INTEGRAL TYPE ASYMPTOTIC CONDITIONS FOR THE SOLVABILITY OF A PERIODIC FOURTH ORDER BOUNDARY VALUE PROBLEM

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Integral Type Asymptotic Conditions For the Solvability of a Periodic Fourth Order Boundary Value Problem. *

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Abstract:

The fourth order periodic boundary value problem

$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = e(x), \quad x \in [0,2\pi],$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

is studied when the nonlinearity g satisfies a more general sign condition of integral type instead of the usual sign condition, namely, there exists a $\rho > 0$ such that $g(x,u)\, u \geq 0$ for a.e. $x \in [0,2\pi]$ and all $u \in \mathbf{R}$ with $|u| \geq \rho$.

1. Introduction.

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, $g : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions and $e(x) \in L^1[0,2\pi]$. This paper is concerned with a study of the boundary value problem

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$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = e(x), \quad x \in [0,2\pi], \tag{1.1}$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0.$$

The author has studied (1.1) in [1], [2] where, among other conditions, it is assumed that the function g satisfy the following sign condition:

$$\text{''}\textit{there exists a } \rho > 0 \textit{ such that } g(x,u)u \geq 0 \text{ for} \tag{1.2}$$
$$\textit{a.e. } x \in [0,2\pi] \text{ and } \textit{all } u \in \mathbf{R} \textit{ with } |u| \geq \rho.\text{''}$$

The purpose of this paper is to replace (1.2) with the following more general sign condition of integral type, "there exists a $\rho > 0$ and a C^3-function $m:\mathbf{R}\ [-\rho,\rho] \to \mathbf{R}$ with $um(u) > 0$, $m'(u) \leq 0$, $m'''(u) \geq 0$. such that

$$\int_0^{2\pi} g(x,u(x))m(u(x))dx \geq 0 \tag{1.3}$$

for all C^3-real valued function $u(x)$ on $[0,2\pi]$, with u''' absolutely continuous on $[0,2\pi]$, $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$ and $\min_{x \in [0,2\pi]} |u(x)| \geq \rho$." We may note that (1.2) implies (1.3) with $m(u) = sgn\ u = u/|u|$ for $u \in \mathbf{R}\ [-\rho,\rho]$.

The results of this paper are motivated by some of the results of Mawhin ([3]) for the second order Lienards equation. The statement and proof of the basic result depend on three lemmas proved in Gupta [2] for which we need the following notations. Besides using the classical spaces $C[0,2\pi]$, $C^k[0,2\pi]$, $L^k[0,2\pi]$ and $L^\infty[0,2\pi]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable or measurable functions that are essentially-bounded on $[0,2\pi]$, we shall use the Sobolev spaces $H^k[0,2\pi]$, ($k = 2,3,or\,4$) defined by

$$H^k[0,2\pi] = \{u:[0,2\pi] \to \mathbf{R} \mid u^{(j)}\ abs.cont.on\ [0,2\pi],\ j=0,1,\ldots,k-1,\ u^{(k)} \in L^2[0,2\pi]\}$$

with the inner product defined by

$$(u,v)_{H^k} = \sum_{j=1}^{k} \frac{1}{2\pi}\int_0^{2\pi} u^{(j)}(x)v^{(j)}(x)dx + (\frac{1}{2\pi}\int_0^{2\pi} u(x)dx)(\frac{1}{2\pi}\int_0^{2\pi} v(x)dx),$$

and the corresponding norm denoted by $|\cdot|_{H^k}$. We also define, for the sake of convenience, the norm in $L^k[0,2\pi]$ by

$$|u|_{L^k} = (\frac{1}{2\pi}\int_0^{2\pi} |u(x)|^k dx)^{\frac{1}{k}}$$

We also use the Sobolev-space $W^{4,1}[0,2\pi]$ defined by

$$W^{4,1}[0,2\pi] = \{u:[0,2\pi] \rightarrow \mathbf{R} \mid u,u',u'',u''' \textit{ abs. cts on } [0,2\pi]\}$$

with norm

$$|u|_{W^{4,1}} = \sum_{j=0}^{4} \int_0^{2\pi} |u^{(j)}(x)| dx.$$

For $u \in L^1[0,2\pi]$, let us write

$$\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(x)dx \text{ and } \tilde{u}(x) = u(x) - \bar{u},$$

so that

$$\int_0^{2\pi} \tilde{u}(x)dx = 0.$$

Let $\tilde{H}^2[0,2\pi] = \{u \in H^2[0,2\pi] \mid \bar{u} = 0\}$.

The following lemmas are proved in Gupta [2].

