THE SECOND AND THE THIRD SMALLEST DISTANCES
ON THE SPHERE

By

Zoltán Füredi

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THE SECOND AND THE THIRD SMALLEST DISTANCES ON THE SPHERE

ZOLTÁN FÜREDI†

Abstract. Let \( s_1(n) \) denote the largest possible minimal distance among \( n \) distinct points on the unit sphere \( S^2 \). In general, let \( s_k(n) \) denote the supremum of the \( k \)-th minimal distance. In this paper we prove and disprove the following conjecture of A. Bezdek and K. Bezdek: \( s_2(n) = s_1([n/3]) \). This equality holds for \( n > n_0 \) however \( s_2(12) > s_1(4) \).

We set up a conjecture for \( s_k(n) \), that one can always reduce the problem of the \( k \)-th minimum distance to the function \( s_1 \). We prove this conjecture in the case \( k = 3 \) as well, obtaining that \( s_3(n) = s_1([n/5]) \) for sufficiently large \( n \).

1. Introduction, Results. Let \( \mathcal{P} \) be a finite pointset on the 2-dimensional unit sphere \( S^2 \) in \( \mathbb{R}^3 \). The spherical distance between the points \( x, y \in S^2 \) is denoted by \( d(x, y) \). Consider the set of distances between the points of \( \mathcal{P} \),

\[
D(\mathcal{P}) = \{d(x, y) : x, y \in \mathcal{P}, x \neq y\}.
\]

Order the elements in \( D(\mathcal{P}) = \{d_1, \ldots, d_t\} \) such that \( d_1 < d_2 < \cdots < d_t \). Then \( d_t \) is the diameter of \( \mathcal{P} \) and the \( k \)-th smallest distance, \( d_k \) is denoted by \( s_k(\mathcal{P}) \). (If \( t > k \), define \( s_k(\mathcal{P}) = \infty \).) So \( s_1(\mathcal{P}) \) is the minimum distance.

Let \( s_k(n) \) denote the supremum of the \( k \)-th smallest distance in \( n \)-point sets on the sphere, i.e.,

\[
s_k(n) = \sup\{s_k(\mathcal{P}) : |\mathcal{P}| = n, \quad \mathcal{P} \subset S^2\}.
\]

The problem of finding \( s_1(n) \) was raised by Tammes [T] in 1930. The exact value of \( s_1(n) \) and the extremal arrangements are known for a couple of small values of \( n \). \( n = 5, 7, 8 \) by Schütte and Van der Waerden [SW], \( n = 10, 11 \) by Danzer [D], \( n = 11 \) by Böröczky [Bö], \( n = 24 \) by Robinson [R] and \( n = 3, 4, 6, 12 \) by L. Fejes Tóth [FT]. More on these results see [D2].) Here we will use the asymptotic result

\[
s_1(n) = (1 + o(1))\sqrt[4]{\frac{8\pi}{\sqrt{3}} n},
\]

(1.1)

This formula means that \( \lim_{n \to \infty} ns_1(n)^2 = 8\pi/\sqrt{3} \). L. Fejes Tóth proved the following

\[
s_1(n) \leq \arccos \left( \frac{1}{2} \left( \cot^2 \frac{n}{n-2} \frac{\pi}{6} - 1 \right) \right)
\]

(1.2)

which yields the upper bound in (1.1). The lower bound can be obtained a hexagonal like packing of circles.

The problem of \( s_2(n) \) was recently proposed by A. Bezdek and K. Bezdek [BB]. They proved that \( s_2(9) = 2\pi/3 \) and that

\[
s_2(n) > s_1([n/3]) - \epsilon
\]

(1.3)

†Mathematical Institute, Hung. Acad. Sci., 1364 Budapest, P.O.B. 127, Hungary
holds for all $\epsilon > 0$. (Here $[x]$ denotes the smallest integer not smaller than $x$.) The construction giving (1.3) is obtained from an $s_1$-extremal arrangement $\mathcal{P}'$ with $|\mathcal{P}'| = [n/3]$, i.e., $s_1(\mathcal{P}') = s_1([\mathcal{P}'])$. Then replace each point $p \in \mathcal{P}'$ by a regular triangle of side length $\epsilon/2$ and with a vertex in $p$.

