u(-\Delta u - f) = 0 \quad \text{for } (x,y) \in \Omega \quad (2.17)$$

$$u = 0 \quad \text{for } (x,y) \in \partial\Omega \ . \quad (2.18)$$

LEMMA 2.3. Under the assumption of Lemma 2.1, we have, for any $2 < p < \infty$,

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\Omega, N, p) \tag{2.19}$$

$$\|u\|_{W^{2,\infty}(\Omega')} \leq C(\Omega, \Omega', N) \tag{2.20}$$

$$\|h\|_{W^{1,\infty}(\Omega)} \leq C(\Omega, N) \tag{2.21}$$

$$\|h\|_{W^{2,p}(\Omega')} \leq C(\Omega, \Omega', N) \tag{2.22}$$

where $\Omega' \subset\subset \Omega$, and the constants in (2.19)- (2.22) depend on k, R and λ.

Proof: We shall use C to denote various constants depending on Ω and N and use C_p to denote various constants depending on Ω, N and p.

First let $1 < p < 2$. Then by (2.14)

$$|f|^p \leq C|\nabla h|^p(|\nabla u|^p + 1) \ . \tag{2.23}$$

Applying Hölder's inequality we get

$$\int_\Omega |f|^p dxdy \leq C\left(\int_\Omega |\nabla h|^{\frac{2p}{2-p}} dxdy\right)^{\frac{2-p}{2}} \left(\int_\Omega (|\nabla u|^2 + 1) dxdy\right)^{\frac{p}{2}} ; \tag{2.24}$$

since $1 < p < 2$, $2 < \frac{2p}{2-p} < \infty$, and by Lemmas 2.1 and 2.2 it then follows that

$$\|f\|_{L^p(\Omega)} \leq C_p \qquad (1 < p < 2) \ . \tag{2.25}$$

Thus, by L^p estimates for the variational inequality (2.15)-(2.18) give

$$\|u\|_{W^{2,p}(\Omega)} \leq C_p \qquad (1 < p < 2) \ . \tag{2.26}$$

Using the Sobolev embedding theorem we conclude that

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p \tag{2.27}$$

for any $1 < p < \infty$.

Now from (2.9) and Hölder's inequality it follows that

$$\|\nabla h\|_{L^\infty(\Omega)} \leq C + C\|\nabla u\|_{L^3(\Omega)} \left(\int_{B_K(0)} (s^2 + t^2)^{-3/4} dsdt\right)^{\frac{2}{3}} \tag{2.28}$$

if K is large enough so that $\Omega \subset B_{K/2}(0)$. Using (2.27) it follows that

$$\|\nabla h\|_{L^\infty(\Omega)} \leq C \tag{2.29}$$

Next, using (2.14), (2.27) and (2.29) we find that

$$\|f\|_{L^p(\Omega)} \leq C_p \tag{2.30}$$

for any $1 < p < \infty$, and thus by a similar argument as above we obtain the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C_p \tag{2.31}$$

for any $1 < p < \infty$.

In order to get higher regularity, we differentiate (2.9) using a similar argument as in [5, p53-55], and obtain:

$$h_{xx} = \frac{N^2}{R} + N\delta \int_\Omega \frac{u_{xx}(s,t)dsdt}{\sqrt{(x-s)^2+(y-t)^2}} - N\delta \int_{\partial\Omega} \frac{u_x(s,t)\cos(\vec{n},x)}{\sqrt{(x-s)^2+(y-t)^2}} d\sigma \ . \tag{2.32}$$

Similar expressions can be derived for h_{xy} and h_{yy}.

By the Sobolev embedding theorem and (2.31), it follows that

$$\|u\|_{W^{1,\infty}(\Omega)} \leq C \tag{2.33}$$

and thus if $(x,y) \in \Omega'$, where $\Omega' \subset\subset \Omega$, then

$$\left| \int_{\partial\Omega} \frac{u_x(s,t)\cos(\vec{n},x)}{\sqrt{(x-s)^2+(y-t)^2}} d\sigma \right| \leq C \ . \tag{2.34}$$

Next, applying Young's inequality (Lemma 7.12 of [5]), and using (2.31), (2.34), we conclude that

$$\|h\|_{W^{2,p}(\Omega')} \leq C_p + C_p\|u\|_{W^{2,2}(\Omega)} \leq C_p \qquad \text{for any } 2 < p < \infty \ , \tag{2.35}$$

the constants in (2.35) depend on $dist(\Omega', \partial\Omega)$.

