CONSTRUCTING UNIFORM ORIENTED MATROIDS WITHOUT THE ISOTOPY PROPERTY

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CONSTRUCTING UNIFORM ORIENTED MATROIDS WITHOUT THE ISOTOPY PROPERTY*

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Abstract. A simple procedure is given for producing a uniform rank 3 oriented matroid with disconnected realization space from a non–uniform example.

Very recently B. Jaggi and P. Mani-Levitska [4] solved the longstanding isotopy problem for simple line arrangements [7]. Using the well-known correspondence between line arrangements, order types in the projective plane [5] and realizable oriented matroids [2,3,6], their main theorem is stated as follows. We write $\mathcal{R}(M)$ for the space of all vector realizations $(x_1, \ldots, x_n) \in (\mathbf{R}^3)^n$ of a rank 3 oriented matroid M on n points. (In other words, $\mathcal{R}(M)$ is the set of $3 \times n$-matrices whose maximal minors have signs given by the alternating map $M : \{1, 2, \ldots, n\}^3 \to \{-, 0, +\}$.) If M is uniform (i.e. all minors are non-zero) then $\mathcal{R}(M)$ is an open subset of $\mathbf{R}^{3n}$. N. White's earlier paper [8] gives a non-uniform oriented matroid M_W with $\mathcal{R}(M_W)$ disconnected and $n = 42$, while the new uniform oriented matroid M_{JM} of Jaggi and Mani-Levitska [4] has $n = 17$ and $\mathcal{R}(M_{JM})$ disconnected. It is the objective of the present note to describe an easy general construction for uniform oriented matroids without the isotopy property.

A rank 3 oriented matroid M is said to be *constructible* if (x_1, x_2, x_3, x_4) is a projective basis and the point x_t is incident to at most two lines spanned by $\{x_1, x_2, \ldots, x_{t-1}\}$ for $t = 5, 6, \ldots, n$. Using the configuration $\lambda_1 = \Omega(17, 15, 13)[\lambda_0]$ in [4] or a similar modification of White's example [8], we easily get a constructible oriented matroid whose realization space has two connected components. For example, the space $\mathcal{R}(\lambda_1)$ modulo the connected group $PGL(\mathbf{R}^3)$ equals the set of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 5 & 5 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 2 & 2 & 2 & 4 & 6 & 0 & -1 & t & -t & t \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 6 & 5 & 1 & 0 & t-1 \end{pmatrix}$$

with $\frac{1}{5} < t < \frac{1}{2}(1 - \frac{1}{\sqrt{5}})$ or $\frac{1}{2}(1 + \frac{1}{\sqrt{5}}) < t < \frac{4}{5}$. Hence it suffices to prove the following

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THEOREM. *Let M be a constructible rank 3 oriented matroid on n points. Then there exists a uniform rank 3 oriented matroid $\widetilde{M}$ on at most $4(n-3)$ points and a continuous surjective map $\mathcal{R}(\widetilde{M}) \to \mathcal{R}(M)$. Hence $\mathcal{R}(\widetilde{M})$ is disconnected whenever $\mathcal{R}(M)$ is disconnected.*

Proof. We define a sequence $M =: M_n, M_{n-1}, M_{n-2}, \ldots, M_5, M_4 =: \widetilde{M}$ of oriented matroids and maps between their realization spaces. Let $n \geq t \geq 5$. Then M_{t-1} is constructed from M_t as follows. First suppose that x_t is incident to exactly two lines $x_i \vee x_j$ and $x_k \vee x_l$ with $1 \leq i, j, k, l < t$. Using the notation of Billera & Munson [1], we let M_t' be the oriented matroid obtained from M_t by the four successive *principal extensions*

(1) $x_{t,1} := [x_t^+, x_i^+, x_k^+]$, $x_{t,2} := [x_t^+, x_i^+, x_k^-]$, $x_{t,3} := [x_t^+, x_i^-, x_k^-]$, $x_{t,4} := [x_t^+, x_i^-, x_k^+]$.

These extensions can be carried out for every vector realization of M_t by setting $x_{t,1} := x_t + \epsilon_1 x_i + \epsilon_2 x_k$, $x_{t,2} := x_t + \epsilon_3 x_i - \epsilon_4 x_k$, $x_{t,3} := x_t - \epsilon_5 x_i - \epsilon_6 x_k$, $x_{t,4} := x_t - \epsilon_7 x_i + \epsilon_8 x_k$ where $1 \gg \epsilon_1 \gg \epsilon_2 \gg \ldots \gg \epsilon_8 > 0$. This implies that the *deletion map* $\Pi : \mathcal{R}(M_t') \to \mathcal{R}(M_t)$ is surjective. Geometrically speaking, in every affine realization of M_t', the intersection point x_t is "caught" in the quadrangle $(x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4})$. Define $M_{t-1} := M_t' \setminus x_t$ by deletion of that point, and let $\pi : \mathcal{R}(M_t') \to \mathcal{R}(M_{t-1})$ denote the corresponding map.