In [BB] an upper bound (which was twice the right hand side of (1.2)) was proved and they asked whether equality holds in (1.3) for all $n > 4$. The aim of this paper is to answer this question.

The 12 vertices of the regular icosahedron show that

$$s_2(12) > s_1(4).$$

However for large $n$ Bezdek's conjecture is true:

**Theorem 1.1.** For $n > n_0$ one has $s_2(n) = s_1([n/3])$.

Let $f(k)$ denote the largest integer $f$ such that for all $\epsilon > 0$ there exists a $k$-distance set of size $f$ and of diameters less than $\epsilon$. We have $f(0) = 1$, $f(1) = 3$, $f(2) = 5$, $f(3) = 7$. For large $k$ the best known upper bound is $O(k^{2/4})$ due to Chung, Szemerédi and Trotter [CSzT], and it is still a challenging problem to decide whether $f(k) = O(k)$, or not. (Recently, Erdős, Hickerson and Pach [EHP] proved some results which give some support to the conjecture that $\lim f(k)/k = \infty$.) Replacing the points of an $s_1$-extremal set on the sphere by congruent small copies of a $(k-1)$-distance set we obtain

$$s_k(n) \geq s_1([n/f(k-1)]).$$

**Conjecture 1.2.** For $n > n_0(k)$ one has $s_k(n) = s([n/f(k-1)])$.

**Theorem 1.3.** For $n > n_0$ one has $s_3(n) = s_1([n/5])$.

In general we can only prove a weaker upper bound.

**Theorem 1.4.** For $n > n_0(k)$ one has $s_k(n) \leq s_1([n/6f(k-1)])$.

2. **A Lemma on the Ratio of $s_1$ and $s_2$.** Let $\Delta \geq 0$ be an integer, $0 < s < \pi/2$. Define the regular $\Delta$-gon (on the unit sphere $S^2$) with center $c$ and inscribed radius $s$ as follows:

- for $\Delta = 0$ the whole sphere,
- for $\Delta = 1$ halfsphere including $c$ such that the distance from $c$ to the boundary is $s$,
- for $\Delta = 2$ a digon with symmetry center $c$ whose distance from the sides is $s$,
- for $\Delta \geq 3$ as usual.
We can extend these definitions to the Euclidean plane, in the cases $\Delta = 0, 1, 2$ the regular $\Delta$-gon is the whole plane, a halfplane or an infinite strip of width $2s$. Define the function $A(\Delta, D, s)$ as the area of the intersection of a regular $\Delta$-gon with inscribed radius $s/2$ and a circle of diameter $D$ with the same center. The same function on the plane is denoted by $A_\infty(\Delta, D, s)$. Clearly,

$$A_\infty(\Delta, D, s) = s^2 A_\infty(\Delta, \frac{D}{s}, 1),$$

and if $\Delta$ and $D/s$ are given then

$$\lim_{s \to 0} \frac{A(\Delta, D, s)}{s^2} = A_\infty(\Delta, \frac{D}{s}, 1).$$

For brevity we use $A(\Delta, x)$ for $A_\infty(\Delta, x, 1)$. E.g., $A(0, x) = x^2 \pi/4$, $A(4, \infty) = 1$, $A(6, \infty) = \sqrt{3}/2$.

Let $\mathcal{P}$ be an $n$-element set on $S^2$, $s_i = s_i(\mathcal{P}), (n > 4)$. Define the minimum distance graph $\mathcal{G} = \mathcal{G}(\mathcal{P})$ with vertex set $\mathcal{P}$ as follows: two points are connected if their distance is $s_1$. Obviously, every point has at most 5 neighbours, so for the maximum degree, $\Delta(\mathcal{G})$, of $\mathcal{G}$ we have

$$\Delta(\mathcal{G}) \leq 5.$$

**Lemma 2.1.** Let $0.1 > \epsilon > 0$ and suppose that $n > n_1(\epsilon), s_2 < \epsilon$. Then

$$s_1(\mathcal{P}) < s_1(n)(1 + \epsilon)\sqrt{\frac{\sqrt{3}/2}{A(\Delta, s_2/s_1)}}.$$