By the Sobolev embedding theorem, (2.31) and (2.35), for any $0 < \alpha < 1$ and $\Omega'' \subset\subset \Omega'$,

$$\|h\|_{C^{1,\alpha}(\Omega'')} \leq C' \tag{2.36}$$

and

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq C'' \ ; \tag{2.37}$$

where C' depends on α and Ω'' and C'' depends on α. Therefore, using (2.14) it follows that

$$\|f\|_{C^\beta(\Omega'')} \leq C''' \tag{2.38}$$

where C''' depends on β and Ω''. Hence, by elliptic estimates for variational inequalities,

$$\|u\|_{W^{2,\infty}(\Omega''')} \leq C \tag{2.39}$$

where $\Omega''' \subset\subset \Omega''$, and $0 < \alpha < 1$. (see [4]) □

§3. The existence of a solution

THEOREM 3.1. Suppose that Ω is a smooth bounded domain, then there exists a solution (u, h) of (1.7)–(1.11) such that

$$u \in W^{2,p}(\Omega) \bigcap W^{2,\infty}_{loc}(\Omega), \ h \in C^1(\bar{\Omega}) \bigcap C^{1,\alpha}_{loc}(\Omega) \tag{3.1}$$

for any $2 < p < \infty$ and $0 < \alpha < 1$.

Proof: Take $2 < p < \infty$ (fixed) and let

$$B = W^{1,p}(\Omega) \bigcap W^{1,2}_0(\Omega) \ . \tag{3.2}$$

For each $u \in B$, define

$$Hu = k + \frac{x^2 + y^2}{2R} + \delta \int_\Omega \frac{u^+(s,t)dsdt}{\sqrt{(x-s)^2 + (y-t)^2}} \ . \tag{3.3}$$

Then, by Hölder's inequality,

$$\|Hu\|_{C^1(\bar{\Omega})} \leq C + C\|u\|_{W^{1,p}(\Omega)} \tag{3.4}$$

$$\|Hu_1 - Hu_2\|_{C^1(\bar{\Omega})} \leq C\|u_1 - u_2\|_{W^{1,p}(\Omega)} \ . \tag{3.5}$$

Now define Tu to be the solution of (1.7)–(1.10) with $h = Hu$ (it is unique for fixed h).

From (3.4), L^p estimates for the variational inequalities and the compactness of the inclusion $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$, it follows that $T: \ B \to B$ is compact.

The compactness of T, the uniqueness of the solution of the variational inequality and (3.5) altogether imply that $T: \ B \to B$ is continuous.

Next, for any $0 < \sigma < 1$, consider any fixed point u of the operator σT:

$$u = \sigma T u \ . \tag{3.6}$$

Notice that since $u/\sigma = Tu$, u is a solution to the following problem:

$$\begin{aligned} -\nabla(h^3\nabla u) &\geq -\lambda\sigma\frac{\partial h}{\partial x} \quad \text{for } (x,y)\in\Omega \\ u &\geq 0 \quad \text{for } (x,y)\in\Omega \\ u[-\nabla(h^3\nabla u)+\lambda\sigma\frac{\partial h}{\partial x}] &= 0 \quad \text{for } (x,y)\in\Omega \\ u &= 0 \quad \text{for } (x,y)\in\partial\Omega \end{aligned}$$

and

$$h(x,y) = k + \frac{x^2+y^2}{2R} + \delta\int_\Omega \frac{u(s,t)dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \ . \tag{3.7}$$

Hence by Lemma 2.3 (with $N=1$),

$$\|u\|_{W^{1,p}(\Omega)} \leq C \tag{3.8}$$

where C is a constant independent of σ.

From this fact and the previous properties of T it follows that the Laray-Schauder fixed point theorem ([5], theorem 10.3) can be applied. Thus, there is a fixed point u for T, that is, there exists a solution (u,h) to the problem (1.7)-(1.11).

Finally by Lemma 2.3,

$$u\in W^{2,\infty}_{loc}(\Omega), \quad h\in C^{1,\alpha}_{loc}(\Omega) \ , \tag{3.9}$$

and the theorem is proved. □

Remark: If the domain Ω is symmetric with respcet to x-axis, then we may take in (3.2)

$$B = W^{1,p}(\Omega)\bigcap W^{1,2}_0(\Omega)\bigcap\{u|u(x,y)=u(x,-y)\} \tag{3.10}$$

and the preceeding argument shows that there exists a solution symmetric with respect to y.

§4. Estimate on the support

Now let us turn our attention to the case $\Omega = B_M(0)$ in (1.7)-(1.11), where M is large. we shall prove:

THEOREM 4.1. For any $\epsilon > 0$, there exists $K > 0$ such that if $M > K$, then

$$u(x,y) > 0 \qquad \text{for} \quad -M < x < -\epsilon M, \;\; (x,y) \in B_M(0) \ . \tag{4.1}$$

To prove this theorem, we start with a scaling:

$$u_M(x,y) = u(Mx, My) \qquad \text{for} \;\; x^2 + y^2 \le 1 \tag{4.2}$$

$$h_M(x,y) = h(Mx, My) \qquad \text{for} \;\; x^2 + y^2 \le 1 \ . \tag{4.3}$$

A simple calculation shows

$$-\nabla(h_M^3 \nabla u_M) \ge -\lambda M \frac{\partial h_M}{\partial x} \qquad \text{for } (x,y) \in B_1 \tag{4.4}$$

and

$$h_M(x,y) = k + \frac{M^2(x^2+y^2)}{2R} + M\delta \int_{B_1} \frac{u(s,t)dsdt}{\sqrt{(x-s)^2 + (y-t)^2}} \tag{4.5}$$

This shows that (u_M, h_M) satisfies (2.1)–(2.5) with $\Omega = B_1(0)$.

