NECESSARY CONDITIONS AT THE BOUNDARY FOR MINIMIZERS IN FINITE ELASTICITY

By

Henry C. Simpson

and

Scott J. Spector

IMA Preprint Series # 390

January 1988

Necessary Conditions at the Boundary
for Minimizers in Finite Elasticity

by

Henry C. Simpson & Scott J. Spector

1. Introduction

In this paper we derive pointwise algebraic conditions on the elasticity tensor $\mathbf{C}(\mathbf{x}, \nabla\mathbf{f}(\mathbf{x}))$ that are necessary for a deformation $\mathbf{f}$ to be a local minimizer of the energy of an elastic body that is subjected to dead loads. In particular we show that the Legendre-Hadamard condition, Agmon's condition, and the new condition: if, for some vector $\mathbf{e}$,

$$\mathbf{e} \otimes \mathbf{n}_0 \cdot \mathbf{C}_0[\mathbf{e} \otimes \mathbf{n}_0] = 0$$

then (1.1)

$$\mathbf{C}_0[\mathbf{e} \otimes \mathbf{n}_0] = \mathbf{0}$$

are necessary conditions. Here $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$ is the outward unit normal to the boundary at any point $\mathbf{x}_0$ where the deformation is not prescribed and $\mathbf{C}_0 = \mathbf{C}(\mathbf{x}_0, \nabla\mathbf{f}(\mathbf{x}_0))$ where $\mathbf{C} = \partial^2 W/\partial(\nabla\mathbf{f})^2$; the second derivative of the stored energy W.

We also show that the three aforementioned conditions are sufficient for the nonnegativity of the second variation of the energy of a homogeneous deformation of a homogeneous body that, in the reference configuration, has the shape[1] of a half-

[1]The crucial geometric assumption is that the boundary of the body contains a portion of a hyperplane that is the only surface upon which the deformation is not prescribed.

ball with the deformation prescribed on the curved surface of the body.

A long standing problem[2] in the calculus of variations is that of finding pointwise algebraic conditions on $\underset{\sim}{C}$ that are necessary and sufficient for the nonnegativity of the second variation, i.e.,

$$\delta^2 E_{\underset{\sim}{f}}(\underset{\sim}{u}) := \int_\Omega \nabla\underset{\sim}{u}(\underset{\sim}{x})\cdot\underset{\sim}{C}(\underset{\sim}{x},\nabla\underset{\sim}{f}(\underset{\sim}{x}))[\nabla\underset{\sim}{u}(\underset{\sim}{x})]d\underset{\sim}{x} \geq 0 \qquad (1.2)^3$$

for all variations $\underset{\sim}{u}$. It is well-known that a necessary condition for (1.2) to be satisfied is the Legendre-Hadamard condition, that is,

$$\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\sim}{C}_0[\underset{\sim}{a} \otimes \underset{\sim}{b}] \geq 0 \qquad (1.3)$$

for all points $\underset{\sim}{x}_0 \in \bar{\Omega}$ and all vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. A well-known condition that is sufficient for (1.2) is that $\underset{\sim}{C}_0$ be positive semi-definite, that is,

$$\underset{\sim}{M} \cdot \underset{\sim}{C}_0[\underset{\sim}{M}] \geq 0 \qquad (1.4)$$

for all points $\underset{\sim}{x}_0 \in \bar{\Omega}$ and matrices $\underset{\sim}{M}$.

In the special case when $C(\underset{\sim}{x},\nabla\underset{\sim}{f}(\underset{\sim}{x})) \equiv \underset{\sim}{C}_0$ is independent of $\underset{\sim}{x}$ more precise results are known. In particular van Hove [29] proved[4] that (1.3) is necessary and sufficient for the nonnegativity of the second variation provided that all

---

[2]See Hadamard [11, p. 252] who referred to (1.2) as "la condition de stabilité de l'équilibre interne."

[3]In the elasticity literature this condition is usually called infinitesimal stability.

[4]When $\Omega \subset \mathbb{R}^2$ this result is due to Terpstra [26] or Hestenes & McShane [14].

variations satisfy $\underset{\sim}{u} = \underset{\sim}{0}$ on the boundary of Ω. If all smooth functions are admitted as variations then the choice $\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{M}\underset{\sim}{x}$ shows that (1.4) is necessary and sufficient for (1.2). Thus the boundary conditions imposed upon the variations are a crucial part of this problem.

We note that our results are related to the existing work on surface instabilities in a half-space.[5] In such problems one is interested in finding algebraic conditions on the elasticity tensor such that the complementing (Lopatinsky-Shapiro) condition fails[6]. Thus the linearized equilibrium equations in a half-space $\mathcal{H}$ admit a nontrivial bounded exponential solution, i.e.,

$$\begin{aligned} \operatorname{div} \underset{\sim}{C}_0[\nabla \underset{\sim}{w}] &= \alpha^2 \underset{\sim}{w} && \text{in } \mathcal{H}, \\ \underset{\sim}{C}_0[\nabla \underset{\sim}{w}]\underset{\sim}{n} &= \underset{\sim}{0} && \text{on } \partial\mathcal{H}, \end{aligned} \tag{1.5}$$

where $\alpha = 0$, $\underset{\sim}{n}$ is the outward unit normal to $\partial\mathcal{H}$, and

$$\underset{\sim}{w}(\underset{\sim}{x}) = \underset{\sim}{z}(-\underset{\sim}{x}\cdot\underset{\sim}{n})\exp(\, i\, \underset{\sim}{x}\cdot\underset{\sim}{t})$$

with $\underset{\sim}{z}$ bounded and $\underset{\sim}{t}$ tangent to $\partial\mathcal{H}$.

