A TWO-POINT BOUNDARY VALUE PROBLEM OF DIRICHLET TYPE WITH RESONANCE AT INFINITELY MANY EIGENVALUES

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1. INTRODUCTION.

Let $g : [0,\pi] \times \mathbf{R} \to \mathbf{R}$ be a given function satisfying Caratheodory's conditions and $h : [0,\pi] \to \mathbf{R}$ be a given function in $L^1[0,\pi]$. This paper is devoted to the study of the boundary value problem

$$u''(x) + u(x) + g(x,u(x)) = h(x), \quad x \in [0,\pi], \qquad (1.1)$$
$$u(0) = u(\pi) = 0,$$

when the asymptotic values of $u^{-1}g(x,u)$ cross infinitely many eigen-values of the linear eigen-value problem

$$u''(x) + \lambda u(x) = 0, \quad x \in [0,\pi], \qquad (1.2)$$
$$u(0) = u(\pi) = 0.$$

We may note that the boundary value problem (1.1) is at resonance, since the linear eigen-value problem

$$u''(x) + u(x) = 0, \ x \in [0,\pi],$$
$$u(0) = u(\pi) = 0,$$

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has $u(x) = A \sin x$, $A \in \mathbf{R}$, as non-trivial solutions. The boundary value problem (1.1) has recently been studied by several authors, [2], [4], [5], [9] [10] in case of some resonance at the second eigen-value $\lambda = 4$, (but not crossing it), for (1.2) under various conditions on $g(x,u)$ and $h(x)$. Also the author with Mawhin, [3], studied the periodic boundary value problem

$$u''(x) + u(x) + f(u(x))u'(x) + g(x,u(x)) = h(x),\ x \in [0,2\pi],$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0,$$

when the asymptotic values of $u^{-1}g(x,u)$ cross infinitely many eigen-values of the linear periodic boundary value problem

$$u''(x) + \lambda u(x) = 0,\ x \in [0,2\pi],$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0.$$

So, motivated by the results of [3], we obtain the existence of a solution for (1.1) when $h(x) \in L^1[0,\pi]$ satisfy the condition

$$\int_0^\pi h(x)\sin x \, dx = 0, \tag{1.3}$$

and $g(x,u)$ satisfies the following supplementary conditions

$$g(x,u)u \geq 0, \tag{1.4}$$

for a.e. $x \in [0,\pi]$ and all $u \in \mathbf{R}$, and there exists a function $\Gamma(x) \in L^1[0,\pi]$ with

$$\|\Gamma\|_{L^1[0,\pi]} \leq \frac{1}{15\cdot 87}, \tag{1.5}$$

and

$$\limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \Gamma(x), \tag{1.6}$$

uniformly for a.e. $x \in [0,\pi]$. We note that conditions (1.5), (1.6) allow the crossing of infinitely many eigen-values of the linear eigen-value problem (1.2) by the asymptotic values of $u^{-1}g(x,u)$.

Our methods use the version of Leray-Schauder continuation theorem as given by Mawhin in [6], [7], [8] and some of the ideas used earlier by the author in [1].

2. Main Results.

Let X,Y denote the Banach spaces $X = C[0,\pi]$ and $Y = L^1[0,\pi]$ with their usual norms. Let Y_2 be the subspace of Y spanned by the function $\sin x$, i.e.,

$$Y_2 = \{u \in Y \mid u(x) = \alpha \sin x, \ \ a.e. \ on \ [0,\pi], \alpha \in \mathbf{R}\}, \tag{2.1}$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. We note that for $u \in Y$ we can write

$$u(x) = u(x) - (\frac{2}{\pi}\int_0^\pi u(t)\sin t \ dt)\sin x + (\frac{2}{\pi}\int_0^\pi u(t)\sin t \ dt)\sin x, \tag{2.2}$$

$x \in [0,\pi]$. We define the canonical projection operators $P : Y \to Y_1$; $Q : Y \to Y_2$ by

$$P(u) = u(x) - (\frac{2}{\pi}\int_0^\pi u(t)\sin t \ dt)\sin x,$$

$$Q(u) = (\frac{2}{\pi}\int_0^\pi u(t)\sin t \ dt)\sin x, \tag{2.3}$$

for $u \in Y$. We note that, for $u \in Y$,

$$\|Pu\|_Y \leq (1 + \frac{4}{\pi}) \|u\|_Y. \tag{2.4}$$

Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projection $P \mid X : X \to X_1$ and $Q \mid X : X \to X_2$ are continuous. In the following, X, Y, P, Q will refer to the Banach spaces and the projections as defined above and we shall not distinguish between $P, P|X$ (resp. $Q, Q|X$) and depend on the context for proper meaning.