It clearly suffices to show that

$$u_M(x,y) > 0 \qquad \text{for} \quad -1 < x < -\epsilon \tag{4.6}$$

for M large enough; for simplicity, we drop the subscripts M for u_M and h_M.

Since $h \ge \frac{\epsilon^2 M^2}{2R}$ if $x^2 + y^2 \ge \epsilon^2$, we get from (2.6),

$$\int_{B_1 \setminus B_\epsilon} |\nabla u|^2 dxdy \le \frac{\lambda^2 \pi}{k\epsilon^6} M^{-4} \ . \tag{4.7}$$

We shall use C to denote various constants independent of M (although may depend on ϵ, k, λ, R and δ).

We are going to carry out a proof similar to that in Lemma 2.3, but this time we shall use the fact that $h \ge \frac{\epsilon^2 M^2}{2R}$ if $x^2 + y^2 \ge \epsilon^2$ to find a better estimate on h when $x^2 + y^2 \ge (4\epsilon)^2$.

By Hölder's inequality, for $1 < p < 2$

$$\left\| \frac{\nabla h}{h} \nabla u \right\|_{L^p(B_1 \setminus B_\epsilon)} \le \left(\frac{2R}{\epsilon^2 M^2}\right) \|\nabla h\|_{L^{\frac{2p}{2-p}}(B_1 \setminus B_\epsilon)} \|\nabla u\|_{L^2(B_1 \setminus B_\epsilon)} \ . \tag{4.8}$$

By using (4.7) to estimate ∇u and Lemma 2.2 to estimate ∇h, we get

$$\left\|\frac{\nabla h}{h}\nabla u\right\|_{L^p(B_1\setminus B_\epsilon)} \le \frac{2R}{\epsilon^2 M^2}(C_p M^2)(CM^{-2}) \le \frac{C_p}{M^2} \ . \tag{4.9}$$

By Lemma 2.2, also

$$\begin{aligned}\left\|\frac{\lambda M}{h^3}\frac{\partial h}{\partial x}\right\|_{L^p(B_1\setminus B_\epsilon)} &\le \frac{8R^3\lambda M}{(\epsilon^2 M^2)^3}\|\nabla h\|_{L^p(B_1)} \\ &\le (CM^{-5})(C_p M^2) \le C_p M^{-3} \ . \end{aligned}\tag{4.10}$$

From (2.14),(4.9),(4.10) it now follows that

$$\|f\|_{L^p(B_1\setminus B_\epsilon)} \le C_p M^{-2} \qquad (1 < p < 2) \ . \tag{4.11}$$

Thus we proved

LEMMA 4.2. For f defined in (2.14) (with $N = M$),

$$\|f\|_{L^p(B_1\setminus B_\epsilon)} \le C_p M^{-2} \qquad (1 < p < 2) \ . \ \square \tag{4.12}$$

Notice that u satisfies the variational inequality

$$-\Delta u \ge f, \ u \ge 0, \ u(-\Delta u - f) = 0 \quad \text{in} \quad B_1 \tag{4.13}$$

$$u = 0 \quad \text{on} \quad \partial B_1 \tag{4.14}$$

where f is given by (2.14) (with $N = M$); by Lemmas 4.2 and Poincare's inequality

$$\|v\|_{L^2(B_1\setminus B_\epsilon)} \le C\|\nabla v\|_{L^2(B_1\setminus B_\epsilon)} \qquad \text{(actually } C = 1 \text{ here)}, \tag{4.15}$$

we get:

$$\|u\|_{L^2(B_1\setminus B_\epsilon)} \le CM^{-2} \qquad (1 < p < 2) \tag{4.16}$$

$$\|\nabla u\|_{L^2(B_1\setminus B_\epsilon)} \le CM^{-2} \qquad (1 < p < 2) \tag{4.17}$$

$$\|f\|_{L^p(B_1\setminus B_\epsilon)} \le C_p M^{-2} \qquad (1 < p < 2) \ . \tag{4.18}$$

LEMMA 4.4. For $1 < p < 2$

$$\|u\|_{W^{2,p}(B_1\setminus B_{2\epsilon})} \le C_p M^{-2} \qquad (1 < p < 2) \ , \tag{4.19}$$

and hence, by the Sobolev embedding theorem,

$$\|u\|_{W^{1,p}(B_1\setminus B_{3\epsilon})} \le C_p M^{-2} \qquad (1 < p < \infty) \ . \tag{4.20}$$

