THE BOUNDED DIAMETER TWO EDGE–DISJOINT SPANNING TREES PROBLEM IS NP–COMPLETE

By

Paul Lemke

IMA Preprint Series # 427

July, 1988

THE BOUNDED DIAMETER TWO EDGE-DISJOINT SPANNING TREES PROBLEM IS NP-COMPLETE

PAUL LEMKE*

Abstract. Complexity results are given for the BOUNDED DIAMETER TWO EDGE-DIJOINT SPANNING TREES problem, which is related to communication network reliability.

The BOUNDED DIAMETER TWO EDGE-DISJOINT SPANNING TREES problem is derived from a model for the reliability of a communications network, where one would like to estimate the probability of the failure of the network given the failure probabilities of each of its links. One measure of the smallness of this probability is the minimum value of an integer d such that the underlying graph contains two edge-disjoint spanning trees of diameter d or less [1].

I will show that the problem:

INSTANCE: A graph G, and positive integers d_1 and d_2 with $d_2 \geq d_1$.

QUESTION: Does G posess two edge-disjoint spanning trees T_1 and T_2 such that the diameter of T_k is less than or equal to d_k for $k = 1, 2$?

is NP-complete for fixed values of d_1 and d_2 whenever $d_1 \geq 4$ (assuming edge lengths of 1). The existence of a polynomial-time algorithm will be demonstrated when $d_1 \leq 3$.

The NP-completeness will be shown by reduction from SET SPLITTING:

INSTANCE: Positive integers n, m, subsets $S_1, S_2, \ldots, S_m$ of $\{1, 2, \ldots, n\}$.

QUESTION: Do there exist sets A_1 and A_2 such that $A_1 \cup A_2 = \{1, 2, \ldots, n\}$, $A_1 \cap A_2 = \phi$, and $A_1 \cap S_i \neq \phi$ and $A_2 \cap S_i \neq \phi$ for $i = 1, 2, \ldots, m$?

which is NP-complete ([2]).

Given an instance of SET SPLITTING, we construct a graph G as follows (see Figure 1a):

The vertex set V of G contains the vertices $u_1, u_2, \ldots, u_m$, $v_1, v_2, \ldots, v_n$, w_1, w_2, z_1, z_2, c, and $d_1 + d_2 - 5$ "tail vertices" ($d_1 + d_2 - 4$ if $d_1 = 4$).

The edge set E of G is defined as follows:

$$(u_i, v_j) \in E \text{ if and only if } j \in S_i, \ \forall i, j$$
$$(v_j, w_k) \in E \quad \forall j, k$$
$$(v_j, c) \in E \quad \forall j$$
$$(w_1, w_2), (z_1, z_2), (w_1, z_1), (w_2, z_2) \in E$$
$$(w_1, c), (w_2, c), (z_1, c), (z_2, c) \in E$$

*Institute for Mathematics and its Applications, 514 Vincent Hall, 206 Church St. S.E., University of Minnesota, Minneapolis, MN 55455

Finally, the edges between c and the tail vertices, and those between tail vertices are as indicated in Figures 1 a-c (the edge *colorings* given for the tails and for the edges involving the w or z vertices will be explained later).

I will now show that G posesses two edge-disjoint spanning trees T_1 and T_2 with diameters less than or equal to d_1 and d_2, respectively, if and only if the answer to the SET SPLITTING question is "yes".

If the answer to SET SPLITTING is "yes", then let the sets A_1 and A_2 be as indicated in SET SPLITTING, and define (disjoint) sets $E_1, E_2 \subseteq E$ as follows:

$$(v_j, c) \in E_k \;\; \forall\, j \in A_k, \;\; k = 1, 2;$$
$$(u_i, v_j) \in E_k \text{ if } (u_i, v_j) \in E \text{ and } j \in A_k, \;\; \forall\, i, \;\; k = 1, 2.$$

Finally, for edges involving tail, w, and z vertices, assign those edges colored blue in Figure 1 to E_1 and those colored red to E_2.

Denote by G_k the graph with vertex set V and edge set E_k, for $k = 1, 2$. Since $S_i \cap A_k \neq \phi$ for each $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2\}$, there exist $j_1, j_2 \in \{1, 2, \ldots, n\}$ for each i such that $j_1, j_2 \in S_i$, and $j_k \in A_k$ for $k = 1, 2$. Therefore we have (u_i, v_{j_k}), $(v_{j_k}, c) \in E_k$ for $k = 1, 2$, and thus there is a length 2 path between u_i and c for each i in each of the (edge disjoint) graphs G_1, G_2. Also, there is a path of length 2 between c and v_j (Passing through w_k) in G_k, $\forall j$ and k, and there are paths of length 2 or less in G_k between c and each of the four vertices w_1, w_2, z_1 and z_2 for $k = 1, 2$. From the way the tails are edge-colored we see that the distance in G_k from c to any tail point is $\leq d_k - 2$, $k = 1, 2$. Finally, we can see by inspection that the distance in G_k between any two tail points is always $\leq d_k$, for $k = 1, 2$. It is well-known that every connected graph posesses a spanning tree which preserves distances to any given vertex (in our case, c). Thus we can find spanning trees T_1 and T_2 of G_1 and G_2, respectively, which preserve distances to c. Moreover, since there are no cycles involving edges incident on tail vertices in either G_1 or G_2, no tail edges can occur in G_k that do not occur in T_k, for $k = 1, 2$, so that distances between tail vertices are preserved. The desired diameter bounds on T_1 and T_2 then follow.