Also for $u \in X$, $v \in Y$, let $(u,v) = \int_0^\pi u(x)v(x)dx$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$, so that $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u,v) = (Pu,Pv) + (Qu,Qv). \tag{2.5}$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X \mid u'(x) \in AC[0,\pi],\ u(0) = u(\pi) = 0\}, \tag{2.6}$$

and for $u \in D(L)$,

$$Lu = u'' + u. \tag{2.7}$$

(Her $AC[0,\pi]$ denotes the space of real-valued absolutely continuous functions on $[0,\pi]$).

Let, now, for $h \in Y_1$, i.e., $h \in L^1[0,\pi]$ with $\int_0^\pi h(t)\sin t\ dt = 0$, Kh denote the unique solution of the problem

$$u''(x) + u(x) = h(x),$$

$$u(0) = u(\pi) = 0,$$

such that $\int_0^\pi u(x)\sin x dx = 0$. Indeed, for $x \in [0,\pi]$,

$$(Kh)(x) = -(\int_0^x h(t)\sin t\ dt)\cos x + (\int_0^x h(t)\cos t\ dt)\sin x$$

$$-\frac{1}{\pi}(\int_0^\pi h(t)(\pi - t)\cos t\ dt)\sin x. \tag{2.8}$$

Also,

$$\|Kh\|_X \leq 3\|h\|_Y, \tag{2.9}$$

so that

$$\|K\| \leq 3. \tag{2.10}$$

Hence $K : Y_1 \to X_1$ is a bounded linear mapping and is such that for

$$u \in Y,\ KP(u) \in D(L) \ \text{ and } \ LKP(u) = P(u). \tag{2.11}$$

Definition 2.1: $g : [0,\pi] \times \mathbf{R} \to \mathbf{R}$ satisfies Caratheodory's conditions if $g(x,\cdot)$ is continuous for a.e. $x \in [0,\pi]$, $g(\cdot,u)$ is measurable on $[0,\pi]$ for each $u \in \mathbf{R}$, and for each $r \in \mathbf{R}$, $r \geq 0$,there is a function $\alpha_r(x) \in L^1[0,\pi]$ such that $|g(x,u)| \leq \alpha_r(x)$ for a.e. $x \in [0,\pi]$, $u \in \mathbf{R}$, $|u| \leq r$.

Let $g : [0,\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions. Let $N : X \to Y$ be the nonlinear mapping defined by

$$(Nu)(x) = g(x,u(x)), \ \ x \in [0,\pi], \tag{2.12}$$

for $u \in X$.

For $h(x) \in Y = L^1[0,\pi]$ with $\int_0^\pi h(x)\sin x \, dx = 0$, the boundary value problem

$$\begin{aligned} &u'' + u + g(x,u) = h(x), \ \ x \in [0,\pi], \\ &u(0) = u(\pi) = 0, \end{aligned} \tag{2.13}$$

now reduces to the functional equation

$$Lu + Nu = h, \tag{2.14}$$

in X, with $h \in Y_1$, given.

Notation 2.2: We denote by $\|P\|$ the norm of the continuous projection $P : Y \to Y_1$ defined in (2.3), so that

$$\|P\| = sup \left\{ \frac{\|Pu\|_Y}{\|u\|_Y} : u \in Y, \ u \neq 0 \right\}. \tag{2.15}$$

Also we denote by $\|K\|$ the norm of the bounded linear mapping $K : Y_1 \to X_1$, defined in (2.8), so that

$$\|K\| = sup \left\{ \frac{\|Kh\|_X}{\|h\|_Y} : h \in Y_1, \ h \neq 0 \right\}. \tag{2.16}$$

We observe from (2.4) and (2.9) that

$$\|P\| \leq 1 + \frac{4}{\pi}, \tag{2.17}$$

and

$$\|K\| \leq 3. \tag{2.18}$$

<u>Theorem 2.3:</u> Let $\Gamma \in L^1[0,\pi]$ be such that

$$\|\Gamma\|_{L^1[0,\pi]} \ < \ \|P\|^{-2} \cdot \|K\|^{-1}. \tag{2.19}$$

Let $g : [0,\pi] \times \mathbf{R} \to \mathbf{R}$ be a given function satisfying Caratheodory's conditions and

(i) $g(x,u)u \geq 0$ for a.e. $x \in [0,\pi]$, $u \in \mathbf{R}$

(ii) $\lim\limits_{|u| \to \infty} sup \dfrac{g(x,u)}{u} \leq \Gamma(x)$,

uniformly for a.e. $x \in [0,\pi]$.