Proof: Take a cutoff function $\zeta \in C^\infty$ so that

$$\begin{aligned} \zeta &= 1 \qquad \text{for } 2\epsilon \le \sqrt{x^2+y^2} \le 1 \\ &= 0 \qquad \text{for } \sqrt{x^2+y^2} \le \epsilon \end{aligned}$$

and $0 \le \zeta \le 1$. Let $w = \zeta u$. Then

$$\Delta w = u\Delta\zeta + 2\nabla\zeta\nabla u + \zeta\Delta u \tag{4.21}$$

and w satisfies the variational inequality

$$\begin{aligned} -\Delta w &\ge F \qquad \text{for } \epsilon < \sqrt{x^2+y^2} < 1 \\ w &\ge 0 \qquad \text{for } \epsilon < \sqrt{x^2+y^2} < 1 \\ w(-\Delta w - F) &= 0 \qquad \text{for } \epsilon < \sqrt{x^2+y^2} < 1 \\ w &= 0 \qquad \text{for } \sqrt{x^2+y^2} = \epsilon, \ \sqrt{x^2+y^2} = 1 \end{aligned}$$

where

$$F = -u\Delta\zeta - 2\nabla\zeta\nabla u - \zeta f \ . \tag{4.22}$$

By (4.16), (4.17), (4.18), we get

$$\|F\|_{L^p(B_1\setminus B_\epsilon)} \le C_p M^{-2} \qquad (1 < p < 2) \ . \tag{4.23}$$

Thus by L^p estimates for the variational inequality,

$$\|w\|_{W^{2,p}(B_1\setminus B_\epsilon)} \le C_p M^{-2} \qquad (1 < p < 2) \tag{4.24}$$

and (4.19) follows. □

Next, we prove

LEMMA 4.5. There exists a constant C such that

$$\delta \int_{B_1} u(x,y)\,dx\,dy \le C M^{-1/5} \tag{4.25}$$

uniformly for large M.

Proof: If (4.25) is not true, then there exists a sequence $M_n \to \infty$ such that

$$\delta \int_{B_1} u_{M_n}(x,y)dxdy > nM_n^{-1/5} \ . \tag{4.26}$$

Thus

$$\delta \int_{B_1} \frac{u_{M_n}(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt \geq \frac{1}{2}\delta \int_{B_1} u_{M_n}(x,y)dxdy > \frac{1}{2}nM_n^{-1/5} \ ; \tag{4.27}$$

hence

$$h_{M_n} \geq \frac{1}{2}nM_n^{1-\frac{1}{5}} > \frac{n}{2}M_n^{4/5} \ . \tag{4.28}$$

By Lemma 2.1,

$$\int_{B_1} (\frac{n}{2}M_n^{4/5})^3 |\nabla u_{M_n}|^2 dxdy \leq CM_n^2 \ , \tag{4.29}$$

and thus

$$\int_{B_1} |\nabla u_{M_n}|^2 dxdy \leq Cn^{-3}M_n^{2-\frac{12}{5}} = Cn^{-3}M_n^{-2/5} \ . \tag{4.30}$$

Using Hölder's inequality and (4.15) with $\epsilon \to 0$, we get

$$\begin{aligned} \int_{B_1} u_{M_n}(x,y)dxdy &\leq \pi^{1/2}\|u_{M_n}\|_{L^2(B_1)} \\ &\leq \pi^{1/2}\|\nabla u_{M_n}\|_{L^2(B_1)} \\ &\leq Cn^{-3/2}M_n^{-1/5} \ . \end{aligned} \tag{4.31}$$

From (4.26) and (4.31), it follows that

$$nM_n^{-1/5} < Cn^{-3/2}M_n^{-1/5} \tag{4.32}$$

or

$$n < Cn^{-3/2} \tag{4.33}$$

which is a contradiction. □

Proof of theorem 4.1. From (2.9)

$$h_x = \frac{xM^2}{R} + M\delta I \tag{4.34}$$

where

$$I = \int_{B_1} \frac{u_x(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt \ . \tag{4.35}$$

Take a cutoff function $\zeta \in C^\infty$ such that

$$\begin{aligned} \zeta &= 1 \qquad \text{for } \sqrt{x^2+y^2} \geq 3\epsilon \\ &= 0 \qquad \text{for } \sqrt{x^2+y^2} \leq 2\epsilon \end{aligned} \tag{4.36}$$

and $0 \leq \zeta \leq 1$. Then

$$\begin{aligned} I &= \int_{B_1} \frac{(u\zeta + u(1-\zeta))_x(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt \\ &= \int_{B_1} \frac{(u\zeta)_x(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt + \int_{B_1} \frac{(u(1-\zeta))_x(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt \\ &\equiv J_1 + J_2 \ . \end{aligned} \tag{4.37}$$

Assume that $(x,y) \in B_1 \setminus B_{4\epsilon}$. Then

$$\sqrt{(x-s)^2+(y-t)^2} \geq \epsilon \qquad \text{for } (s,t) \in B_{3\epsilon} \ . \tag{4.38}$$

Since $u(1-\zeta) = 0$ on $\partial B_{3\epsilon}$, no boundary term will appear when we use integration by parts for J_2. Hence

$$\begin{aligned} J_2 &= \int_{B_{3\epsilon}} \frac{(u(1-\zeta))_x(s,t)}{\sqrt{(x-s)^2+(y-t)^2}} dsdt \\ &= -\int_{B_{3\epsilon}} u(1-\zeta) \left(\frac{\partial}{\partial s} \frac{1}{\sqrt{(x-s)^2+(y-t)^2}} \right) dsdt \end{aligned} \tag{4.39}$$

and thus

$$\begin{aligned} |J_2| &\leq C \int_{B_{3\epsilon}} |u| dsdt \\ &\leq CM^{-1/5} \ , \end{aligned} \tag{4.40}$$

the last inequality is obtained by Lemma 4.5.