**Proof.** By (1.1) for every $\epsilon > 0$ there exists an $n_1(\epsilon)$ such that

$$\epsilon s_2(n) \frac{\sqrt{3}}{2} n > \text{Area } S^2 = 4\pi.$$

On the other hand for every $p \in \mathcal{P}$ define its Dirichlet cell, $C(p) = \{q \in S^2 : d(p, q) = d(\mathcal{P}, q)\}$. Let $p_1, \ldots, p_t$ be the neighbors of $p$ in $\mathcal{G}$, and let $H_i$ be the half sphere containing $p$ which perpendicularly bisects $pp_i$. Then $C(p)$ contains the intersection of $H_i - s$ and a spherical circle of radius $s_2/2$ around $p$. Hence Area $C(p) \geq A(\Delta, s_2, s_1)$. Obviously, $A(\Delta, s_2, s_1) > A(\Delta, s_2/s_1)s_1^2/(1 + \epsilon)$. Then

$$4\pi \geq \sum \text{Area } C(p) > n A(\Delta, s_2/s_1)s_1^2/(1 + \epsilon).$$

Finally, (2.2) and (2.3) imply the Lemma. []

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3. The Proof of Theorem 1.4. Let \( P \) be a finite point-set on \( S^2 \). Consider the minimum \((k - 1)\)-distance graph \( G^{k-1} = G(P) \), i.e., two points \( x, y \) in \( P \) are connected if \( d(x, y) \leq s_{k-1}(P) \). Let \( f(k - 1, \varepsilon) \) denote the maximum size of a \((k - 1)\)-distance set of diameter at most \( \varepsilon \).

**Proposition 3.1.** Every point in \( G^{k-1} \) is connected by less than \( 6f(k - 1, s_{k-1}) - 6 \) other points.

**Proof.** Let \( p \in P \), and consider a closed circle \( C \) with radius \( s_{k-1} \) and center \( p \). Divide \( C \) into 6 congruent pieces with 3 diagonals through \( p \), any two of them have an angle \( \pi/3 \). Then the diameters of each piece is \( s_{k-1} \), so it contains at most \( f(k - 1, s_{k-1}) \) elements of \( P \).  

**Proof of Theorem 1.4.** There exists an \( \varepsilon > 0 \) such that \( f(k - 1, \varepsilon) = f(k - 1) \). We have an \( n_0(k) \) such that for \( n > n_0(k) \) \( s_{k-1}(n) < \varepsilon \) holds. Then \( G^{k-1} \) does not contain a complete subgraph of \( 6f(k - 1) - 5 \) vertices \( (k \geq 2) \), but every degree is not larger than \( 6f(k - 1) - 6 \). One can use Brook's theorem (see, e.g., in Bollobás' book [Bo]), that the chromatic numbers of \( G^{k-1} \) is at most \( 6f - 6 \). So there exists a \( P' \subset P \) with \( |P'| \geq |P|/(6f - 6) \) such that \( s_1(P') \geq s_k(P) \).  

4. The Proof of Theorem 1.1. By (1.1) we have an \( n_2 \) such that for all \( n > n_2 \)

\[
\frac{s_1([n/3])}{s_1(n)} > 1.71
\]

holds. Suppose that \( P \) is an arbitrary \( n \)-set on the sphere with \( n > n_2 \). To prove the theorem we have to show that

\[
s_2(P) < s_1([n/3]).
\]

We may suppose that

\[
s_2(P) > 1.71s_1(n),
\]

otherwise (1.1) implies (2.2).

