A MINIMAL CUTSET OF THE BOOLEAN LATTICE
WITH ALMOST ALL MEMBERS

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Abstract. Two almost explicit constructions are given satisfying the title.

1. Preliminaries. Let \([n]\) denote the set of the first \(n\) positive integers, \(2^{[n]}\) its power set. Sometimes \(2^{[n]}\) will be called the Boolean lattice and denoted by \(B_n\). The collection of all \(k\)-subsets of a set \(S\) is denoted by \({S \choose k}\). A family \(L = \{L_0, L_1, \ldots, L_t\} \subset 2^{[n]}\) is called a chain if its members contain each other, \(L_0 \subset L_1 \subset \cdots \subset L_t\). Such a chain is maximal if \(t = n\), in which case \(|L_i| = i\) for all \(i\). The family \(C \subset 2^{[n]}\) is a cutset of the Boolean lattice if \(C \cap L \neq \emptyset\) for all maximal chains \(L\). A minimal cutset \(C\) is a cutset with the property that for every \(C \subset C\) some maximal chain avoids \(C \setminus \{C\}\). For example the whole \(k\)-th level of the Boolean lattice \({[n] \choose k}\) is a minimal cutset. But there are minimal cutsets of much larger size, e.g. the following family

\[(1.1) \quad \{C \subset [n] : |C \cap \{1, 2\}| = 1\}

has size \(2^{n-1}\). Denote the maximum size of a minimal cutset of \(B_n\) by \(c(n)\). Ko-Wei Lih asked whether \(c(n) = 2^{n-1}\) in general. It is easy to see that

\[(1.2) \quad c(n + 1) \geq 2c(n).

(Indeed, if \(C\) is a minimal cutset of \(B_n\) then \(C \cup \{C \cup \{n + 1\} : C \subset C\}\) is a minimal cutset of \(B_{n+1}\).) The inequality \((1.2)\) implies that there is a limit of the sequence \(c(n)/2^n\) whenever \(n\) tends to infinity. This limit is at least 1/2 by \((1.1)\). In [L] Ko-Wei Lih gives a construction for \(n = 6\) due to Jin-Fa Chern in which \(|C| = 33 > 2^{n-1}\). (Unfortunately, his example contains a misprint. To fix it, the set \(\{1, 2, 4, 5, 6\}\) should be replaced by \(\{1, 2, 3, 5, 6\}\).) It is natural to ask whether the answer is asymptotic to \(2^n\). In this note we give an almost explicit construction proving that

\[(1.3) \quad \lim_{n \to \infty} \frac{c(n)}{2^n} = 1.

"Almost explicit" means that we will define a large cutset (of size \((1 - o(1))2^n\)) and prove that by deleting only \(o(2^n)\) members of it one can obtain a minimal cutset.

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4Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA. Research supported in part by NSF grant DMS 86-06225 and Airforce Grant OSR-86-0076.
2. An Almost Deterministic Construction. Let \( k \geq 3 \) be an integer, and suppose that \( n \) is divisible by \( k \). Let \( S_1 \cup \ldots \cup S_{n/k} \) be a partition of \( [n] \) into \( k \)-element parts. Define the family \( \mathcal{C} \) as follows.

\[
\mathcal{C} = \{ C \subseteq [n] : 0 < |S_i \cap C| < k \text{ for all } S_i \} \cup \{ C \subseteq [n] : \exists S_i, S_j \text{ with } |S_i \cap C| = 1, |S_j \cap C| = k - 1 \}.
\]

We claim that \( \mathcal{C} \) is a cutset. Indeed, if \( \emptyset = L_0 \subset L_1 \subset \ldots \subset L_n \) is a maximal chain then define \( t \) as the largest integer such that \( L_t \) is still disjoint from some \( S_i \). Then \( L_{t+1} \) intersects all \( S_i \). If \( L_{t+1} \) does not contain any \( S_j \), then \( L_t \) belongs to the first part of \( \mathcal{C} \). If \( L_{t+1} \) contains some \( S_j \), then \( L_t \) belongs to the second part of \( \mathcal{C} \).