By Hölder's inequality

$$
\begin{aligned}
\|J_1\|_{L^\infty(B_1)} &\le \|(u\zeta)_x\|_{L^3(B_1\setminus B_{2\epsilon})}\left[\sup_{(x,y)\in B_1}\left\|\frac{1}{\sqrt{(x-\cdot)^2+(y-\cdot)^2}}\right\|_{L^{3/2}(B_1)}\right] \\
&\le C\|\nabla(u\zeta)\|_{L^3(B_1\setminus B_{2\epsilon})} \qquad (4.41)
\end{aligned}
$$

where the last inequality is obtained by using (4.20).

Thus for $(x,y)\in B_1\setminus B_{4\epsilon}$

$$|I|\le|J_1|+|J_2|\le CM^{-1/5}\ , \qquad (4.42)$$

and thus, for $x\le -4\epsilon$, we have

$$
\begin{aligned}
h_x &\le \frac{xM^2}{R}+C\delta M^{1-\frac{1}{5}} \\
&\le -\frac{2\epsilon M^2}{R}+CM^{2/5}<0 \qquad (4.43)
\end{aligned}
$$

if $M>K(\epsilon)$, and hence $u(x,y)>0$ for $x<-4\epsilon$. □

§5. Uniqueness

In this section we prove uniqueness provided δ is small. The estimates that we obtained in §2 and §3 are uniform for δ, that is, if $0\le\delta\le D$, then constants C depend only on D. Ω will be a fixed smooth domain.

THEOREM 5.1. There exists $\delta_1>0$ such that the solution of (1.7)–(1.11) is unique if $0\le\delta\le\delta_1$, where δ_1 depends on Ω, λ, k and R.

Proof: We shall use C to denote constants which do not depend on δ (but may depend on D).

If (u,h),$(\tilde u,\tilde h)$ are two solutions, then we have

$$[-\nabla(h^3\nabla u)+\lambda h_x](\tilde u-u)\ge 0 \qquad (5.1)$$

$$[-\nabla(\tilde h^3\nabla\tilde u)+\lambda\tilde h_x](u-\tilde u)\ge 0\ . \qquad (5.2)$$

Thus

$$\int_\Omega h^3\nabla u\nabla(\tilde u-u)\ge\lambda\int_\Omega h(\tilde u-u)_x \qquad (5.3)$$

$$\int_\Omega \tilde{h}^3 \nabla\tilde{u}\nabla(u-\tilde{u}) \geq \lambda \int_\Omega \tilde{h}(u-\tilde{u})_x \ ; \tag{5.4}$$

hence

$$\int_\Omega h^3|\nabla(u-\tilde{u})|^2 + (\tilde{h}^3 - h^3)\nabla\tilde{u}\nabla(\tilde{u}-u) \leq \lambda \int_\Omega |\nabla(u-\tilde{u})||h-\tilde{h}| \ . \tag{5.5}$$

Using (2.19) and (2.21), we get

$$\begin{aligned} k^3 \int_\Omega |\nabla(u-\tilde{u})|^2 &\leq C\int_\Omega |\nabla(u-\tilde{u})||h-\tilde{h}| \\ &\leq C\|\nabla(u-\tilde{u})\|_{L^2(\Omega)}\|h-\tilde{h}\|_{L^2(\Omega)} \ . \end{aligned} \tag{5.6}$$

By Young's inequality ([5], Lemma 7.12) with $p = q = 2$, we get

$$\|h-\tilde{h}\|_{L^2(\Omega)} \leq \delta C\|u-\tilde{u}\|_{L^2(\Omega)} \ . \tag{5.7}$$

Now (5.6), (5.7) together with Poincare's inequality give:

$$\int_\Omega |\nabla(u-\tilde{u})|^2 \leq C\|\nabla(u-\tilde{u})\|_{L^2(\Omega)}\|h-\tilde{h}\|_{L^2(\Omega)} \leq \delta C\int_\Omega |\nabla(u-\tilde{u})|^2 \ . \tag{5.8}$$

Thus, if δ is small, then

$$\int_\Omega |\nabla(u-\tilde{u})|^2 = 0 \tag{5.9}$$

which implies that $u = \tilde{u}$. □

§6. The case $\mu = \mu_0 e^{\alpha u}$

In the previous sections we studied the case when the viscosity is constant. We now study the case when the viscosity is given by (1.13).

