VECTOR FIELDS IN THE VICINITY OF A COMPACT INVARIANT MANIFOLD*

by

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I. Statement of Problem

Let us consider two vector fields

(1) $X' = F(X)$

(2) $Y' = G(Y)$

defined on a given Euclidean space E where F and G are of class C^{N+1}. Furthermore assume that there is a smooth compact manifold M smoothly imbedded in E and that M is invariant for both vector fields. Also that F and G agree on M, ie. $F|M = G|M$.

We wish to study the question of C^s - conjugacies between (1) and (2). Let s be a nonnegative integer. We shall say that F and G are C^s - <u>conjugate</u> near M if there are open neighborhoods V_1 and V_2 of M and a homeomorphism $H:V_1 \to V_2$ such that

A) H is a C^s-diffeomorphism for $s > 1$,

B) $H(X) = X$ for $x \in M$,

C) If $X(t)$ is a solution of (1) and $X(t) \in V_1$, for t in some interval I, then $Y(t) = H(X(t))$ is a solution of (2) for $t \in H$, and

D) Statement C) holds for $H^{-1}:V_2 \to V_1$.

It is easy to see that for $s > 1$ the conditions C) and D) for a C^s - conjugacy can be restated as

$$DH(X)F(X) = G(H(X)) \qquad , \quad X \in V_1$$

$$DH^{-1}(Y)G(Y) = F(H^{-1}(Y)) \qquad , \quad Y \in V_2 .$$

More specifically we want to study the question of a C^s-conjugacy between (1) and (2) when G is the "linearized" vector field near M . We will define the linearized vector field shortly. Our general approach will be to find sufficient conditions in terms of $J(X) = DF(X)$,the Jacobian matrix of F , for $X \varepsilon M$ that guarantee that (1) and (2) are C^s - conjugate near M .

In the case that M is a fixed point or a periodic orbit, then the C^s - conjugacy question fits into the classical theory or ordinary differential equations and answers can be found in several sources including Belickii (1973, 1978), Grobman (1959, 1962), Hartman (1960, 1963, 1964), Nelson (1969), Palmer (1980) and Sternberg (1957, 1958).

Our interest here is primarily in the case that $\dim M > 2$. In this setting, rather little is known about any of these problems, however some contributions are especially relevant.

First there is the theorem of Pugh and Shub (cf. Hirsch et. al (1977) and Pugh-Shub (1970)) who give sufficient conditions for a C^0-conjugacy between (1) and (2). Specifically they show that if M is asymptotically stable and the flow near M is normally hyperbolic, then (1) and (2) are C^0-conjugate. (Incidentally, it is not difficult to show that their assumption of asymptotic stability can be dropped.)

Next there is the theorem of Robinson (1971) which can be applied to the question of a smooth conjugacy between (1) and (2). If, in addition to an assumption about the normal hyperbolicity, one assumes that F and G satisfy

$$D_{\perp}^{p}(F,G) = (0,0) \qquad \text{(at } X = 0)$$

for $0 < P < N$, where N is sufficiently large, then there is a C^s - conjugacy between (1) and (2). Robinson also describes a fairly complicated formula relating N , s and the spectral properties of (1) .

As noted above, we seek conditions in terms of the Jacobian matrix $J(X)$ alone which guarantee that there is a C^k - conjugacy between (1) and (2). This differs in an important way from Robinson's approach since he also made assumptions about the Taylor series expansion of the nonlinear part of F near M . Nevertheless one can take advantage of Robinson's Theorem.

It is convenient to simplify the discussion and assume that M is smoothly imbedded in E and that M has a trivial normal bundle. (The general problem can easily be reduced to this case.) It then follows that one can introduce curvilinear local coordinates so that in the vicinity of M the vector field (1) becomes

$$\begin{aligned} x' &= A(\theta)x + F(x,\theta) \\ \theta' &= g(\theta) + G(x,\theta) \end{aligned} \tag{3}$$

where θ represents local coordinates on M and $x \in R^k$ represents a normal vector to M . Furthermore F and G satisfy

$$(F,D_1F,G)(0,\theta) = (0,0,0)$$

where $D_1 = \partial/\partial x$. Also $A(\theta)$ is the linear part of F projected in the normal x-direction at the point $\theta \in M$. The equation $\theta' = g(\theta)$ describes the flow on the manifold M .

The linearized vector field near M is defined as the vector field

$$(4)\qquad \begin{aligned} y' &= A(\phi)y \\ \phi' &= g(\phi) \end{aligned}$$

where $\phi \in M$ and $y \in R_k$. The linearized flow in the tangent bundle TM is given (in these coordinates) by

$$(5)\qquad \begin{aligned} v' &= B(\theta)v \\ \theta' &= g(\theta) \end{aligned}$$

where $B = D_2 g$, $D_2 = \partial/\partial\theta$ and $v \in R^p$ where $p = \dim M$.

The specific problem we are interested in here concerns the behavior of the flow in the vicinity of M . Specifically we seek sufficient conditions in terms of the matrices $A(\theta)$ and $B(\theta)$ in order that there exists a C^s-conjugacy H of the form

$$(6)\qquad y = x + u(x,\theta) \quad , \phi = \theta + v(x,\theta)$$

which maps Eq. (3) to Eq. (4) in the vicinity of M . The restriction that H = identity on M means that $u(o,\theta) = 0$ and $v(0,\theta) = 0$.