<u>Lemma 1:</u> Let $\Gamma \in L^1[0,2\pi]$ be such that for a.e. $x \in [0,2\pi]$,

$$\Gamma(x) \leq 1,$$

with strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Then there exists a $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) = \frac{1}{2\pi} \int_0^{2\pi} [(\tilde{u}''(x))^2 - \Gamma(x)\tilde{u}^2(x)]dx \geq \delta |\tilde{u}|_{H^2}^2 .$$

Lemma 2: Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty[0,2\pi]$, $\Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ is such that $\Gamma_0(x) \leq 1$ for a.e. $x \in [0,2\pi]$ with strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Let $\delta(\Gamma_0) > 0$ be given by Lemma 1. Then for every $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) \geq [\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] |\tilde{u}|_{H^2}.$$

Lemma 3: Let $\gamma \in L^1[0,2\pi]$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ be as in lemma 2 and $\delta(\Gamma_0)$ be given by lemma 1. Then for all measurable functions $p(x)$ on $[0,2\pi]$ with $\bar{\gamma} \leq \bar{p}$, $p(x) \leq \Gamma(x)$ for a.e. $x \in [0,2\pi]$, all continuous functions $f: \mathbf{R} \to \mathbf{R}$ and all $u \in W^{4,1}[0,2\pi]$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} [\bar{u} - \tilde{u}(x)] [-\frac{d^4u}{dx^4} + f(u(x))u'(x) + p(x)u(x)]dx$$

$$\geq \bar{\gamma}\, \bar{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] |\tilde{u}|_{H^2}^2$$

2. Main Results.

Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and let $g : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions, namely,

(i) for each $u \in \mathbf{R}$, the function $x \in [0,2\pi] \to g(x,u) \in \mathbf{R}$ is measurable on $[0,2\pi]$,

(ii) for a.e. $x \in [0,2\pi]$, the function $u \in \mathbf{R} \to g(x,u) \in \mathbf{R}$ is continuous on $\mathbf{R}$, and

(iii) for each $r > 0$, there exists a function $\alpha_r(x) \in L^1[0,2\pi]$ such that $|g(x,u)| \leq \alpha_r(x)$ for a.e. $x \in [0,2\pi]$ and all $u \in \mathbf{R}$ with $|u| \leq r$.

We prove the following existence theorem for the boundary value problem (1.1).

Theorem 1: Let $\gamma \in L^1[0,2\pi]$ with $\bar{\gamma} = 0$ and let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1[0,2\pi]$, $\Gamma_\infty \in L^\infty[0,2\pi]$, Γ_0 measurable on $[0,2\pi]$, $\Gamma_0(x) \le 1$ for a.e. $x \in [0,2\pi]$ with strict inequality holding on a subset of $[0,2\pi]$ of positive measure and $\frac{\pi^2}{3} |\Gamma_1| + |\Gamma_\infty| < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by Lemma 1. Assume that the inequalities

$$\gamma(x) \le \liminf_{|u| \to \infty} u^{-1} g(x,u) \le \limsup_{|u| \to \infty} u^{-1} g(x,u) \le \Gamma(x), \tag{2.1}$$

hold uniformly for a.e. $x \in [0,2\pi]$.

Also assume that there exists a $\rho > 0$ and a C^3-function $m: \mathbf{R}\ [-\rho,\rho] \to \mathbf{R}$ with $u\ m(u) > 0$, $m'(u) \le 0$, $m'''(u) \ge 0$, such that

$$\int_0^{2\pi} g(x,u(x)) m(u(x)) dx \ge 0 \tag{2.2}$$

for all C^3-real-valued functions $u(x)$ on $[0,2\pi]$ with u''' absolutely-continuous on $[0,2\pi]$, $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, and $\min_{x \in [0,2\pi]} |u(x)| \ge \rho$.

Then for every given $e(x) \in L^1[0,2\pi]$ with

$$\int_0^{2\pi} e(x) m(u(x)) dx \le 0, \tag{2.3}$$

for all C^3-real-valued functions $u(x)$ on $[0,2\pi]$ as above, the boundary-value problem (1.1) has at least one solution.

Proof: Let $\eta_0 = \frac{1}{2}[\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] > 0$. Then, for each η, $0 < \eta \le \eta_0$, we can find $r(\eta) > 0$ such that for a.e. $x \in [0,2\pi]$ and all u with $|u| \ge r(\eta)$, we have

$$\gamma(x) - \eta \le u^{-1} g(x,u) \le \Gamma(x) + \eta.$$

Define, $g_\eta{:}[0,2\pi] \times \mathbf{R} \to \mathbf{R}$ by $g_\eta(x,u) = \gamma_\eta(x,u)u$, where,

$$\gamma_\eta(x,u) = \begin{cases} u^{-1}g(x,u) \text{ if } |u| \ge r(\eta), \\ [r(\eta)]^{-1}g(x,r(\eta))[\frac{x}{r(\eta)}] + (1 - \frac{x}{r(\eta)})\Gamma(x) \text{ if } 0 \le x < r(\eta) \\ [r(\eta)]^{-1}g(x,-r(\eta))[\frac{x}{r(\eta)}] + (1 + \frac{x}{r(\eta)})\Gamma(x) \text{ if } -r(\eta) < x < 0, \end{cases}$$