Using the transformation

$$w = 1 - e^{-\alpha u} \ , \tag{6.1}$$

we obtain from (1.1)–(1.5) the variational inequality:

$$\begin{aligned} -\nabla(h^3\nabla w) &\geq -\lambda\alpha h_x \quad \text{for } (x,y)\in\Omega & (6.2)\\ w &\geq 0 \quad \text{for } (x,y)\in\Omega & (6.3)\\ w[-\nabla(h^3\nabla u) + \lambda\alpha h_x] &= 0 \quad \text{for } (x,y)\in\Omega & (6.4)\\ w &= 0 \quad \text{for } (x,y)\in\partial\Omega & (6.5) \end{aligned}$$

and

$$h(x,y) = k + \frac{x^2+y^2}{2R} + \delta \int_\Omega \frac{1}{\alpha}\left(\log \frac{1}{1-w(s,t)}\right) \frac{dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \ . \tag{6.6}$$

The main difficulty is to show that $1-w$ stays uniformly positive. To overcome this difficulity, let us fix $\epsilon \in (0,1)$ and take a cutoff function ζ such that $\zeta \in C^\infty$,$0 \le \zeta \le 1$ and

$$\begin{aligned} \zeta(w) &= 1 \quad \text{for } w \le 1-\epsilon \\ &= 0 \quad \text{for } w \ge 1-\epsilon/2 \ . \end{aligned}$$

Define

$$G(w) = \zeta(w) \log \frac{1}{1-w}, \quad \text{for } 0 \le w < \infty \tag{6.7}$$

then $G \in C^\infty$. Next, instead of (6.6), take

$$h(x,y) = k + \frac{x^2+y^2}{2R} + \delta \int_\Omega \frac{1}{\alpha} G(w(s,t)) \frac{dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \tag{6.8}$$

Let us consider the system (6.2)–(6.5) and (6.8). From (6.2)–(6.5), it follows that

$$\int_\Omega |\nabla w|^2 dxdy \le \frac{\lambda^2\alpha^2|\Omega|}{k^4} \ . \tag{6.9}$$

Differentiating (6.8), we get

$$\nabla h = \frac{(x,y)}{R} + \delta \int_\Omega G'(w(s,t)) \frac{\nabla w(s,t)}{\alpha} \frac{dsdt}{\sqrt{(x-s)^2+(y-t)^2}} \ . \tag{6.10}$$

Note that $|G'(w(s,t))|$ is bounded uniformly, and thus, by Young's inequality ([5], Lemma 7.12), we get, for $2 < p < \infty$,

$$\|\nabla h\|_{L^p(\Omega)} \le C + \delta C \|\frac{1}{\alpha}\nabla w\|_{L^2(\Omega)} \ ; \tag{6.11}$$

by (6.9)

$$\|\nabla h\|_{L^p(\Omega)} \le C \tag{6.12}$$

where the constant C is indepedent of α. Thus, if

$$f = \frac{3\nabla h}{h}\nabla w - \frac{\lambda\alpha}{h^3}\frac{\partial h}{\partial x} \tag{6.13}$$

then, as in the proof of Lemma 2.3, (using (6.9)),

$$\|f\|_{L^p(\Omega)} \le C_p\alpha \qquad (1 < p < 2) \ . \tag{6.14}$$

From (6.9) and (6.14), it follows by using the L^p estimates for the variational inequality that

$$\|w\|_{W^{2,p}(\Omega)} \le C_p\alpha \qquad (1 < p < 2) \tag{6.15}$$

Applying the Sobolev embedding theorem to (6.15), we obtain

$$w \le C\alpha \le 1 - \epsilon \tag{6.16}$$

provided α is small, and then the expressions (6.8) and (6.6) coincide. As explained above, the existence of the solution now follows as before.

§7. The one dimensional problem

Consider the one dimensional variational inequality:

$$\begin{aligned} -(h^3\frac{u'}{e^{\alpha u}})' &\ge -\lambda h' \quad \text{for } x \in [-M, M] && (7.1)\\ u &\ge 0 \qquad \text{for } x \in [-M, M] && (7.2)\\ u[-(h^3\frac{u'}{e^{\alpha u}})' + \lambda h'] &= 0 \qquad \text{for } x \in [-M, M] && (7.3)\\ u(\pm M) &= 0 && (7.4) \end{aligned}$$

and

$$h = k + \frac{x^2}{2R} + \delta\int_{-M}^{M} u(s)\log\frac{2M}{|x-s|}ds \tag{7.5}$$

where M, R, λ, K, α and δ are positive constants; and α is positive constants which shall be assume to be sufficiently small in proving the existence of the solution, both α and δ will be assume to be sufficiently small in proving the uniqueness of the solution.