As \( s_1(n) \to 0 \) if \( n \to \infty \) we have an \( n_3 \) such that \( s_1([n/4]) < 0.01 \) holds for all \( n > n_3 \). Then by Proposition 1.4 we have

\[
s_2(P) < 0.01.
\]

So we may apply Lemma 2.1 to \( P \) with \( n > \max\{n_2, n_3\} \), \( \Delta = 5 \) and \( \varepsilon = 0.01. \) We have

\[
A(5, 1.71) = A(5, \infty) = \frac{5}{4} \tan 36^\circ \sim 0.908 \ldots \text{ so by Lemma 2.1}
\]

\[
s_1(P) < s_1(n) \cdot 0.986 \ldots
\]
Then (4.5) and (4.3) imply that

\[(4.6) \quad s_2(\mathcal{P})/s_1(\mathcal{P}) > 1.733 \cdots > \sqrt{3}.\]

**Claim 4.1.** \(\Delta(\mathcal{G}) \leq 3.\)

**Proof.** Suppose on the contrary that \(p \in \mathcal{P}, \ q_1, \ldots, q_4 \in \mathcal{P}\) with \(d(p, q_i) = s_1.\) If the distances \(d(q_i, q_j)\) are all at least \(\sqrt{3}s_1,\) then each angle \(q_ipq_{i+1}\) is at least 120°, a contradiction. So we have, say, \(d(q_1, q_2) = s_1.\) If \(d(q_i, q_1) \quad (i = 3, 4)\) is less than \(\sqrt{3}s_1\) then it is also \(s_1,\) but then \(s_1 < d(q_2, q_i) < \sqrt{3}s_1 < s_2,\) a contradiction. Hence \(d(q_i, q_j) \geq \sqrt{3}s_1\) for \(i = 1, 2, j = 3, 4.\) Then we obtain the contradiction \(d(q_3, q_4) < s_1.\)

In the same way we can obtain:

**Fact 4.2.** If \(\Delta(\mathcal{G}) = 3\) and \(s_2 \geq \sqrt{3}s_1\) then \(s_2 < 2s_1 \sin 75° < s_1 1.931\ldots.\)

Apply again Lemma 2.1 to \(\mathcal{P}\) with \(\Delta = 3, \ s_2/s_1 > \sqrt{3},\) and \(\epsilon = 0.01.\) Then \(A(3, \sqrt{3}) = 1.267\ldots,\) hence

\[(4.7) \quad s_1(\mathcal{P}) < s_1(n)0.834\ldots\]

Then (4.7) and (4.3) imply

\[(4.8) \quad s_2(\mathcal{P})/s_1(\mathcal{P}) > 2.048 \cdots > 2.\]

Then, obviously, we have

\[(4.9) \quad \Delta(\mathcal{G}) \leq 2.\]

Finally, we are going to use the following (simple) fact (see, e.g., in Bollobás’ Book [Bo]): If for a graph \(\mathcal{H}\) on the vertex set \(V\) and with \(\Delta(\mathcal{H}) \leq 2\) then there exists a \(V' \subset V, \ |V'| \geq |V|/3\) such that \(V'\) does not contain any edge of \(\mathcal{H}\) (i.e., \(V'\) is an empty or independent set of vertices). So by (4.9) there exists a \(\mathcal{P}' \subset \mathcal{P}, \ |\mathcal{P}'| \geq \lceil n/3 \rceil,\) such that \(s_1(\mathcal{P}') > s_1(\mathcal{P}).\) Then we have

\[s_1(\lceil n/3 \rceil) \geq s_1(\mathcal{P}') \geq s_2(\mathcal{P}),\]

and the proof of 1.1 is complete.

**5. The Proof of Theorem 1.3.** We are going to use the method of the proof of Theorem 1.1 but we have to investigate more subcases. We will use the following simple facts on 2-distance sets \(\mathcal{R}\) on the sphere. Suppose that the distances are \(a < b < 0.001.\)
FACT 5.1. $|\mathcal{R}| \leq 5$ with equality if and only if $\mathcal{R}$ is a regular pentagon. Then

\[(5.1) \quad 1.61801 \cdots = 2 \sin 54^\circ < \frac{b}{a} < 1.62.\]

FACT 5.2. If $|\mathcal{R}| = 4$ then one of the following six cases holds (see Fig. 1):

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{}
\end{figure}

(i) $\mathcal{R}$ consists of 2 regular triangles of side length $a$, with a common side. Then $(1 + \cos a)(1 + \cos b) = 4 \cos^2 a$, hence

\[1.732 \cdots = \sqrt{3} < \frac{b}{a} < 1.733.\]

(ii) $\mathcal{R}$ is a regular quadrilateral. Then $1.414 \cdots = \sqrt{2} < \frac{b}{a} < 1.415.$

(iii) $\mathcal{R}$ consists of four vertices of a regular pentagon. Then (5.1) holds.