Conversely, assume that the answer to the SET SPLITTING question is "no". Let T_1 and T_2 be edge-disjoint spanning trees of G. Let E_1 be the edge set of T_1, and let $E_2 = E - E_1$ (so that E_2 contains the edges of T_2). Define:

$$A_k = \{j \,|\, (v_j, c) \in E_k\} \text{ for } k = 1, 2.$$

Then we have $A_1 \cup A_2 = \{1, 2, \ldots, n\}$ and $A_1 \cap A_2 = \phi$, and since the answer to the SET SPLITTING question is "no", there must exist $i_0 \in \{1, 2, \ldots, m\}$ and $k_0 \in \{1, 2\}$ such that $S_{i_0} \cap A_{k_0} = \phi$. Defining $G_k = (V, E_k)$ for $k = 1, 2$, we see that there is no path of length 2 from u_{i_0} to c in G_{k_0}, since every vertex adjacent to u_{i_0} in G_{k_0} is contained in $\{v_j | j \in S_{i_0}\}$ and therefore not adjacent to c in G_{k_0}. Thus the distance from u_{i_0} to c in G_{k_0} is at least 3, while the distance from u_{i_0} to c in G_{3-k_0} is clearly at least 2. Let T denote the set of tail vertices of G, and for any vertices x and y in V let $D_k(x, y)$ denote the distance between x and y in G_k, for $k = 1, 2$. From the way the tails are constructed it is an easy exercise to verify that either:

$$\begin{aligned}
&\max_{t \in T} D_1(c, t) > d_1 - 2, \\
\text{or } &\max_{t \in T} D_2(c, t) > d_2 - 2, \\
\text{or } &\max_{t \in T} D_1(c, t) \geq d_1 - 2 \text{ and } \max_{t \in T} D_2(c, t) \geq d_2 - 2,
\end{aligned}$$

regardless of how the tail edges are distributed among E_1 and E_2. Since any path between u_{i_0} and a tail vertex must pass through c, there must exist $t \in T$ such that either $D_1(u_{i_0}, t) > d_1$ or $D_2(u_{i_0}, t) > d_2$. Since T_k is a spanning tree of G_k for $k = 1, 2$, these distance inequalities hold on T_1 and T_2 as well, so that one of the diameter bounds is violated. □ □

I will now show that a polynomial time algorithm exists to solve the BOUNDED DIAMETER TWO EDGE-DISJOINT SPANNING TREES problem when $d_1 \leq 3$.

Given the graph $G = (V, E)$, let E_k denote the edge set of T_k, $k = 1, 2$, where T_1 and T_2 are the desired spanning trees. First we note that if $d_1 = 1$ or 2 and G has no multiple edges, there can be no solution unless G is the trivial graph of one vertex. Otherwise, for $d_1 = 3$, T_1 must contain two "center vertices" u_1 and u_2 such that every vertex in $V - \{u_1\} - \{u_2\}$ is adjacent to either u_1 or u_2 in T_1, and u_1 and u_2 are adjacent to each other in T_1. Given the vertices u_1 and u_2, we may partition $V - \{u_1\} - \{u_2\}$ into 3 sets, V_1, V_2, and V_3, defined by:

$$\begin{aligned}
V_1 &= \{v | (u_1, v) \in E,\ (u_2, v) \notin E\}; \\
V_2 &= \{v | (u_1, v) \notin E,\ (u_2, v) \in E\}; \\
\text{and } V_3 &= \{v | (u_1, v), (u_2, v) \in E\}.
\end{aligned}$$

Clearly we must have $(u_k, v) \in E_1$ whenever $v \in V_k$, for $k = 1, 2$, in order to insure that every vertex in $V_1 \cup V_2$ is adjacent to either u_1 or u_2 in T_1. Also, if $v \in V_3$, then we must have either (u_1, v) or (u_2, v) in E_1 but not both (since $(u_1, u_2) \in E_1$, and T_1 cannot have a cycle). This accounts for all edges in E_1, so we are left with $2^{|V_3|}$ possibilities for T_1, all of which have the desired diameter of 3.

The tree T_2 must also have a center consisting of either one vertex, or two vertices adjacent in T_2, depending on whether d_2 is even or odd, respectively. Every vertex in V will be within a distance of $\lfloor \frac{d_2}{2} \rfloor$ of some vertex in the center of T_2. Given the tree T_1 and the center of the tree T_2 we can readily resolve the problem by calculating the distance in $(V, E - E_1)$ of all vertices in V from the center of T_2. If all distances are $\leq \lfloor \frac{d_2}{2} \rfloor$, then a spanning tree of $(V, E - E_1)$ must exist realizing these distances, and we have a solution. If one of the distances is $> \lfloor \frac{d_2}{2} \rfloor$ there is no such solution for that particular center of T_2.