Introduces the transformation

$$w = 1 - e^{-\alpha u} \ , \tag{7.6}$$

as before. Then

$$u = \frac{1}{\alpha} \log \frac{1}{1-w} \ , \tag{7.7}$$

and (7.1)–(7.5) is transformed into the following problem:

$$-(h^3 w')' \geq -\lambda\alpha h' \quad \text{for } x \in [-M, M] \tag{7.8}$$

$$w \geq 0 \quad \text{for } x \in [-M, M] \tag{7.9}$$

$$w[-(h^3 w')' + \lambda\alpha h'] = 0 \quad \text{for } x \in [-M, M] \tag{7.10}$$

$$w(\pm M) = 0 \tag{7.11}$$

and

$$h = k + \frac{x^2}{2R} + \frac{\delta}{\alpha} \int_{-M}^{M} \left(\log \frac{1}{1 - w(s)} \right) \log \frac{2M}{|x-s|} ds \ . \tag{7.12}$$

As we did in §6, instead of (7.12), we consider

$$h = k + \frac{x^2}{2R} + \frac{\delta}{\alpha} \int_{-M}^{M} (\log G(w(s))) \log \frac{2M}{|x-s|} ds \tag{7.13}$$

where $G(w)$ is defined in (6.7). The system (7.1)–(7.5) is equivalent to (7.8)–(7.11) and (7.13) if we can establish the bound:

$$w \leq 1 - \epsilon \tag{7.14}$$

but this follows, for small α, from the following lemma.

LEMMA 7.1. Suppose that $\lambda \leq \Lambda$. If w satisfies (7.8)–(7.11), then

$$|w(y)| \leq 2M \frac{\Lambda\alpha}{k^2} \quad \text{for } |y| \leq M \ . \tag{7.15}$$

Proof: Integrating (7.10) over $[-M, M]$, we get

$$\int_{-M}^{M} h^3 |w'|^2 = -\lambda\alpha \int_{-M}^{M} h' w = \lambda\alpha \int_{-M}^{M} h w' \tag{7.16}$$

$$\leq \lambda\alpha \left(\int_{-M}^{M} h^2 |w'|^2 \right)^{1/2} (2M)^{1/2} \tag{7.17}$$

$$\leq \frac{\lambda\alpha}{k^{1/2}}(2M)^{1/2}\left(\int_{-M}^{M} h^3|w'|^2\right)^{1/2} \quad , \tag{7.18}$$

and hence

$$\int_{-M}^{M} h^3|w'|^2 \leq \frac{\lambda^2\alpha^2}{k}(2M) \ . \tag{7.19}$$

It follows that

$$\|w'\|_{L^2[-M,M]} \leq \frac{\lambda\alpha}{k^2}(2M)^{1/2} \ , \tag{7.20}$$

so that

$$|w(y)| \leq \int_{-M}^{M} |w'(s)|ds \leq (2M)^{1/2}\|w'\|_{L^2[-M,M]} \tag{7.21}$$

Using (7.20), (7.15) follows. □

Lemma 7.1 tells us that for $0 < \epsilon < \frac{1}{2}$, there exists an $A > 0$ (we may take $A = \frac{k^2}{2M\Lambda}(1-\epsilon)$ such that

$$w(y) \leq 1 - \epsilon \qquad \text{for } \ 0 < \alpha < A \ , \tag{7.22}$$

and hence (7.12) and (7.13) coincide. Suppose next that $0 \leq \delta \leq D$. We shall use constants C to denote various constants depending only on M, R, k, A, D and Λ. Since we are going to use the same method as in §3 to prove existence, care has to be taken so that the constants will not change when we replace λ by $\sigma\lambda$ $(0 \leq \sigma \leq 1)$.

Note that, by (7.15),

$$\limsup_{\alpha\to 0} \frac{1}{\alpha} \log \frac{1}{1-w(s)} \leq \lim_{\alpha\to 0} \log \frac{1}{1-\alpha 2M\Lambda/k^2} = \frac{2M\Lambda}{k^2} \ . \tag{7.23}$$

and, consequently,

$$\sup_{0<\alpha\leq A} \sup_{|s|\leq M} \left|\frac{1}{\alpha}\log\frac{1}{1-w(s)}\right| \leq C \ . \tag{7.24}$$

By (7.12),

$$h \leq C + C\int_{-M}^{M} \log\frac{2M}{|x-s|}ds \leq C \ . \tag{7.25}$$

In order to derive an estimate for $w'(x)$ for all x, it suffices to derive an estimate for $w'(x)$ for every interval $[a,b]$ such that $w(a) = w(b) = 0$ and $w(x) > 0$ for $x \in (a,b)$.

