THE $q$-LOG-CONCAVITY OF $q$-BINOMIAL COEFFICIENTS

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IMA Preprint Series # 418
April 1988
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Abstract. The number \(\binom{n}{k}_q\) of \(k\)-dimensional subspaces of an \(n\)-dimensional vector space over the field with \(q\) elements is a polynomial in \(q\) with nonnegative coefficients. We establish that \(\binom{n}{k}_q\), \(0 \leq k \leq n\), is a log-concave sequence of polynomials. That is, the polynomial \(\binom{n}{k}_q^2 - \binom{n}{k-1}_q \binom{n}{k+1}_q\) has nonnegative coefficients for \(0 < k < n\).

Our proof is simple and combinatorial. Our result generalizes the easily seen fact that \(\binom{n}{k}_q\), \(0 \leq k \leq n\), is a log-concave sequence of numbers when \(q \geq 0\), and it strengthens our two year old result that the polynomial \(\binom{n}{k}_q^2 - q \binom{n}{k-1}_q \binom{n}{k+1}_q\) has nonnegative coefficients for \(0 < k < n\). We discuss related results and questions.

Key words. unimodal, log-concave, binomial coefficient, Gaussian coefficient, Stirling number

AMS(MOS) subject classifications. Primary 05A15; Secondary 11B65.

1. Introduction. A polynomial \(a_k(q)\) is said to be a \(q\)-analogue of an integer \(a_k\) if \(a_k(1) = a_k\). For example, the \(q\)-binomial coefficient \(\binom{n}{k}_q\) is a \(q\)-analogue of the binomial coefficient \(\binom{n}{k}\) and Gould's \(q\)-Stirling number of the second kind \(\mathcal{S}_q(n,k)\) is a \(q\)-analogue of the Stirling number of the second kind \(S(n,k)\). With these examples in mind, for a given property of numbers we search for an analogous \(q\)-property of polynomials; so that whenever the polynomials \(a_k(q)\) satisfy the \(q\)-property, the numbers \(a_k(1) = a_k\) satisfy the given property of numbers. For example, a \(q\)-analogue of \(a_k \geq 0\) is \(a_k(q)\) has nonnegative coefficients as a polynomial in \(q\).

Motivated by the main result in [3] and a conjecture in Garsia and Remmel[4], we make the following definition of \(q\)-log-concave. The reader can easily supply a definition of \(q\)-unimodal.

**Definition 1.1.** A sequence of polynomials \(a_k(q), 0 \leq k \leq n\), is \(q\)-log-concave if 
\[(a_k(q))^2 - (a_{k-1}(q))(a_{k+1}(q))\]
has nonnegative coefficients.

The main theorem of this paper implies that, for each \(n\), the sequence of \(q\)-binomial coefficients \(\binom{n}{k}_q, 0 \leq k \leq n\), is \(q\)-log-concave, settling one of the conjectures in [3].

In section 2 we briefly state two well-known combinatorial descriptions of \(\binom{n}{k}_q\), which will be used to obtain the results of sections 3 and 4. In section 3, to put our main result in proper perspective, we give the easy combinatorial argument from [2] which shows that 
\[\binom{n}{k}_q^{(q)} - q^{\ell-k+1} \binom{n}{\ell}_q^{(q)}\]
has nonnegative coefficients if \(k \leq \ell\). We mention an algebraic proof of this result subsequently supplied by Stanton. In section 4, we give a beautiful combinatorial proof of the sharper result that \(\binom{n}{k}_q^{(q)} - \binom{n}{k-1}_q^{(q)} \binom{n}{\ell+1}_q\) has nonnegative coefficients if \(k \leq \ell\). In section 5 we mention related results and problems.

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2. Two combinatorial descriptions of $q$-binomial coefficients. In [6] Knuth provides a direct proof that the number of $k$-dimensional subspaces of an $n$-dimensional vector space over the field $F_q$ with $q$ elements is the polynomial in $q$ obtained by $q$-counting partitions whose Ferrers diagram fits in a $k \times (n-k)$ rectangle. That is,

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \sum_{\lambda \in (n-k)^k} q^{\mid \lambda \mid}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, and $\mid \lambda \mid = \sum \lambda_i$.