Then, for each $h \in Y = L^1[0,\pi]$ with $\int_0^\pi h(x) \sin x \, dx = 0$, the boundary value problem

$$\begin{aligned} u'' + u + g(x,u) &= h(x), \ x \in [0,\pi], \\ u(0) = u(\pi) &= 0, \end{aligned} \tag{2.20}$$

has at least one solution u in $X = C[0,\pi]$.

We need the following lemma in the proof of this theorem.

Lemma 2.4: Let Γ, g be as in Theorem 2.3, and let $\epsilon > 0$ be given. Then there exists constants $c_1 \geq 0$, $c_2 \geq 0$ such that for $u \in Y$,

$$(Nu, u) \geq \frac{1}{\|\Gamma\|_{L^1[0,\pi]} + \epsilon\pi} \cdot \|Nu\|_Y^2 - C_1 \|Nu\|_Y - C_2, \tag{2.21}$$

where N is defined by (2.12).

Proof of lemma: We first have from our assumptions on g that there exists a $\rho \geq 0$, depending on ϵ, such that

$$|g(x,u)| \leq (\Gamma(x) + \epsilon)\,|u(x)| \tag{2.22}$$

for a.e. $x \in [0,\pi]$, and $u \in \mathbf{R}$ with $|u| \geq \rho$.

Now, for $u \in Y$,

$$\begin{aligned}(Nu, u) &= \int_0^\pi g(x,u(x))u(x)dx = \int_0^\pi |g(x,u(x))|\ |u(x)|dx \\ &\geq \int_{|u(x)|\geq\rho} |g(x,u(x))| \cdot |u(x)|dx \\ &\geq \int_{|u(x)|\geq\rho} \frac{1}{(\Gamma(x)+\epsilon)} |g(x,u(x))|^2 dx,\end{aligned} \tag{2.23}$$

in view of (2.22). We next note that

$$\int_{|u(x)|\geq\rho} |g(x,u(x))|dx = \int_{|u(x)|\geq\rho} \frac{|g(x,u(x))|}{(\Gamma(x)+\epsilon)^{1/2}} \cdot (\Gamma(x)+\epsilon)^{1/2}dx$$

$$\leq \Big(\int_{|u(x)|\geq\rho} \frac{1}{(\Gamma(x)+\epsilon)} \cdot |g(x,u(x))|^2 dx\Big)^{1/2} \Big(\int_{|u(x)|\geq\rho} (\Gamma(x)+\epsilon)dx\Big)^{1/2}$$

$$\leq \Big(\int_{|u(x)|\geq\rho} \frac{1}{(\Gamma(x)+\epsilon)} \cdot |g(x,u(x))|^2 dx\Big)^{1/2} \cdot \big(\|\Gamma\|_{L^1[0,\pi]} + \epsilon\pi\big)^{1/2}.$$

Using this in (2.23) we then get that

$$(Nu, u) \geq \frac{1}{\|\Gamma\|_{L^1[0,\pi]} + \epsilon\pi} \left(\int_{|u(x)| \geq \rho} |g(x, u(x))| \, dx\right)^2$$

$$\geq \frac{1}{\|\Gamma\|_{L^1[0,\pi]} + \epsilon\pi} \cdot \|Nu\|_Y^2 - C_1\|Nu\|_Y - C_2$$

for some constants $C_1 \geq 0$, $C_2 \geq 0$, depending on Γ, ϵ only.

This completes the proof of the lemma.//.