By Rolle's theorem, there is a number $c \in (a,b)$ such that $w'(c) = 0$. From (7.10), it follows that

$$h^3(x)w'(x) = \lambda\alpha(h(x) - h(c)) \ . \tag{7.26}$$

And hence, by (7.25),

$$\frac{1}{\alpha}|w'(x)| \leq \Lambda \left|\frac{h(x) - h(c)}{h^3(x)}\right| \leq C \ . \tag{7.27}$$

Differentiating (7.13) (which is now the same as (7.12)), we get

$$h'(x) = \frac{x}{R} + \delta \int_{-M}^{M} \frac{1}{\alpha} \frac{w'(s)}{1 - w(s)} \log \frac{2M}{|x - s|} ds \ , \tag{7.28}$$

and, by (7.22) (7.27),

$$|h'(x)| \leq \frac{M}{R} + \delta C \int_{-M}^{M} \log \frac{2M}{|x - s|} ds \leq C \ . \tag{7.29}$$

By (7.10),

$$h^3 w'' + 3h^2 h' w' = \lambda \alpha h' \qquad \text{if } w(x) > 0 \ , \tag{7.30}$$

and thus, by (7.27), (7.29),

$$\sup_{|x| \leq M} \frac{1}{\alpha} |w''(x)| \leq C \ . \tag{7.31}$$

Using these estimates, we can prove the existence of the solution by using the same method as in §3. In the case $\alpha = 0$, we may work with u instead of w. We have proved:

THEOREM 7.2. If α is sufficiently small such that

$$\frac{2M\lambda\alpha}{k^2} \leq 1 - \epsilon \ , \tag{7.32}$$

then there exists a solution to (7.1)–(7.5). □

Next, we prove that the free boundary consists of at most one free boundary point.

THEOREM 7.3. If $0 \le \alpha \le A$, $0 \le \delta \le \delta_0$, then there is at most one free boundary point. *Proof*: We can rewrite h' as:

$$h'(x) = \frac{x}{R} + \delta \int_{-M-x}^{M-x} \frac{1}{\alpha} \frac{w'(t+s)}{1-w(t+s)} \log \frac{2M}{|t|} dt \ , \tag{7.33}$$

then

$$\begin{aligned} h''(x) &= \frac{1}{R} + \delta \int_{-M}^{M} \frac{1}{\alpha} \left[\frac{w''(s)}{1-w(s)} + \frac{(w'(s))^2}{(1-w(s))^2} \right] \log \frac{2M}{|x-s|} ds \\ &\quad -\delta \frac{1}{\alpha} w'(M) \log \frac{2M}{M-x} + \delta \frac{1}{\alpha} w'(-M) \log \frac{2M}{M+x} \\ &= \frac{1}{R} - \delta I \ . \end{aligned}$$

By (7.27) (7.31), we get

$$h''(x) \ge \frac{1}{R} - \delta C_1 \qquad \text{for } |x| \le \frac{M}{2}. \tag{7.34}$$

so that if δ is small

$$h''(x) > 0 \qquad \text{for } |x| \le \frac{M}{2} \ . \tag{7.35}$$

By (7.28), if δ is small enough, then

$$h'(x) \ge \frac{1}{R}\frac{M}{2} - \delta C_2 > 0 \qquad \text{for } x \ge \frac{M}{2} \tag{7.36}$$

$$h'(x) \le \frac{1}{R}(-\frac{M}{2}) + \delta C_2 < 0 \qquad \text{for } x \le -\frac{M}{2} \ . \tag{7.37}$$

Take δ_0 so that (7.35), (7.36) and (7.37) hold for $\delta \le \delta_0$. Then $h'(x)$ has only one zero in $[-M, M]$, say at d. Then

$$h'(x) < 0 \qquad \text{for } -M \le x < d \tag{7.38}$$

$$h'(x) > 0 \qquad \text{for } d < x \le M \ . \tag{7.39}$$

Suppose now that b is the first free boundary point, i.e.,

$$w(x) > 0 \qquad \text{for } -M < x < b \tag{7.40}$$

$$w(b) = 0 \qquad \text{for } b < M \ . \tag{7.41}$$

By regularity, $w'(b) = 0$. By (7.38), $w > 0$ on $(-M, d)$. Hence $b \geq d$.

Thus if we define

$$\begin{aligned} \tilde{w}(x) &= w(x) \qquad \text{for } -M \leq x < b \\ &= 0 \qquad \text{for } b \leq x \leq M \ . \end{aligned} \tag{7.42}$$

Since $h(x) > 0$ for $x > b$, $\tilde{w}$ is also a solution of (7.8)–(7.11). Since the solution is unique (for fixed h), we have $w = \tilde{w}$, and hence b is the only free boundary point. □

We also have uniqueness if δ is small:

THEOREM 7.4. There exists a number $\delta_1 > 0$ so that the solution (w, h) to the problem (7.8)–(7.12) is unique provided that $0 \leq \delta \leq \delta_1$.

Proof: The proof is essentially the same as that in Theorem 5.1. One needs only to check that

$$\|h - \tilde{h}\|_{L^2[-M,M]} \leq C\delta \|u - \tilde{u}\|_{L^2[-M,M]} \tag{7.43}$$

which is obviously true. □

Remark: Several problems remain open: the uniqueness of the solution is unique without assuming that δ is small, the shape of the free boundary in the two dimensional case, and the existence of the solution when $\mu = \mu_0 e^{\alpha u}$ without assuming α small.

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