(iv) $\mathcal{R}$ is a convex quadrilateral with diagonals of length $b$, and sides $a, a, b, b$. Then

\[\cos a = \cos^2 b + \sin^2 b \sqrt{\frac{1 + 2 \cos b}{2 + 2 \cos b}},\]

hence \((b/a) \sim \sqrt{2 + \sqrt{3}}\), i.e.,

\[(5.2) \quad 1.931 < \frac{b}{a} < 1.932\]

(v) $\mathcal{R}$ is a triangle of side lengths $b, b$, and $a$ and its center with circumscribed radius $a$. Then

\[\cos b = \cos^2 a - \sin^2 a \sqrt{\frac{1 + 2 \cos a}{2 + 2 \cos a}},\]

hence $b/a \sim \sqrt{2 + \sqrt{3}}$, i.e., (5.2) holds.
(vi) $\mathcal{R}$ is a regular triangle of side length $b$ and its center. Then $\cos b = 1 - 1.5 \sin^2 b$, hence
\[
1.732 < \frac{b}{a} < \sqrt{3} = 1.732 \ldots \quad \Box
\]

By (1.1) the limit of $s_1(\lfloor n/5 \rfloor)/s_1(n)$ is $\sqrt{5} = 2.236 \ldots$, so there exists an $n_4$ such that for all $n > n_4$
\[(5.3) \quad \frac{s_1(\lfloor n/5 \rfloor)}{s_1(n)} > 2.236 \]
holds. Suppose that $\mathcal{P}$ is an arbitrary $n$-set on the sphere with $n > n_2$. To prove our theorem we have to show that
\[(5.4) \quad s_3(\mathcal{P}) < s_1(\lfloor n/5 \rfloor). \]

We may suppose that
\[(5.5) \quad s_3(\mathcal{P}) > 2.236s_1(n), \]
otherwise (5.3) implies (5.4).

By Proposition 1.4 there exists an $n_5$ such that $s_3(n) < 0.001$ holds for all $n > n_5$. From now on we suppose that $n > \max\{n_4, n_5 \}$. We need a definition. Let $\mathcal{G}^2 = \mathcal{G}^2(\mathcal{P})$ be a graph with vertex set $\mathcal{P}$ defined in the following way: connect two points $p, q \in \mathcal{P}$ with an arc (with the shortest path on $S^2$) if $d(p, q) = s_1$ or $s_2$. If these arcs form a planar representation of $\mathcal{G}^2$ then (by the four color theorem) there exists a $\mathcal{P}' \subset \mathcal{P}$, $|\mathcal{P}'| \geq |\mathcal{P}|/4$ such that $s_3(\mathcal{P}) \leq s_1(\mathcal{P}') \leq s_1(\lfloor |\mathcal{P}'|/4 \rfloor)$.

From now on we suppose that there exist two arcs $ac$ and $bd$ such that $\{a, b, c, d\} \subset \mathcal{P}$, $d(ac)$ and $d(bd) \in \{s_1, s_2\}$ and $\text{int } \widehat{ac} \cap \text{int } \widehat{bd} \neq \emptyset$.

**FACT 5.3.** If $a, b, c, d$ from a convex quadrilateral with diagonals $ac, bd$ then $d(ab) + d(cd) \leq d(ac) + d(bd)$ and $d(ad) + d(bc) \leq d(ac) + d(bd)$. Here equality can hold if $a, b, c, d$ are lying on a great circle. $\Box$

Now Fact 5.3 implies that the total sum of opposite sides is less than $2s_2$, so one of them is $s_1$. We may suppose that, e.g., $d(a, b) = d(a, d) = s_1$. We distinguish 4 cases (Fig. 2).

![Diagram](image)

**Figure 2.**

a) $d(a, c) = d(b, d) = s_1$, i.e., both diagonals have length $s_1$. Then a side of $\{a, b, c, d\}$ is
shorter than \( s_1 \), by Fact 5.3, a contradiction.