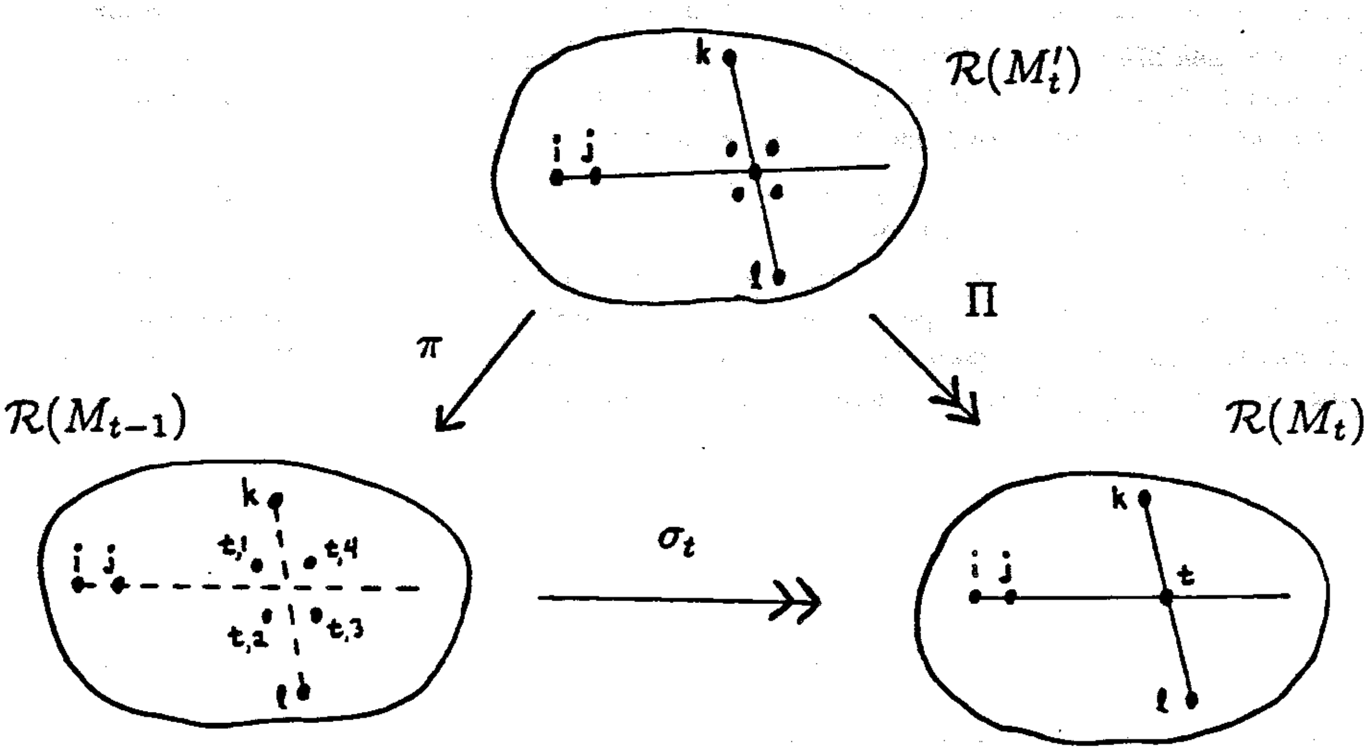

FIGURE. Illustration of the oriented matroids M_t, M_t' and M_{t-1}.

Next consider an arbitrary realization $X := (x_1, \dots, x_{t-1}, x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4}, x_{t+1,1}, \dots, x_{n,4})$ of M_{t-1}. As a consequence of the principal extension construction used in (1), $x_i \vee x_j$ and $x_k \vee x_l$ are the only lines spanned by $\{x_1, \dots, x_{t-1}, x_{t+1,1}, \dots, x_{n,4}\}$ which intersect the quadrangle $(x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4})$. For any other such line the intersection point $x_t := (x_i \vee x_j) \wedge (x_k \vee x_l)$ is on the same side as $x_{t,1}, \dots, x_{t,4}$. Therefore $\sigma_t(X) := (x_1, \dots, x_t, x_{t+1,1}, \dots, x_{n,4}) \in \mathcal{R}(M_t)$.

Hence we have a well-defined continuous map $\sigma_t : \mathcal{R}(M_{t-1}) \to \mathcal{R}(M_t)$, $X \mapsto \sigma_t(X)$. Moreover, σ_t is surjective because $\Pi = \sigma_t \circ \pi$ is surjective.

It remains to define M_{t-1} and σ_t when x_t is incident to less than two lines in M_t. If x_t is on no such line, then we define $M_{t-1} := M_t$ and σ_t as the identity map. Finally, suppose that x_t is on only one line $x_i \vee x_j, 1 \leq i,j < t$. In that case we replace (1) by setting $x_{t,1} := [x_t^+, x_k^+]$, $x_{t,2} := [x_t^+, x_i^+, x_k^-]$, $x_{t,3} := [x_t^+, x_i^-, x_k^-]$ for some $x_k \notin x_i \vee x_j$, and in the definition of the map σ_t we set $x_t := (x_i \vee x_j) \wedge (x_k \vee x_{t,1})$.

Iterating these constructions resolves all previous dependencies, and we obtain a uniform oriented matroid $\widetilde{M} := M_4$ on $4(n-3)$ or fewer points. Moreover, we have a continuous surjection $\sigma := \sigma_n \circ \sigma_{n-1} \circ \dots \circ \sigma_5$ from $\mathcal{R}(\widetilde{M})$ onto $\mathcal{R}(M)$. □

Some remarks.

(1) Using a fairly straightforward procedure for doubling oriented matroids, one gets the following corollary: *Given any integer C, there exists a uniform rank 3 oriented matroid $\widetilde{M}_C$ with $4(n-3)C$ points such that $\mathcal{R}(\widetilde{M}_C)$ has at least 2^C connected components.*

(2) The local modification $M_t \mapsto M_{t-1}$ is quite similar to a twofold application of the Ω-"opening" operation of Jaggi and Mani-Levitska which would produce precisely one of the points $x_{t,i}$. The crucial difference: the above opening operation splits each intersection point into four new points and thereby ensures the existence of a well-defined closing map $\sigma_t : \mathcal{R}(M_{t-1}) \to \mathcal{R}(M_t)$. For the Ω-operation the desired closing map $\mathcal{R}(\Omega(i,j,k)[M]) \to \mathcal{R}(M)$ does not exist in general.

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