so that g_η and γ_η satisfy Caratheodory's conditions and

$$\gamma(x) - \eta \le \gamma(x,u) \le \Gamma(x) + \eta, \tag{2.4}$$

for a.e. $x \in [0,2\pi]$ and all $u \in \mathbf{R}$. Let us, next, define $h_\eta{:}[0,2\pi] \times \mathbf{R} \to \mathbf{R}$ by

$$h_\eta(x,u) = g(x,u) - g_\eta(x,u).$$

Then, there exists $\alpha_\eta(x) \in L^1[0,2\pi]$, depending only on γ,Γ, and $\alpha_{r(\eta)}$, such that for a.e. $x \in [0,2\pi]$ and all $u \in \mathbf{R}$, we have

$$|h_\eta(x,u)| \le \alpha_\eta(x).$$

Now, the equation in (1.1) is equivalent to

$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + \gamma_\eta(x,u(x))u(x) + h_\eta(x,u(x)) = e(x)$$

We shall use the same degree arguments as the ones used in [2,3] to prove our theorem. Accordingly, it suffices to show that the set of all possible solutions of the family of equations

$$-\frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x)+\eta) + \lambda\gamma_\eta(x,u(x))]u(x) \tag{2.5}$$

$$+ \lambda h_\eta(x,u(x)) - \lambda e(x) = 0,$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

is, a priori, bounded in $C^1[0,2\pi]$ independently of $\lambda \in [0,1[$. Let, now, $u(x)$ be a possible solution of (2.5) for some $\lambda \in [0,1[$. Multiplying the equation in (2.5) by $(\overline{u} - \tilde{u}(x))$ we obtain, on integrating the resulting equation on $[0,2\pi]$ and using (2.4) along with Lemma 3, with Γ_∞ replaced by $\Gamma_\infty + \eta$ and γ replaced by $\gamma - \eta$,

$$0 = \frac{1}{2\pi}\int_0^{2\pi} [\overline{u} - \tilde{u}(x)]\{-\frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x)+\eta) + \lambda\gamma_\eta(x,u(x))]u(x)$$

$$+ \lambda h_\eta(x,u(x)) - \lambda e(x)\}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \{(\tilde{u}''(x))^2 - [(1-\lambda)(\Gamma(x)+\eta) + \lambda\gamma_\eta(x,u(x))]\tilde{u}^2(x)$$

$$+ [(1-\lambda)(\Gamma(x)+\eta) + \lambda\gamma_\eta(x,u(x)]\overline{u}^2$$

$$+ \lambda(\overline{u} - \tilde{u}(x))(h_\eta(x,\eta(x)) - e(x))\}dx$$

$$\geq \eta_0 \ |\tilde{u}|_{H^2}^2 - \eta\overline{u}^2 - (|\alpha_\eta|_{L^1} + |e|_{L^1})(|\overline{u}| + |\tilde{u}|_{L^\infty})$$

$$\geq \eta_0 \ |\tilde{u}|_{H^2}^2 - \eta\overline{u}^2 - \beta_\eta(|\overline{u}| + |\tilde{u}|_{H^2}),$$

where $\beta_\eta > 0$, is a constant depending on η and independent of $\lambda \in [0,1]$.

We next show that there exists a $\tau \in [0,2\pi]$ such that $|u(\tau)| < \rho$. Suppose, on the other hand, that $|u(x)| \geq \rho$ for all $x \in [0,2\pi]$. Integrating the equation in (2.4) over $[0,2\pi]$ after multiplying it by $m(u(x))$, we obtain, after noticing that

$$-\int_0^{2\pi} \frac{d^4u}{dx^4} m(u(x))dx = -\int_0^{2\pi} m'(u(x))(u''(x))^2dx + \int_0^{2\pi} m'''(u(x))\frac{(u'(x))^4}{3}dx \geq 0,$$

$$(1-\lambda)\int_0^{2\pi} (\Gamma(x) + \eta)u(x)m(u(x))dx + \lambda\int_0^{2\pi} g(x,u(x))m(u(x))dx - \lambda\int_0^{2\pi} e(x)m(u(x))dx \leq 0,$$

which is impossible, because the first term is positive, the second one is non-negative and the third non-positive, in view of our assumptions. Thus $|u(\tau)| < \rho$ for some

$\tau \in [0,2\pi]$, and if $\xi \in [0,2\pi]$ is such that $\overline{u} = u(\xi)$, we obtain

$$
\begin{aligned}
|\overline{u}| = |u(\xi)| &= |u(\tau) + \int_{\tau}^{\xi} u'(x)dx| \\
&< \rho + \sqrt{2\pi}(\int_0^{2\pi} |u'(x)|^2 dx)^{1/2} \qquad (2.7) \\
&\leq \rho + 2\pi |\tilde{u}|_{H^2}.
\end{aligned}
$$

Combining (2.6) and (2.7) with sufficiently small $\eta > 0$, we obtain that there is a constant C, independent of $\lambda \in [0,1[$ such that

$$|u|_{H^2} \leq C,$$

which implies that $|u|_{C^1[0,2\pi]} \leq C_1$, for some constant C_1, independent of $\lambda \in [0,1[$.