Proof of Theorem 2.3: As noted above in (2.13), (2.14), the boundary value problem (2.20) reduces to the functional equation

$$Lu + Nu = h, \tag{2.24}$$

in X, with $h \in Y_1$. Now to solve the functional (2.24) it suffices to solve the system of equations

$$\begin{aligned} Pu + KPNu &= h_1, \\ QNu &= 0, \end{aligned} \tag{2.25}$$

$u \in X$, $h_1 = Kh$, (note that since $h \in Y_1$, $Ph = h$, $Qh = 0$). Indeed, if $u \in X$ is a solution of (2.25) then $u \in D(L)$ and

$$LPu + LKPNu = Lu + PNu = Lh_1 = h,$$

$$QNu = 0,$$

which gives on adding that $Lu + Nu = h$.

Now, (2.25) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = h_1, \tag{2.26}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can, therefore, apply the version given in [6] (Theorem 1, Corollary 1) or [7]

(Theorem IV.4) or [8] of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (2.26) if the set of all possible solutions of the family of equations

$$Pu + (1-\lambda)Qu + \lambda QNu + \lambda KPNu = \lambda h_1, \tag{2.27}$$

$\lambda \in]0,1[$, is a priori bounded, independantly of λ. Notice that (2.27) is then equivalent to the system of equations

$$Pu + \lambda KPNu = \lambda h_1,$$
$$(1-\lambda)Qu + \lambda QNu = 0. \tag{2.28}$$

If $u_\lambda \in X$ is a solution of (2.28) for some $\lambda \in]0,1[$, then $u_\lambda \in D(L)$ and

$$(Pu_\lambda, PNu_\lambda) + \lambda(KPNu_\lambda,\ PNu_\lambda) = \lambda(h_1, PNu_\lambda),$$
$$(1-\lambda)(Qu_\lambda, QNu_\lambda) + \lambda(QNu_\lambda, QNu_\lambda) = 0.$$

Consequently, we have

$$\begin{aligned}(Pu_\lambda, PNu_\lambda) &= -\lambda(KPNu_\lambda, PNu_\lambda) + \lambda(h_1, PNu_\lambda)\\ &\leq \lambda\|KPNu_\lambda\|_X \cdot \|PNu_\lambda\|_Y + \lambda\|h_1\|_X \cdot \|PNu_\lambda\|_Y\\ &\leq \|K\| \cdot \|P\|^2 \cdot \|Nu_\lambda\|_Y^2 + \|P\| \cdot \|h_1\|_X \cdot \|Nu_\lambda\|_Y,\end{aligned}$$
$$(Qu_\lambda, QNu_\lambda) \leq 0,$$

which gives, on adding, that

$$(Nu_\lambda, u_\lambda) \leq \|K\| \cdot \|P\|^2 \cdot \|Nu_\lambda\|_Y^2 + \|P\| \cdot \|h_1\|_X\ \|Nu_\lambda\|_Y. \tag{2.29}$$

Let, now, $\epsilon > 0$ be such that

$$\|\Gamma\|_{L^1[0,\pi]} + \epsilon\pi < \|P\|^{-2} \cdot \|K\|^{-1}. \tag{2.30}$$

Using lemma 2.4 and (2.29) we get that

$$\left(\frac{1}{\|\Gamma\|_{L^1[0,\pi]}+\epsilon\pi}-\|K\|\cdot\|P\|^2\right)\|Nu_\lambda\|_Y^2\le(C_1+\|P\|\,\|h_1\|_X)\|Nu_\lambda\|_Y+C_2.$$

Accordingly, there exists a constant C_3, independent of $\lambda\in]0,1[$, such that

$$\|Nu_\lambda\|_Y\le C_3. \tag{2.31}$$

Using (2.31) in the first equation of (2.28) we get

$$\begin{aligned}\|Pu_\lambda\|_X&\le\|K\|\cdot\|P\|\,\|Nu_\lambda\|_Y+\|h_1\|_X\\&\le\|K\|\cdot\|P\|C_3+\|h_1\|_X\equiv C_4.\end{aligned} \tag{2.32}$$

Note that C_4 is a constant, independent of $\lambda\in]0,1[$.

It only remains to prove that there is a constant C, independent of $\lambda\in]0,1[$ such that $\|Qu_\lambda\|_X\le C$. Let us suppose, on the other hand, that the set

$$\{\|Qu_\lambda\|_X : u_\lambda \;\; a\ solution\ of\ (2.28),\;\; \lambda\in]0,1[\} \tag{2.33}$$

is unbounded.