A member \( C \) of a cutset \( \mathcal{C} \) is essential if \( \mathcal{C} \setminus \{ C \} \) is not cutset. Define

\[
\mathcal{C}_0 = \{ C \subseteq [n] : 0 < |S_i \cap C| < k \text{ for all } S_i \text{ and } \exists S_i, S_j \text{ with } |S_i \cap C| = 1, |S_j \cap C| = k - 1 \}.
\]

We claim that every member of \( \mathcal{C}_0 \) is essential in \( \mathcal{C} \). Indeed, if \( C \in \mathcal{C}_0 \) with \( |S_i \cap C| = 1 \) and \( |S_j \cap C| = k - 1 \), then every maximal chain containing \( \mathcal{C} \setminus \{ C_i, C \} \) and \( C \cup S_j \) avoids \( \mathcal{C} \setminus \{ C \} \).

Starting with an arbitrary cutset one can always obtain a minimal cutset by deleting the unnecessary members one by one. But we can never delete an essential set. So all minimal cutsets contained in \( \mathcal{C} \) contain \( \mathcal{C}_0 \). We have

\[
|\mathcal{C}_0| = (2^k - 2)^{n/k} - 2(2^k - k - 2)^{n/k} + (2^k - 2 - 2)^{n/k} > 2^n((1 - \frac{2}{2^k})^{n/k} - 2(1 - \frac{k + 2}{2k})^{n/k} > 2^n(1 - \frac{k}{2^n} - 2\exp[-\frac{n}{2^n}]).
\]

Here we used the inequalities \( (1 - x)^y \leq \exp[-xy] \), which holds for \( -\infty < x \leq 1 \) and \( y \geq 0 \), and \( 1 - xy \leq (1 - x)^y \), which holds for \( 0 \leq x \leq 1 \) and \( y \geq 1 \). If \( n \sim 2^k \log k \), (i.e., \( k \sim \log n - \log \log \log n \)) then the (2.1) gives the following.

**Corollary 2.1.** For sufficiently large \( n \)

\[
c(n) > 2^n(1 - \frac{4\log \log n}{\log n}).
\]

We shall improve this result in Theorem 4.1.

3. Filters and Ideals. A subfamily \( \mathcal{F} \) of \( 2^{[n]} \) is called a filter if \( F \in \mathcal{F} \) and \( F \subseteq F' \subseteq [n] \) imply \( F' \in \mathcal{F} \). Starting with any subfamily \( S \subseteq 2^{[n]} \) one can obtain a filter \( \mathcal{F}(S) \) as follows. \( \mathcal{F}(S) = \{ F \subseteq [n] : \exists S \subseteq S \text{ such that } S \subseteq F \} \). \( \mathcal{F}(S) \) is the filter induced by \( S \).

A family \( \mathcal{I} \) is called an ideal if \( I \in \mathcal{I} \) and \( I' \subseteq I \) imply \( I' \in \mathcal{I} \) as well. For an arbitrary family \( S \subseteq 2^{[n]} \) we associate an ideal \( \mathcal{I}(S) \) in the following way. \( \mathcal{I}(S) = \{ I \subseteq [n] : \exists S' \subseteq S \text{ such that } I \cap S = \emptyset \} \). \( \mathcal{I}(S) \) is the ideal induced by \( S \). (Warning! This definition differs from the usual one.) In this way \( \mathcal{F}(S) \) and \( \mathcal{I}(S) \) consist of complementary pairs, i.e. \( A \in \mathcal{F}(S) \) if and only if \( ([n] \setminus A) \in \mathcal{I}(S) \).