\( \beta \) \( d(a, c) = s_1, \ d(b, d) = s_2 \). Then the point \( a \) lies in the center of the triangle \( bcd \) and the circumscribed radius is \( s_1 \). Then all distances in \( \{a, b, c, d\} \) is not more than \( 2s_1 \), i.e., it is a 2 distance set. Only the case 5.2 (i) can hold, i.e., \( b/a \sim \sqrt{3} \).

\( \gamma \) \( d(a, c) = s_2, \ d(b, d) = s_1 \). The bad angle is less than 61° so the distance of \( c \) from \( b \) and \( d \) is less than \( s_2 \). Again we have obtained the case 5.2 (i).

\( \delta \) The lengths of both diagonals are \( s_2 \). Then \( s_2 \leq 2s_1 \). We obtained that in all the four cases \( \alpha, \ldots, \delta \) we have

\[
(5.6) \quad s_2 \leq 2s_1.
\]

We claim that in \( \mathcal{G} \) (not in \( \mathcal{G}^2 \) !) every degree is small:

**Claim 5.4.** \( \Delta(\mathcal{G}) \leq 3 \).

Indeed if \( p \in \mathcal{P} \) and \( \Gamma(p) \) is the set of its neighbors then \( p \cup \Gamma(p) \) is a 2 distance set by (5.5) and (5.6). Then \( |p \cup \Gamma(p)| \leq 5 \) by Fact 5.1. Moreover, if \( |p \cup \Gamma(p)| = 5 \) then it is a regular pentagon, but in that case \( |\Gamma(p)| = 2 \), a contradiction. \( \square \)

**Proposition 5.5.** \( s_3 > s_1 + s_2 \).

**Proof.** We will prove that for every Dirichlet cell \( D \) we have

\[
(5.7) \quad \text{Area } D > 0.1734(s_1 + s_2)^2.
\]

This implies the Proposition as follows:

\[
1.001 \frac{\sqrt{3}}{2} - \frac{1}{2.236^2} (s_1 + s_2)^2 < 0.1734(s_1 + s_2)^2 \leq \frac{4\pi}{n} < 1.001 \frac{\sqrt{3}}{2} s_1(n)^2,
\]

i.e., \( s_1 + s_2 < 2.236s_1(n) \). Then (5.5) implies Proposition 5.5. To prove (5.7) we have two cases. Let \( p = \mathcal{P} \cap D \).

- If \( \deg_{\mathcal{G}}(p) \leq 2 \) (i.e., \( p \) has at most 2 neighbours in \( \mathcal{G} \)) then we can apply Lemma 2.1 with \( \Delta = 2 \), i.e.,

\[
\text{Area } D > A(2, s_2, s_1)
\]

\[
(5.8) \quad > 0.999 \left( s_1 \sqrt{s_2^2 - s_1^2} + s_2 \arcsin \frac{s_1}{s_2} \right).
\]

Here, the right hand side is larger than \( 0.1734(s_1 + s_2)^2 \) for \( 0 \leq s_1 \leq s_2 \leq 2s_1 \).

- If \( \deg_{\mathcal{G}}(p) \geq 3 \) then \( \deg_{\mathcal{G}}(p) = 3 \) by Claim 5.4. Let \( \Gamma(p) = \{u, v, w\} \). Then every distance in \( \{p, u, v, w\} \) is either \( s_1 \) or \( s_2 \). Then one of the following three subcases holds (by Fact 5.2)
-- \{p, u, v, w\} is isomorphic to 5.2 (i). Then \(D\) contains the intersection of 3 halfspheres and a circle of radius \(s_2/2\). (see Fig. 3). Hence \(\sqrt{3} s_1 < s_2 < 1.733s_1\) and

\[
\text{Area } D > 0.999 \left( \frac{\sqrt{3}}{6} + \frac{\sqrt{2}}{4} + \frac{3}{8} \left( \frac{4}{3} \pi - 2 \arctan \sqrt{2} \right) \right) s_1^2 > 1.48 s_1^2
\]

\[0.1734(s_1 + 1.733s_1)^2 > 0.1734(s_1 + s_2)^2.\]