This description is often rewritten in terms of multiset permutations

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \sum_{\omega \in S(1^k 2^{n-k})} q^{\text{inv} \ \omega}$$

where $S(1^k 2^{n-k})$ is the set of words $\omega = \omega_1 \omega_2 \cdots \omega_n$ in which $k$ of the letters are 1 and $n-k$ of the letters are 2. The inversion number of $\omega$, $\text{inv} \ \omega$, is the number of $i < j$ such that $\omega_i > \omega_j$.

3. Proof of the centered result. The first $q$-log-concavity result for $q$-binomial coefficients was that $\left[ \begin{array}{c} n \\ k \end{array} \right]_q^2 - q \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_q \left[ \begin{array}{c} n \\ k+1 \end{array} \right]_q$ has nonnegative coefficients as a polynomial in $q$ (see [3]). The following proposition can be found in the author's thesis[2, Ch. 2, Sec. 8]. We offer two proofs. The first is our simple combinatorial proof using (2.2); the second is an algebraic proof later furnished by Stanton[13].

**Proposition 3.1.** For $0 < k \leq \ell < n$,

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} n \\ \ell \end{array} \right]_q - q^{\ell-k+1} \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_q \left[ \begin{array}{c} n \\ \ell+1 \end{array} \right]_q$$

has nonnegative coefficients as a polynomial in $q$.

**Proof #1.** We describe an injection $\varphi$

$$\varphi : S(1^k 2^{2n-k+1}) \times S(1^{\ell+1} 2^{n-\ell-1}) \hookrightarrow S(1^k 2^{n-k}) \times S(1^{\ell+2n-\ell})$$

such that if $\varphi(\pi, \sigma) = (\nu, \omega)$ then $(\ell - k + 1) + \text{inv} \ \pi + \text{inv} \ \sigma = \text{inv} \ \nu + \text{inv} \ \omega$.

Given $\pi \in S(1^k 2^{2n-k+1})$ and $\sigma \in S(1^{\ell+1} 2^{n-\ell-1})$, consider pairs of multiset permutations $(\nu^{(i)}, \omega^{(i)})$ with $0 \leq i \leq n$ defined by

$$\nu^{(i)} = \pi_1 \pi_2 \cdots \pi_i \sigma_{i+1} \cdots \sigma_n$$

$$\omega^{(i)} = \sigma_1 \sigma_2 \cdots \sigma_i \pi_{i+1} \cdots \pi_n$$
Notice

(1) \( \nu^{(0)} = \sigma \) has \( \ell + 1 \) 1's
(2) \( \nu^{(n)} = \pi \) has \( k - 1 \) 1's
(3) the number of 1's in \( \nu^{(i)} \) differs from the number of 1's in \( \nu^{(i+1)} \) by at most one.

Since \( \ell + 1 > k > k - 1 \), it follows from the three observations above that there is some \( i, 0 < i < n \), such that the number of 1's in \( \nu^{(i)} \) is \( k \). Choose the smallest such \( i \), and define \( \varphi(\pi, \sigma) = (\nu^{(i)}, \omega^{(i)}) \). It is easy to see that this procedure defines an injection \( \varphi \).

We write, more descriptively, that \( \varphi(\pi, \sigma) = (\pi_L \sigma_R, \sigma_L \pi_R) \), where \( \pi = \pi_L \pi_R, \sigma = \sigma_L \sigma_R \), and \( |\pi_L| = |\sigma_L| \) is this minimal \( i \). We claim that \( (\ell - k + 1) + \text{inv } \pi + \text{inv } \sigma = \text{inv } \pi_L \sigma_R + \text{inv } \sigma_L \pi_R \). Let \( m_1 \omega \) denote the number of 1's in \( \omega \), and let \( m_2 \omega \) denote the number of 2's in \( \omega \).