Given the center of T_2, denote by $D_2(v)$ the distance in $(V, E - E_1)$ from the center of T_2 to any vertex $v \in V$. Let $k \in \{1,2\}$ be such that $D_2(u_k) \geq D_2(u_{3-k})$. If $(c, \ldots, p, u_k)$ is the sequence of vertices in a shortest path in $(V, E - E_1)$ from a center vertex of T_2 to u_k, then since this implies $(p, u_k) \in E - E_1$ we must have $p \in V_3$, since $(v, u_k) \in E_1$ for $v \in V - V_3 - \{u_k\}$ (we are assuming that G has no multiple edges). Choose $p' \in V_3 - \{p\}$, and assume that $(p', u_k) \in E - E_1$, so that $(p', u_{3-k}) \in E_1$. Now define E_1' by:

$$E_1' = E_1 \cup \{(p', u_k)\} - \{(p', u_{3-k})\},$$

and define $D_2'(v)$ to be the distance in $(V, E - E_1')$ from the center of T_2 to any $v \in V$. I will now show that $D_2'(v) \leq D_2(v)$ for all $v \in V$. Since $E - E_1'$ has all the elements of $E - E_1$ except (p', u_k), we need only show that a path of length ℓ in $(V, E - E_1)$ from the center of T_2 to v which uses (p', u_k) implies the existence of a path in $(V, E - E_1')$ of length $\leq \ell$ from the center of T_2 to v. If the path in $(V, E - E_1)$ using (p', u_k) passes through p' and u_k in that order going from the center of T_2 to v, then we may reroute the first part of this path (the part from the center of T_2 to u_k) through p using the minimum length path from the center of T_2 to v_k postulated above to exist in $(V, E - E_1)$ (this path clearly does not use (p', u_k) and so also exists in $(V, E - E_1')$. If the path in $(V, E - E_1)$ using (p', u_k) passes through u_k and p' in that order going from the center of T_2 to v, then we may reroute the first part of this path (the part from the center of T_2 to p') through u_{3-k} using (u_{3-k}, p') and the path from the center of T_2 to u_{3-k} in $(V, E - E_1)$ (which has length no longer than the path in $(V, E - E_1)$ from the center of T_2 to u_k and which does not use (p', u_k), because $D_2(u_{3-k}) \leq D_2(u_k)$). Either way we get a path from the center of T_2 to v_k in $(V, E - E_1')$ which is no longer than the corresponding path in $(V, E - E_1)$. We therefore have $D_2'(v) \leq D_2(v)$ for all $v \in V$ and we conclude that we can assume without loss of generality that $(p', u_k) \in E_1$ for all $p' \in V_3 - \{p\}$. Since there are only $|v_3|$ possible locations for p, given k, u_1, u_2, and the center of T_2, and since there are only 2 possible values for k, $\binom{|V|}{2}$ possible locations for u_1 and u_2, and at most $\binom{|V|}{2}$ possible locations for the center of T_2, we need consider no more than $2|V_3|\binom{|V|}{2}^2 \leq \frac{1}{2}|V|^5$ cases (each one involving finding all the distances in a graph of $|V|$ vertices from a given one or two vertices). We conclude that the BOUNDED DIAMETER TWO EDGE-DISJOINT SPANNING TREE problem is in P when $d_1 = 3$. □ □

COMMENT. Note that the bound above depends only on $|V|$ and not on d_2.

COMMENT. Of course, the NP-completeness result applies automatically to graphs in which multiple edges are allowed. The polynomial time algorithm can also be extended to such graphs with only minor modifications.

COMMENT. For the NP-completeness proof, the graph G can be fairly sparse because the SET SPLITTING problem is NP-complete even when $|S_1| = |S_2| = \cdots = |S_m| = 3$.

COMMENT. I have a polynomial time algorithm for the following problem:

INSTANCE: A graph G, distinct vertices v_1 and v_2 of G.

QUESTION: Are there edge-disjoint spanning trees T_1 and T_2 of G such that all distances to v_k in T_k are ≤ 2 for $k = 1, 2$?

However, the case where $v_1 = v_2$ is NP-complete by the argument given for the general BOUNDED DIAMETER TWO EDGE-DISJOINT SPANNING TREES problem, as it applies to the graph in Figure 1a, where $v_1 = v_2 = c$.

REFERENCES

[1] F. CHUNG, *Combinatorial problems arising in network optimization*, talk given at Institution for Mathematics and its Applications, Minneapolis, Minnesota, February 26, 1988.

[2] L. LOVASZ, *Coverings and colorings of hypergraphs*, Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Publishing, Winnipeg, 1973, pp. 3–12.