This completes the proof of the theorem.//

We present a few corollaries to Theorem 1 in the following.

Corollary 1: Let the function $m: \mathbf{R} \quad [-\rho,\rho] \rightarrow \mathbf{R}$ in Theorem 1 be given by

$$m(u) = sgn\ u = \frac{u}{|u|} \qquad (2.8)$$

for $u \in \mathbf{R}$, $|u| \geq \rho$.

Then for every $e \in L^1[0,2\pi]$ with $\overline{e} = 0$, the boundary value problem (1.1) has at least one solution.

Proof: The corollary is immediate as it is easy to see that all the assumptions of Theorem 1 remain valid.//

The following Corollary gives a necessary and sufficient condition for the boundary-value problem (1.1) to have a solution for a given $e \in L^1[0,2\pi]$ with $\overline{e} = 0$.

Corollary 2: Assume that $g(x,\cdot): \mathbf{R} \rightarrow \mathbf{R}$ is non-decreasing for a.e. $x \in [0,2\pi]$ and that condition (2.1) hold with γ and Γ as in Theorem 1.

Then for any given $e \in L^1[0,2\pi]$ with $\overline{e} = 0$, the boundary value problem (1.1) has a solution if and only if there exists a $y \in L^\infty[0,2\pi]$ such that

$$\int_0^{2\pi} g(x,y(x))dx = 0. \tag{2.9}$$

Proof: The necessity is immediate if we take for y a solution $u(x)$ and integrate the equation over $[0,2\pi]$. For sufficiency, let $\rho = |y|_{L^\infty}$ and $m(u) = \frac{u}{|u|}$. Then, if $u \in C^3[0,2\pi]$ with u''' absolutely continuous and

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

is such that

$$\min_{u \in [0,2\pi]} |u(x)| > \rho = |y|_{L^\infty} \tag{2.10}$$

we have, by monotonicity

$$\frac{u(x)}{|u(x)|} g(x,u(x)) \geq \frac{u(x)}{|u(x)|} g(x,y(x))$$

for a.e. $x \in [0,2\pi]$, and hence, as $\frac{u(x)}{|u(x)|}$ is a constant when (2.10) holds, we get using (2.9) that

$$\int_0^{2\pi} g(x,u(x)) \frac{u(x)}{|u(x)|} dx \geq 0,$$

for those u, and the result follows from Corollary 1.//

The following Corollary replaces the assumption (2.2) by an integral type asymptotic sign condition on $g(x,u)$.

Corollary 3: Assume that the assumptions of Theorem 1 hold except that (2.2) is replaced by the existence of $m : \mathbf{R}\ \{0\} \rightarrow \mathbf{R}$ of class C^3 with $u\ m(u) > 0$, $m'(u) \leq 0$, $m'''(u) \geq 0$ and of a function $\mu(x) \in L^1[0,2\pi]$ such that

$$\lim_{|u| \to \infty} \inf g(x,u)m(u) \geq \mu(x), \tag{2.11}$$

uniformly a.e. in $x \in [0,2\pi]$ and

$$\int_0^{2\pi} \mu(x)dx > 0.$$

Then for any given $e \in L^1[0,2\pi]$ such that there is a $\rho > 0$ with

$$\int_0^{2\pi} e(x)m(u(x))dx \leq 0, \tag{2.12}$$

for all $u \in C^3[0,2\pi]$ with u''' absolutely-continuous on $[0,2\pi]$, $u(0)-u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$ and $\min_{x \in [0,2\pi]} |u(x)| \geq \rho$; the boundary value problem (1.1) has at least one solution.

Proof: It suffices to check that (2.2) of Theorem 1 holds. This is identical to the proof of Corollary 1 of [3].//

Remark 1: We remark that we have treated the boundary value problem (1.1) with non-zero righthand side, unlike the problem in [3] where the corresponding second order problem is treated with zero right-hand side.

Remark 2: We note that the generality of the integral type asymptotic sign condition (2.2) on $g(x,u)$ imposed by the function $m(u)$ of (2.8), slightly limits the class of $e \in L^1[0,2\pi]$ for which (1.1) is solvable when compared to the class of e allowed by Theorem 2 of [2] under pointwise asymptotic sign condition on $g(x,u)$.

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