\[
\text{inv } \pi = \text{inv } \pi_L + \text{inv } \pi_R + (m_2 \pi_L)(m_1 \pi_R)
\]
\[
\text{inv } \sigma = \text{inv } \sigma_L + \text{inv } \sigma_R + (m_2 \sigma_L)(m_1 \sigma_R)
\]
\[
\text{inv } \pi_L \sigma_R = \text{inv } \pi_L + \text{inv } \sigma_R + (m_2 \pi_L)(m_1 \sigma_R)
\]
\[
\text{inv } \sigma_L \pi_R = \text{inv } \sigma_L + \text{inv } \pi_R + (m_2 \sigma_L)(m_1 \pi_R)
\]

So,

\[
\text{inv } \pi_L \sigma_R + \text{inv } \sigma_L \pi_R - \text{inv } \pi - \text{inv } \sigma
= (m_2 \pi_L)(m_1 \sigma_R) + (m_2 \sigma_L)(m_1 \pi_R) - (m_2 \pi_L)(m_1 \pi_R) - (m_2 \sigma_L)(m_1 \sigma_R)
= (m_2 \pi_L - m_2 \sigma_L)(m_1 \sigma_R - m_1 \pi_R)
= (m_2(\pi_L \sigma_R) - m_2 \sigma)(m_1(\pi_L \sigma_R) - m_1 \pi)
= ((n - k) - (n - \ell - 1))(k - (k - 1))
= \ell - k + 1
\]

\[ \square \]

This injection was inspired by Bhatt and Leiserson's paper[1], however, as the existence of the following algebraic proof suggests, it is better viewed as a Gessel-Viennot style lattice path argument[5].

Proof \#2 (Stanton). The reader is referred to Macdonald[8] for definitions and results employed here. Let \( 0 < k \leq \ell < n \). In \( n \) variables \( x = (x_1, x_2, \ldots, x_n) \) the dual form of the Jacobi-Trudi identity[8, Ch. 1, 3.5], which expands a Schur function in terms of elementary symmetric functions, yields

\[
s(\lambda_1 \ell - k)(x) = e_k(x)e_\ell(x) - e_{k-1}(x)e_{\ell+1}(x).
\]
Using the principal specialization $e_r(1, q, \ldots, q^{n-1}) = q^{r(r-1)/2} [n]_q$ (see, e.g., [8, Ch. 1, Sec. 2, Ex. 3]), we obtain

$$s_{(2^k \ell - k)}(1, q, \ldots, q^{n-1})$$

$$= \left( q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_q \right) \left( q^{\frac{(\ell-1)(\ell)}{2}} \left[\begin{array}{c} n \\ \ell \end{array}\right]_q \right) - \left( q^{\frac{(k-1)(k-2)}{2}} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q \right) \left( q^{\frac{(\ell+1)(\ell)}{2}} \left[\begin{array}{c} n \\ \ell+1 \end{array}\right]_q \right)$$

$$= q^{\frac{k(k-1)}{2} + \frac{(\ell-1)(\ell)}{2}} \left( \left[\begin{array}{c} n \\ k \end{array}\right]_q \left[\begin{array}{c} n \\ \ell \end{array}\right]_q - q^{\ell-k+1} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q \left[\begin{array}{c} n \\ \ell+1 \end{array}\right]_q \right).$$

Our result now is seen to follow from the fact that the Schur function $s_{(2^k \ell - k)}(x)$ has nonnegative coefficients as a polynomial in $x_1, x_2, \ldots, x_n$ (see, e.g., [8, Ch. 1, 5.12]).

The following corollary has a proof employing an injection, as above, but we deduce it from Proposition 3.1.

**COROLLARY 3.2.** For $0 \leq k - r \leq k \leq \ell \leq \ell + r \leq n$,

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q \ell_q \left[\begin{array}{c} n \\ k-r \end{array}\right]_q \left[\begin{array}{c} n \\ \ell+r \end{array}\right]_q$$

has nonnegative coefficients as a polynomial in $q$.

