EXTREMAL CODES ARE HOMOGENEOUS

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Abstract. We show that extremal codes are homogeneous. This implies that each punctured code of an extremal code has the same weight distribution which can be calculated directly from the weight distribution of the parent extremal code.

It has been observed that all punctured codes of a certain extremal code have the same weight distribution even though the extremal code in question does not have a transitive group. Our main theorem shows that this must be so for all extremal codes.

Let $\overline{C}$ be a code of length $n+1$ and let M_i be the matrix whose rows are the vectors in $\overline{C}$ of weight i. Then $\overline{C}$ is called *homogeneous* if, for any i, each column of M_i has the same weight. Any code with a transitive group is homogeneous. We show that there are other homogeneous codes.

The code C of length n obtained from $\overline{C}$ by deleting a fixed coordinate is called a *punctured code* of $\overline{C}$. Let $\underline{A}_i$ denote the number of vectors of weight i in $\overline{C}$ and $\underline{a_i}$ the number of vectors of weight i in C. The following proposition is due to Eugene Prange [3, Thm. 77].

PROPOSITION. *If $\overline{C}$ is homegeneous, then*

$$a_i = \frac{(n+1-i)}{(n+1)}\, A_i + \frac{(i+1)}{(n+1)}\, A_{i+1}\ .$$

From this proposition we can compute the weight distribution of any code which is punctured from a homogeneous code $\overline{C}$ when we know the weight distribution of $\overline{C}$.

An *extremal code* means one of the following codes:

1) An $\left(n+1, \frac{n+1}{2}, 4\left\lfloor \frac{n+1}{24} \right\rfloor + 4\right)$ doubly–even code. These can exist only when $n+1 \equiv 0 \pmod 8$.
2) An $\left(n+1, \frac{n+1}{2}, 3\left\lfloor \frac{n+1}{12} \right\rfloor + 3\right)$ ternary self–dual code. Here $n+1$ must be $\equiv 0 \pmod 4$.
3) An $\left(n+1, \frac{n+1}{2}, 2\left\lfloor \frac{n+1}{6} \right\rfloor + 2\right)$ quaternary self–dual code. Here $n+1$ must be even.

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Orthogonality in the first two cases is with respect to the usual inner product. In the third case the inner product of two vector's $x = (x_1 \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is defined to be $\sum_{i=1}^{n} x_i y_i^2$ and orthogonality is with respect to this inner product. Extremal codes are self-dual codes with the largest possible minimum weights [3, Thm. 81 Cor; 4]. It is well-known that in the first case all weights in a doubly-even code are divisible by 4, in the second case all weights are divisible by 3, and in the third case all weights are even. We demonstrate our main theorem for these three cases.

THEOREM. *Extremal self-dual codes are homogeneous.*

Proof. Case 1) is the easiest. The Assmus–Mattson theorem [3, Thm. 89] shows that the vectors of any weight in an extremal doubly-even code hold a one-design. This is equivalent to the code being homogeneous.

We have to work harder for the other two cases. We show first that every punctured code has the same weight distribution.

Let $\overline{C}$ be a ternary self-dual $(n+1, \frac{n+1}{2}, 3\left\lfloor \frac{n+1}{12} \right\rfloor + 3)$ code and C a punctured $(n, \frac{n+1}{2})$ code. Consider the subcode D of $\overline{C}$ consisting of all vectors in $\overline{C}$ with zero on the deleted coordinate. It is not hard to see that when we puncture D on this coordinate we get $C^{\perp}$. The number of non-zero $A_i, i \neq 0$, in $\overline{C}$ is $\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{12} \right\rfloor$. Let b_i be the number of vectors of weight i in $C^{\perp}$. Then the number of non-zero b_i , $i \neq 0$, in $C^{\perp}$ is at most $\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{12} \right\rfloor$. Clearly, the minimum weight of C is at least $3\left\lfloor \frac{n+1}{12} \right\rfloor + 2$. Since

$$\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{12} \right\rfloor \leq 3\left\lfloor \frac{n+1}{12} \right\rfloor + 2 \quad \text{for} \quad n+1 \equiv 0 \pmod 4$$

there is a unique solution for the weight distribution of $C^{\perp}$ [3, Thm. 80]. Hence there is a unique solution for the weight distribution of C [3, Thms. 78, 79].

If M_i is the matrix of all vectors in $\overline{C}$, then the number of non-zero entries in the column of M_i corresponding to the deleted coordinate is a_{i-1} since all weights in $\overline{C}$ are divisible by 3 and a vector of weight $i-1$ in C could only have come from a vector of weight i in $\overline{C}$ with a non-zero component in the deleted position. Hence $\overline{C}$ is homogeneous.

Finally, let $\overline{C}$ be an $(n+1, \frac{n+1}{2}, 2\left\lfloor \frac{n+1}{6} \right\rfloor + 2)$ quaternary self-dual code. A punctured code C is an $(n, \frac{n+1}{2})$ code. Again $C^{\perp}$ is the punctured code of the subcode D of $\overline{C}$ defined as before. The number of non-zero b_i (defined as above), $i \neq 0$, is at most $\frac{n+1}{2} - \left\lfloor \frac{n+1}{6} \right\rfloor - 1$. The minimum weight of C is at least $2\left\lfloor \frac{n+1}{6} \right\rfloor + 1$. As before we have a unique solution for the weight distribution of $C^{\perp}$, hence for the weight distribution of C, since

$$\frac{n+1}{2} - \left\lfloor \frac{n+1}{6} \right\rfloor - 1 \leqq 2 \left\lfloor \frac{n+1}{6} \right\rfloor + 1 \quad \text{for} \quad n+1 \quad \text{even}.$$

Again a column of M_i has a_{i-1} non–zero elements since all vectors in $\overline{C}$ have even weight and a vector of weight $i-1$ in C must have come from a vector of weight i in $\overline{C}$ with a non–zero component in the deleted coordinate. Hence $\overline{C}$ is homogeneous. □

If a code has a transitive monomial group it follows that every punctured code has the same weight distribution but only if the the code has a transitive permutation group can one deduce that every punctured code has the same complete weight distribution. It was demonstrated in [2] that every punctured code of any extremal ternary (24, 12, 9) code which contains the all–ones vector $\underline{h} = (1, \ldots, 1)$ has the same weight distribution (this follows from our theorem) and also the same complete weight distribution. It was further shown in [2] that one of these codes, the (24, 12, 9) symmetry code does not have a transitive permutation group even though it has a transitive monomial group. In [2] it was demonstrated that every code punctured from the (36, 18, 12) or (48, 24, 15) symmetry and extended quadratic residue codes have the same complete weight distributions. In these cases it is not known whether these codes have transitive permutation groups although they do have transitive monomial groups.

In [1] the authors show that a particular (18, 9, 8) extremal quaternary code $\overline{C}$ does not have a transitive group yet every punctured code of $\overline{C}$ has the same weight distribution (which also follows from our theorem) and the same complete weight distribution. These observations lead to the following conjecture.

Conjecture. 1) Every punctured code of an extremal $(n+1, \frac{n+1}{2}, 3\left\lfloor \frac{n+1}{12} \right\rfloor + 3)$ ternary self–dual code which contains h and where $n+1 \equiv 0 \pmod{12}$ has the same complete weight distribution.

2) Every punctured code of an extremal $(n+1, \frac{n+1}{2}, 2\left\lfloor \frac{n+1}{6} \right\rfloor + 2)$ quaternary self–dual code which contains h and where $n+1 \equiv 0 \pmod 6$ has the same complete weight distribution.

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