II. The Spectra and Normal Hyperbolicity.

We shall use the spectral theory for flows developed in Sacker-Sell (1978, 1980). Let Σ_N denote the normal spectrum of M , that is Σ_N is the collection of all $\lambda \in R$ for which the linear skew-product flow

$$x' = (A(\theta) - \lambda I)x \quad , \quad \theta' = g(\theta)$$

fails to have an exponential dichotomy. Similarly let Σ_T denote the tangent spectrum of M, that is Σ_T is the collection of all $\lambda \in R$ for which

$$v' = (B(\theta) - \lambda I)v \quad , \quad \theta' = g(\theta)$$

fails to have an exponential dichotomy. Recall that if $\dim M \geq 1$, then $0 \in \Sigma_T$.

Next define $a \geq 0$ and $b \geq 0$ by

$$a = \inf\{\lambda \geq 0: \ \Sigma_T \subseteq [-\lambda,\lambda]\}$$

$$b = \sup\{\lambda \geq 0: \ \Sigma_N \subseteq (-\infty,-\lambda] \cup [\lambda,\infty)\}$$

The manifold M is said to be <u>normally hyperbolic</u> in the flow generated by (1) if $a < b$. M is <u>normally hyperbolic of degree</u> r, where r is positive integer, if $ra < b$.

Since the dimension of the normal bundle is k, it follows from the Spectral Theorem, Sacker-Sell (1978) that normal spectrum is the union of q nonoverlapping compact intervals, $I_1,\ldots,I_q$, where $1 \leq q \leq k$. Moreover associated with each spectral interval I_i there is an invariant spectral subbundle V_i of $R^k \times M$ with $\dim V_i(\theta) = n_i$. Furthermore n_i is independent of θ, $n_i \geq 1$ and $n_1 + \ldots + n_q = n$.

Next we wish to define the notion of an admissible k-type $(\lambda_1,\ldots,\lambda_k)$ from the spectrum Σ_N. What this means, in the case that A is a constant matrix with only real eigenvalues, is that the λ_i's are the eigenvalues of A repeated with their multiplicities. More generally we shall say that a given k-tuple of real numbers $(\lambda_1,\ldots,\lambda_k)$ is <u>admissible</u> provided

i) the mapping $j \to \lambda_j$ from $\{1,\ldots,k\}$ to R has its range in Σ_N , and

ii) $\mathrm{Card}\{j:\lambda_j \in I_i\} = n_i$, $1 \leq i \leq q$.

III. Statement of Main Result

In the statement of the smooth linearization theorem which we give below we shall use properties of the normal spectrum generated by $x' = A(\theta)x$. These properties are basically the generalization of the time-varying case of eigenvalue nonresonance conditions which arise in the study of linearization near a fixed point.

Theorem. Consider the equation (3)

$$x' = A(\theta)x + F(x,\theta)$$
$$\theta' = g(\theta) + G(x,\theta)$$

near M where the coefficients are of class C^{N+1} and M is normally hyperbolic of order r . Let a and b be defined as above. Assume that one has

1) $|\lambda - (m_1\lambda_1 + \ldots + m_k\lambda_k)| > ra$

2) $|m_1\lambda_1 + \ldots + m_k\lambda_k| > (r+1)a$

for all $\lambda \in \Sigma_N$, and all admissible k-tuples $(\lambda_1,\ldots,\lambda_k)$ and nonnegative integers $m_1,\ldots,m_k$ that satisfy

$$2 \leq (m_1 + \ldots + m_k) \leq N$$

_If_ $q = \min(r,N)$ _is sufficiently large then there is a_ C^s - _conjugacy between_ (3) _and_ (4) .

The basic approach to this problem is to introduce a preliminary change of variables

$$(7) \qquad z = x + u(x,\theta) \quad , \quad \beta = \theta + v(x,\theta)$$

to reduce Eq. (3) to

$$\dot{z} = A(\beta)z + F(z,\beta)$$

$$\dot{\beta} = g(\beta) + g(z,\beta)$$

where $D^P(F,G) = (0,0)$ at $(0,\beta)$ for $0 < P < N$ and then to use Robinson's Theorem. The function u and v in Eq. (7) are chosen to be appropriate polynomials in the x-variable with coefficients that depend on θ . The smoothness of these coefficients is guaranteed by the following proposition concerning the solutions of inhomogeneous linear differential systems.

Lemma. Let M *be a smooth compact manifold with a flow* $\theta' = g(\theta)$ *given in local coordinates* θ , *and consider the linear inhomogeneous differential system over* M *given by*

$$x' = A(\theta)x + f(\theta) \quad , \quad x \in X$$

$$\theta' = g(\theta)$$

where X *is a finite dimensional Banach space, and* A, F *and* g *are of class* C^N *on* M . *Assume further that the manifold* M *in the vector field*

$$x' = A(\theta)x \quad , \quad \theta' = g(\theta)$$

is normally hyperbolic of degree r . *Then there is a unique continuous function* $x : M \to X$ *such that* $x(\theta \cdot t)$ *is a solution of* $x = A(\theta \cdot t)x + f(\theta \cdot t)$ *and* $\theta \cdot t$ *is a solution of* $\theta' = g(\theta)$, $\theta(0) = \theta$ *on* M . *Moreover* x *is of class* C^s *on* M *where* $s = \min(r,N)$.

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