**Proof.** By the above proposition, each term of the following telescoping sum has nonnegative coefficients.

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q \ell_q - q^{r(\ell-k+r)} \left[\begin{array}{c} n \\ k-r \end{array}\right]_q \left[\begin{array}{c} n \\ \ell+r \end{array}\right]_q$$

$$= \sum_{i=0}^{r-1} q^{i(\ell-k+i)} \left( \left[\begin{array}{c} n \\ k-i \end{array}\right]_q \ell_i_q - q^{\ell-k+2i+1} \left[\begin{array}{c} n \\ k-i-1 \end{array}\right]_q \ell_i_q \right).$$

\square

At the beginning of section 4, we indicate why we refer to the above as centered results.

4. **Proof of the main result.** Since $[n]_q$ is a symmetric, unimodal polynomial (see, e.g., [15]), so are $[\ell]_q$ and $[\ell]_q[n]_q$ and $[k-1]_q[n]_q$. As observed in [2, 2.8.4], since the degree of $[n]_q[n]_q$ exceeds the degree of $[k-1]_q[n]_q$ by $2(\ell-k+1)$, the result of section 2 may be viewed as a corollary of our main result that $[k]_q[n]_q - [k-1]_q[n]_q$ has nonnegative coefficients. In fact, as is easily seen from the geometric representation of symmetric, unimodal polynomials given in the following example, if $[k]_q[n]_q - [k-1]_q[n]_q$ has nonnegative coefficients then $[k]_q[n]_q - q[i]_q[k-1]_q[n]_q$ has nonnegative coefficients for $0 \leq i \leq 2(\ell-k+1)$. The result in section 3 is the weakest of these; our main result of this section is the strongest.
EXAMPLE 4.1. The polynomial $q^2_{1} = 1 + 2q + 3q^2 + 4q^3 + 3q^4 + 2q^5 + q^6$ may be pictured as the curve in Fig. 1(a); The polynomial $q^2_{4} = 1 + q + 2q^2 + q^3 + q^4$ may be pictured as the curve in Fig. 1(b).

Since the curve for $q^2_{4}$ never rises above the curve for $q^2_{1}$, we see that the polynomial $q^2_{4} - q^2_{6}$ has nonnegative coefficients.

The pictures for $q^2_{k} - q^1_{i}$, $i = 0, 1, 2$, are shown in Fig 2(a), (b) and (c) respectively. Multiplication by $q_i$ shifts the curve for $q^2_{4} - q^2_{6}$ to the right $i$ units.

THEOREM 4.2. For $0 < k \leq \ell < n$,

$$[n]_q [k]_q \left[ n \right]_{\ell} - \left[ n \right]_{k-1} \left[ n \right]_{\ell+1}$$

has nonnegative coefficients as a polynomial in $q$.

Proof. We use (2.1) to give a combinatorial proof. Let $P$ denote the set of partitions of nonnegative integers. Let $P(k, n-k)$ denote the set of all partitions whose Ferrers diagrams fit in a $k \times (n-k)$ rectangle. We describe an injection $\varphi$

$$\varphi : P(k-1, n-k+1) \times P(\ell+1, n-\ell-1) \rightarrow P(k, n-k) \times P(\ell, n-\ell)$$

such that if $\varphi(\lambda, \mu) = (\eta, \rho)$ then $|\lambda| + |\mu| = |\eta| + |\rho|$.

The injection $\varphi$ is a composition of two maps. We define maps $A$ and $L$ on $P \times P$ and observe that

1. $A$ and $L$ are involutions on $P \times P$.
2. $A(P(k-1, n-k+1) \times P(\ell+1, n-\ell-1)) \subset P(k-1, n-k) \times P(\ell+1, n-\ell)$.
3. $L(P(k-1, n-k) \times P(\ell+1, n-\ell)) \subset P(k, n-k) \times P(\ell, n-\ell)$.

Finally, we take $\varphi = L \circ A$ with domain restricted to ordered pairs of partitions in $P(k-1, n-k+1) \times P(\ell+1, n-\ell-1)$. 