-- \{p, u, v, w\} is isomorphic to 5.2 (v) (Fig.4). Then \(1.931 < s_2/s_1 < 1.932\) and Area \(D > 0.990(1.497 \ldots s_1^2) > 1.495s_1^2 > 0.1734(s_1 + s_2)^2.\)

-- \{p, u, v, w\} is isomorphic to 5.2 (vi). (See Fig. 5). Then \(\sqrt{3} s_1 > s_2 > 1.732s_1\), hence Lemma 2.1 yields (with \(\epsilon = 0.001\), \(\Delta = 3\), \(s_2 = s_1.1732\)) that

\[s_1(P) < s_1(n)0.828.\]

Thus by (5.5) we have

(5.9) \[s_3 > 2.7s_1.\]

We claim that in this case there are no two crossing edges of \(G^2\), a contradiction to our earlier assumptions. If \{p, u, v, w\} is a 2-distance set of type (vi) then there is no other type of 4-element 2-distance set in \(G^2\). Consider the crossing edges \(ac\) and \(bd\). The cases \(\alpha, \beta, \gamma\) are impossible so we have that \(s_1 = d(a, b) = d(a, d), s_2 = d(a, c) = d(b, d) \sim 1.73s_1\). This is not a 2-distance set so we may suppose that, e.g., \(d(d, c) > s_3 > 2.7s_1\). Then \(d(b, c) \leq s_2\), too. It is easy to check, that such a convex quadrilateral does not exist.

The proof of Proposition 5.5 is complete. \[\square\]

**Proposition 5.6.** If \(ac\) and \(bd\) are two crossing arcs in \(G^2\) then \(\min\{\deg_G(x) : x \in \{a, b, c, d\}\} \leq 4. \]

*Proof.* As we have seen above, we may suppose that \(d(a, b) = d(a, d) = s_1, d(a, c) = s_2 \) and \(d(b, d) = s_1 \) or \(s_2\). Then, by Proposition 5.5, \(\{a, b, c, d\}\) is a 2-distance set. So its type
is among (i)–(iv), by Fact 5.2. We claim that \( \deg_{G^2}(a) \leq 4 \). If \( e \in \mathcal{P} - \{a, b, c, d\} \) and \( d(a, e) = s_1 \), then \( \{a, b, c, d, e\} \) is a 2-distance set with \( \deg_\mathcal{G}(a) \geq 3 \), which contradicts to Fact 5.1. If \( d(a, e) = s_2 \) then \( \{a, b, c, d\} \) and \( \{a, b, c, e\} \) are similar 2-distance sets. This is impossible in the cases (i), (ii) and (iv), and in the case (iii) we obtain a regular pentagon. \( \square \)

**Proposition 5.7.** There exists a \( \mathcal{P}' \subset \mathcal{P} \), \( |\mathcal{P}'| \leq |\mathcal{P}|/5 \) such that \( s_1(\mathcal{P}') \geq s_3(\mathcal{P}) \).

**Proof.** Let \( \mathcal{P}_0 = \mathcal{P} \) and consider two crossing edges. An endpoint of them has degree at most 4 (in \( G^2 \)). Denote this point by \( p_1 \) and let \( \mathcal{P}_1 = \mathcal{P} - \{p\} - \{\Gamma(p)\} \). Repeat this step until we have crossing edges of length at most \( s_2 \) in \( \mathcal{P}_i \). Finally, we have a set \( Q = \{p_1, \ldots, p_t\} \) such that \( d(q, p) > s_2 \) for \( q \in Q \), and \( p \in \mathcal{P}_t \cup Q \), and \( |\mathcal{P}_t| \geq |\mathcal{P}| - 5|Q| \). Then, by the four color theorem we have a \( Q' \subset \mathcal{P}_t \), \( |Q'| \geq |\mathcal{P}_t|/4 \) with \( s_1(Q') > s_2 \). Then let \( \mathcal{P}' = Q \cup Q' \). \( \square \)

Finally, Proposition 5.7 obviously implies (5.4). \( \square \)

**References**


[CSzT] F.R.K. Chung, E. Szemerédi and W.T. Trotter, Jr., *The number of distinct distances determined by a finite point set in the plane.*