The neighborhood \( \mathcal{N}(S) \) of a family \( S \) is defined as the family of those subsets in \( [n] \) whose Hamming distance from \( S \) is exactly 1, i.e. \( \mathcal{N}(S) = \{ N \subseteq [n] : N \notin S \text{ and } \exists G \in S \text{ such that } |N \triangle G| = 1 \} \). Note that \( N \cap \mathcal{N}(S) = \emptyset \). The complement \( \overline{S} \) of the family \( S \) is defined as \( \overline{S} = 2^{[n]} \setminus S \). The following idea underlies the construction in Section 2.
Observation 3.1. Suppose that $\mathcal{I}$ is an ideal and $\mathcal{F}$ is a filter such that there are no two sets $I \in \mathcal{I} \setminus \mathcal{F}$ and $F \in \mathcal{F} \setminus \mathcal{I}$ such that

$$I \subseteq F \text{ and } |F \setminus I| = 1.$$  

Then $\mathcal{C} = (\overline{\mathcal{I}} \cap \overline{\mathcal{F}}) \cup (\mathcal{I} \cap \mathcal{F})$ is a cutset. Moreover, all members of $N(\mathcal{I}) \cap N(\mathcal{F})$ are essential. $\square$

If we use an arbitrary family $\mathcal{S}$ to induce an ideal and a filter, then we obtain

**Lemma 3.2.** If for every $S$ and $S' \in \mathcal{S}$ one has $|S \cap S'| \neq 1$, then the ideal $\mathcal{I}(\mathcal{S})$ and the filter $\mathcal{F}(\mathcal{S})$ fulfill Observation 3.1.

**Proof.** Indeed, if $F \in \mathcal{F}(\mathcal{S}) \setminus \mathcal{I}(\mathcal{S})$ then there exists an $S_1 \in \mathcal{S}$ such that $S_1 \subseteq F$ and $F$ intersects all members of $\mathcal{S}$. Moreover if $I \in \mathcal{I}(\mathcal{S}) \setminus \mathcal{F}(\mathcal{S})$ then there exists an $S_2 \in \mathcal{S}$ such that $S_2 \cap I = \emptyset$ and $I$ does not contain any member of $\mathcal{S}$. So in this case $|F \setminus I| = 1$ would imply $S_1 \cap S_2 = F \setminus I$, a contradiction. $\square$

4. A Random Construction. In view of Lemma 3.2, all that we need in order to construct a large minimal cutset is to find a suitable family $\mathcal{S}$ that has a filter $\mathcal{F}(\mathcal{S})$ with a big neighborhood. In this section we describe a random family $\mathcal{S}$ satisfying

$$|S \cap S'| \neq 1,$$

such that for some positive constant $c$

$$|N(\mathcal{F}(\mathcal{S}))| > 2^n(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}).$$

Of course, the same lower bound holds for $|N(\mathcal{I}(\mathcal{S}))|$ as well, thus

$$|N(\mathcal{F}(\mathcal{S})) \cap N(\mathcal{I}(\mathcal{S}))| > 2^n(1 - 2c\frac{(\log n)^{3/2}}{\sqrt{n}}).$$

So Lemma 3.2 yields that $\mathcal{C} = (\overline{\mathcal{I}(\mathcal{S})} \cap \overline{\mathcal{F}(\mathcal{S})}) \cup (\mathcal{I}(\mathcal{S}) \cap \mathcal{F}(\mathcal{S}))$ is a cutset with a large number of essential sets.

**Theorem 4.1.** There exists a $c > 0$ such that $c(n) > 2^n(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}).$

**Proof.** To find such a family $\mathcal{S}$ our method is a modified version of what was used in [FKK] and in [K] to construct a small filter with large neighborhoods. Suppose that $n$ is
divisible by 8, and let $B_1 \cup \ldots \cup B_{n/2}$ be a partition of the underlying set into pairs. Let $k$ be an integer $k \sim \sqrt{n/\log n}$. For every $K \in \binom{[n/2]}{k}$ let $\xi_K$ be a random variable with

$$\Prob(\xi_K = 1) = \left( \frac{1000 \log n}{\sqrt{n}} \right)^{3/2} \left( \frac{n/8}{k} \right)^{-1} = p$$

$$\Prob(\xi_K = 0) = 1 - p.$$ 

These random variables are to be chosen totally independently. Let $S$ be the random family defined by

$$S = \{ \cup_{i \in K} B_i : \xi_K = 1 \}.$$ 

Of course, $S$ satisfies (4.1). We next show that the expected size of $N(\mathcal{F}(S))$ is as large as it was given in (4.2). This implies the existence of a family $S$ which fulfills both (4.1) and (4.2), proving Theorem 4.1.