We, now, have from the first equation in (2.28) that

$$LPu_\lambda+\lambda LKPNu_\lambda=\lambda Lh_1,$$

i.e.

$$LPu_\lambda+\lambda PNu_\lambda=\lambda h,$$

so that

$$\begin{aligned}\|LPu_\lambda\|_Y&\le\lambda\|PNu_\lambda\|_Y+\lambda\|h\|_Y\\&\le\|P\|\cdot\|Nu_\lambda\|_Y+\|h\|_Y\\&\le\|P\|C_3+\|h\|_Y\equiv C_5.\end{aligned}$$

Since, now,

$$LPu_\lambda = (Pu_\lambda)'' + Pu_\lambda,$$

$\|Pu_\lambda\|_X \le C_4$, we see that $\|(Pu_\lambda)''\|_Y$ is bounded by a constant independent of $\lambda \in]0,1[$. It, now, follows that there is a constant C_6, independent of $\lambda \in]0,1[$, using $u_\lambda(0) = u_\lambda(\pi) = 0$, that

$$\|(Pu_\lambda)'\|_X \le C_6.$$

We next use the well-known estimate

$$\left|\frac{v(x)}{\sin x}\right| \le \frac{\pi}{2} \max_{s \in [0,\pi]} |v'(s)|,$$

for $v \in X$, $v(0) = v(\pi) = 0$; to get

$$|(Pu_\lambda)(x)| \le \frac{\pi}{2} C_6 \sin x, \tag{2.34}$$

for $x \in [0,\pi]$. Now, by (2.33) we see that there is a sequence $\{\lambda_n\}$, $\lambda_n \in]0,1[$, such that

$$\|Qu_\lambda\|_X = \left|\frac{2}{\pi}\int_0^\pi u_{\lambda_n}(x)\sin x \, dx\right| \to \infty, \quad as\ n \to \infty.$$

We may assume, that

$$\int_0^\pi u_{\lambda_n}(x)\sin x \, dx \to \infty, \quad as\ n \to \infty,$$

so that there is an n_0 such that

$$\int_0^\pi u_{\lambda_n}(t)\sin t \, dt \ge \frac{\pi^2}{4} C_6 \ \text{ for } \ n \ge n_0. \tag{2.35}$$

Now, for $n \ge n_0$, $x \in [0,\pi]$ we have using (2.34), (2.35) that

$$\begin{aligned} u_{\lambda_n}(x) &= Qu_{\lambda_n}(x) + Pu_{\lambda_n}(x) \\ &= \left(\frac{2}{\pi}\int_0^\pi u_{\lambda_n}(t)\sin t \, dt\right)\sin x + Pu_{\lambda_n}(x) \\ &\ge \frac{2}{\pi} \cdot \frac{\pi^2}{4} C_6 \sin x - \frac{\pi}{2} C_6 \sin x = 0. \end{aligned}$$

Since, now, $g(x,v)v \geq 0$ for a.e. $x \in [0,\pi]$, $v \in \mathbf{R}$ we have $g(x,u_{\lambda_n}(x)) \geq 0$ for a.e. $x \in [0,\pi]$, $n \geq n_0$ and hence

$$(QNu_{\lambda_n}, Qu_{\lambda_n}) \geq 0, \text{ for } n \geq n_0.$$

It then follows from the second equation in (2.28) that

$$(1-\lambda_n)(Qu_{\lambda_n}, Qu_{\lambda_n}) = (1-\lambda_n) \cdot \frac{2}{\pi} (\int_0^\pi u_{\lambda_n}(t)\sin t \; dt)^2 \leq 0,$$

for $n \geq n_0$, a contradiction. Similarly, assuming $\int_0^\pi u_{\lambda_n}(t) sin\ t\ dt \to -\infty$, leads to a contradiction. Thus the set in (2.33) is, indeed, bounded by a constant independent of $\lambda \in]0,1[$.

This compeltes the proof of the Theorem.//.

Remark 2.5: We notice from (2.17), (2.18) that

$$\|P\|^2 \cdot \|K\| \leq 3(1 + \frac{4}{\pi})^2.$$

Also, it is easy to see that $3(1 + \frac{4}{\pi})^2 < 15{\cdot}87$. So if we assume that

$$\|\Gamma\|_{L^1[0,\pi]} \leq \frac{1}{15{\cdot}87}$$

then

$$\|\Gamma\|_{L^1[0,\pi]} < \|P\|^{-2} \cdot \|K\|^{-1}.$$

Thus, theorem 2.3 immediately proves the assertion made in the introduction.

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