5
We first define $A$.

Given partitions $\lambda$ and $\mu$, let $I$ be largest so that $\lambda_I \geq \mu_{I+1} + (\ell - k + 1)$ else, if no such $I$ exists, let $I = 0$. Define $A(\lambda, \mu) = (\gamma, \tau)$ where

$$
\gamma = (\mu_1 + (\ell - k + 1), \ldots, \mu_I + (\ell - k + 1), \lambda_I, \lambda_{I+1}, \lambda_{I+2}, \ldots)
$$

$$
\tau = (\lambda_1 - (\ell - k + 1), \ldots, \lambda_I - (\ell - k + 1), \mu_I, \mu_{I+1}, \mu_{I+2}, \ldots)
$$

We define $L$ likewise.

Given partitions $\gamma$ and $\tau$, let $\gamma'$ and $\tau'$ denote the conjugate partitions (so that $\tau'_j$ is the length of the $J^{th}$ column of the Ferrers diagram of $\tau$). Let $J$ be largest so that $\tau'_J \geq \gamma'_{J+1} + (\ell - k + 1)$ else, if no such $J$ exists, let $J = 0$. Define $L(\gamma, \tau) = (\eta, \rho)$ where

$$
\eta = (\tau'_1 - (\ell - k + 1), \ldots, \tau'_J - (\ell - k + 1), \gamma'_{J+1}, \gamma'_{J+2}, \ldots)
$$

$$
\rho = (\gamma'_1 + (\ell - k + 1), \ldots, \gamma'_J + (\ell - k + 1), \tau'_{J+1}, \tau'_{J+2}, \ldots)
$$

If we let $r(\lambda, \mu) = (\mu, \lambda)$ and $c(\lambda, \mu) = (\lambda', \mu')$, then $L = r \circ c \circ A \circ c \circ r$. So to show (1) we need only show $A$ is an involution. The verification is straightforward once we show $\gamma$ is a partition. By the choice of $I$, we have $\lambda_{I+1} < \mu_{I+2} + (\ell - k + 1)$. Since $\mu$ is a partition, we have $\mu_{I+2} \leq \mu_I$. Therefore, $\lambda_{I+1} < \mu_I + (\ell - k + 1)$, which is more than we need for $\gamma$ to be a partition. The verifications of (2) and (3) are straightforward. $\square$

**Example 4.3.** We illustrate the injection $\varphi$ of Theorem 4.2. Suppose $n = 20$, $k = 8$, and $\ell = 9$. If $\lambda = 1311109721$ and $\mu = 1010108865443$, then the computation of $(\gamma, \tau) = A(\lambda, \mu)$ is shown in Fig. 3(a), and the computation of $(\eta, \rho) = L(\gamma, \tau)$ is shown in Fig. 3(b).
A special case of the above theorem gives the main result of this paper.

**Corollary 4.4.** The sequence of q-binomial coefficients \( \binom{n}{k}_q \) is q-log-concave in \( k \).

The following corollary has a proof employing an injection, as above, but we deduce it from Theorem 4.2.

**Corollary 4.5.** For \( 0 \leq k - r \leq k \leq \ell \leq \ell + r \leq n \),

\[
\binom{n}{k}_q \binom{n}{\ell}_q - \binom{n}{k-r}_q \binom{n}{\ell+r}_q
\]

has nonnegative coefficients as a polynomial in \( q \).

**Proof.** By the above proposition, each term of the following telescoping sum has non-negative coefficients.

\[
\binom{n}{k}_q \binom{n}{\ell}_q - \binom{n}{k-r}_q \binom{n}{\ell+r}_q
= \sum_{i=0}^{r-1} \left( \binom{n}{k-i}_q \binom{n}{\ell+i}_q - \binom{n}{k-i-1}_q \binom{n}{\ell+i+1}_q \right)
\]

\( \Box \)