Let $N$ be an arbitrary but fixed member of $2^{[n]}$. Denote the number of blocks $B_i$ which are contained in $N$ by $n_2$, and let $N_2 = \{ i : B_i \subset N \}$. Similarly, let $N_1 = \{ i : |B_i \cap N| = 1 \}$, and $|N_1| = n_1$. We give an exact formula for the probability that $N$ belongs to $N(\mathcal{F}(S))$. $N$ belongs to $N(\mathcal{F}(S))$ if and only if $\xi_K = 0$ for all $K \in \binom{N_2}{k}$ and $\xi_K = 1$ for some $k$-set $K$ with $|K \setminus N_2| = 1$ and $(K \setminus N_2) \subset N_1$. Since the variables $\xi_K$ are independent, we obtain that

$$\Prob(N \in N(\mathcal{F}(S))) = (1 - p)^{n_2} (1 - (1 - p)^{n_1}) \binom{n_2}{k}$$

$$\geq (1 - p^{n_2}) (1 - \exp[-pn_1 \binom{n_2}{k-1}])$$

Now suppose that $N$ is a typical member of $B_n$. More exactly, define the collection $\mathcal{N}$ of typical sets $N$ by

$$\mathcal{N} = \{ N \in 2^{[n]} : |n_2(N) - \frac{n}{8}| < \sqrt{n \log n} \text{ and } |n_1(N) - \frac{n}{4}| < 0.1n \}.$$ 

Then the well-known de Moivre-Laplace formula (see, e.g. in [R, p. 151]) gives that

$$|\mathcal{N}| > 2^n (1 - \frac{1}{n}).$$

There exists some positive constant $c$ such that for every typical set $N$,

$$p^{n_2} \left( \frac{n_2}{k} \right) = \left( \frac{1000 \log n}{\sqrt{n}} \right)^{3/2} \left( \frac{n/8}{k} \right)^{-1} < c \frac{(\log n)^{3/2}}{\sqrt{n}}$$

$$4$$
and

\[
(4.6) \quad p_{n_1} \left( \frac{n_2}{k - 1} \right) = \frac{(1000 \log n)^{3/2}}{\sqrt{n}} \frac{kn_1}{n_2 - k + 1} \binom{n/2}{k} > 2 \log n.
\]

(Here we used the inequalities for \((1 - x)^y\) from Section 2.) Then (4.5) and (4.6) imply the following lower bound in (4.3). If \(N \in \mathcal{N}\) then

\[
(4.7) \quad \text{Prob}(N \in N(\mathcal{F}(S))) > 1 - c \frac{(\log n)^{3/2}}{\sqrt{n}}.
\]

Then (4.4) and (4.7) give that the expected size \(E(N(\mathcal{F}(S)))\) fulfills (4.2). Hence there exists a family \(S\) satisfying (4.2). \(\square\)

5. **Problems, Remarks.** It is a natural question how close \(c(n)\) can be to \(2^n\). Obviously, \(2^n - c(n) \geq 2^n/n\). Kostochka [K] proved that for every filter \(\mathcal{F}\) one has \(2^n - |N(\mathcal{F})| > 0.011 \cdot 2^n(\log n)^{3/2}/\sqrt{n}\). So the method presented in this note cannot give a better bound than Theorem 4.1.

Another possible direction for the further research is to extend the investigation to other (popular) posets. (Cf. [GRS], [N], [SW]).

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