5. **Related results and questions.** At the special session in Algebraic Combinatorics at the AMS meeting in East Lansing, Michigan in March 1988, we presented our combinatorial proof that the sequence of q-binomial coefficients \( \binom{n}{k}_q \) is q-log-concave in \( k \) and announced the conjecture that the sequence of q-Stirling numbers of the second kind \( \tilde{S}_q(n,k) \) is q-log-concave in \( k \). Within days after the meeting, Sagan[12] found another proof of q-log-concavity of the sequence of q-binomial coefficients. It employs recursions and makes use of the idea at the beginning of section 4. A week later, Leroux[7] and Sagan[12] found proofs of the conjecture for q-Stirling numbers. The proof due to Leroux uses a combinatorial description of \( \tilde{S}_q(n,k) \) inspired by Knuth[6] and Milne[9]. He employs a refinement of our involution \( \mathcal{A} \) and a modification of our involution \( \mathcal{L} \) to show that the \( p,q \)-Stirling numbers of the second kind defined in Wachs and White[14] satisfy “\( p(S_{p,q}(n,k))^2 - S_{p,q}(n,k-1)S_{p,q}(n,k+1) \) has nonnegative coefficients as a polynomial in the number of variables \( p \) and \( q \).” Log-concavity follows since \( S_{1,q}(n,k) = \tilde{S}_q(n,k) \). Sagan's proof employs the recursion \( \tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + [k]_q \tilde{S}_q(n-1,k) \), where \( \tilde{S}_q(0,0) = 1 \) and \( [k]_q = 1 + q + \cdots + q^{k-1} \).

The main conjecture in [3] is still open. Namely, let \( \alpha_\lambda(k;p) \) denote the number of subgroups of order \( p^k \) in a finite abelian \( p \)-group of type \( \lambda \in \mathcal{P} \). Then \( \alpha_\lambda(k;p) \) is a polynomial in \( p \), described combinatorially in [2]. We conjectured that the sequence \( \alpha_\lambda(k;p), 0 \leq k \leq |\lambda| \), is \( p \)-log-concave. Strauss, on Princeton's Computers and Information
Technology staff has verified this conjecture for $|\lambda| \leq 29$. The main result of the present paper establishes this conjecture for $\lambda = 1^n$. Stanley asked about $p$-log-concavity in $k$ when we first presented our proof that the sequence $\alpha_k(k; p)$, $0 \leq k \leq |\lambda|$, is $p$-unimodal in $k$. Brualdi asked the same question at a later presentation. Rabau[11] found a simpler combinatorial proof that the sequence of $q$-binomial coefficients is $q$-unimodal than the one given in [3]. In [4], Garsia and Remmel conjecture that the sequence of $q$-Stirling numbers of the second kind $\tilde{S}_q(n, k)$ is $q$-unimodal in $k$. Note (see [2]) that a $q$-log-concave sequence of polynomials need not be $q$-unimodal (e.g. $2 + 5q, 4 + 4q, 5 + 2q$).

Finally, we remind the reader of another, more famous, class of conjectures. A polynomial $a_0 + a_1 q + \cdots + a_n q^n$ is said to be unimodal if the sequence of numbers $a_0, a_1, \ldots, a_n$ is unimodal. O'Hara[10] recently found the first combinatorial proof that $\binom{n}{k}_q$ is a unimodal polynomial. Strauss has verified that $\alpha_k(k; p)$ is a unimodal polynomial in $p$ for $|\lambda| \leq 29$. Garsia and Remmel[4] conjecture that $\tilde{S}_q(n, k)$ is a unimodal polynomial. Wachs and White[14] ask whether $\tilde{S}_q(n, k)$ might be a log-concave polynomial. Stanton observed the curious fact that $\binom{n}{k}^2 - q_1 \binom{n}{k-1} \binom{n}{k+1}$ is a unimodal polynomial. (This fact follows immediately from the computation in Proof #2 of Proposition 3.1 and the unimodality of the polynomial $s_{2^*(1, q, \ldots, q^{n-1})}$.) See, e.g., [8, Ch. 1, Sec. 8, Ex. 4]. Strauss had conjectured that $\alpha_k(k; p)^2 - \alpha_k(k-1; p) \alpha_k(k+1; p)$ is a unimodal polynomial. Stanton’s observation and Strauss’s conjecture may be extended to the case $k \leq \ell$ as in Proposition 3.1 and Theorem 4.2.

Acknowledgements. The author would like to thank Richard Stanley who, on hearing the $p$-unimodality result of [3], first asked her about $p$-log-concavity; and Dennis White who pointed out to her the $q$-unimodality conjecture in [4].

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