One of the necessary conditions that we derive, Agmon's condition, is the requirement that (1.5) not admit a nontrivial bounded exponential solution for any $\alpha \neq 0$. The failure of this condition induces a more severe instability than the failure of the complementing condition since it is possible for the

---

[5] See, e.g., Biot [6, pp. 204-216], Nowinski [19, 20], Reddy [21, 22], and Usmani & Beatty [28]. Hutchinson & Tvergaard [15] conjecture that such solutions explain the onset of surface wrinking in aluminum.

[6] Cf. Agmon, Douglis & Nirenberg [2] and Simpson & Spector [24,25].

complementing condition to fail at the unique global minimizer of the energy (see Section 8) while, as a consequence of our results, the failure of Agmon's condition implies that the underlying deformation is not even a local minimizer.

Finally, we note that our results are also related to those of Chen [7] who considered the problem of finding pointwise conditions on the (constant) elasticity tensor $\underset{\sim}{C}_0$ that are necessary and sufficient for the nonnegativity of the second variation when the body is a thin plate and the variations satisfy $\underset{\sim}{u} = \underset{\sim}{0}$ on the edges of the plate. In particular Chen proved that a restricted rank-two convexity condition, i.e.,

$$[\underset{\sim}{a} \otimes \underset{\sim}{b} + \underset{\sim}{c} \otimes \underset{\sim}{n}_0] \cdot \underset{\sim}{C}_0[\underset{\sim}{a} \otimes \underset{\sim}{b} + \underset{\sim}{c} \otimes \underset{\sim}{n}_0] \geq 0 \tag{1.6}$$

for all vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$, is a sufficient condition for (1.2) to be satisfied for such a plate and that (1.6) is also necessary for (1.2) to be satisfied for arbitrarily thin plates. However, (1.6) is not necessary if the size of the plate remains fixed.

One of the necessary conditions that we derive, (1.1), can also be viewed as a restricted rank-two convexity condition since (1.1) together with the Legendre-Hadamard condition, (1.3), imply that if for some vector $\underset{\sim}{c}$,

$$\underset{\sim}{c} \otimes \underset{\sim}{n}_0 \cdot \underset{\sim}{C}_0[\underset{\sim}{c} \otimes \underset{\sim}{n}_0] = 0$$

then (1.6) is satisfied for all vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

2. Notation

We let

Lin^n := space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ with inner product and norm

$$\mathbf{G} \cdot \mathbf{H} := \mathrm{trace}\,(\mathbf{G}\mathbf{H}^T), \quad |\mathbf{G}|^2 := \mathbf{G} \cdot \mathbf{G},$$

where $\mathbf{H}^T$ denotes the transpose of $\mathbf{H}$. We write

$$\mathrm{Lin}^n_+ := \{\mathbf{H} \in \mathrm{Lin}^n \colon \det \mathbf{H} > 0\},$$
$$\mathrm{Sym}^n := \{\mathbf{E} \in \mathrm{Lin}^n \colon \mathbf{E} = \mathbf{E}^T\},$$

where det denotes the determinant. Given any $\mathbf{M} \in \mathrm{Lin}^n$ we extend its domain of definition to $\mathbb{C}^n$ and write $\mathbf{M}\mathbf{z} := \mathbf{M}\mathbf{x} + i\mathbf{M}\mathbf{y}$ where $\mathbf{x} := \mathrm{Re}\{\mathbf{z}\}$ and $\mathbf{y} := \mathrm{Im}\,\{\mathbf{z}\}$ are, respectively, the real and imaginary parts of $\mathbf{z} \in \mathbb{C}^n$. We denote by $\mathbf{a} \otimes \mathbf{b}$ the tensor product of any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$; in components

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j \,.$$

We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^n$: for a vector field $\mathbf{u}$, $\nabla\mathbf{u}$ is the tensor field with components $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$; for a tensor field $\mathbf{S}$, div $\mathbf{S}$ is the vector field with components $\sum_j \partial S_{ij}/\partial x_j$. Given any function $\Phi(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c})$ with vector or tensor arguments we write, e.g., $\partial\Phi/\partial\mathbf{a}$ for the partial Fréchet derivative with respect to $\mathbf{a}$ holding the remaining arguments fixed.

We denote by $L^2((0,\infty),\mathbb{C}^n)$ the space (of equivalence classes) of square-integrable functions $\mathbf{z}\colon(0,\infty) \to \mathbb{C}^n$. For $\mathbf{z} \in C^1([0,\infty),\mathbb{C}^n)$ with $\mathbf{z}, \dot{\mathbf{z}} \in L^2((0,\infty),\mathbb{C}^n)$ we write $\|\mathbf{z}\|_1$ for the H^1

norm of $\underset{\sim}{z}$, i.e.,

$$\|\underset{\sim}{z}\|_1^2 := \int_0^\infty |\underset{\sim}{z}(s)|^2 ds + \int_0^\infty |\dot{\underset{\sim}{z}}(s)|^2 ds \ . \tag{2.1}$$

Throughout this paper $\Omega \subset \mathbb{R}^n$ will denote a bounded open region with C^1 boundary, $\partial\Omega$. Further, we assume that

$$\partial\Omega = \mathcal{D} \cup \mathcal{S}$$

where $\mathcal{D}$ and $\mathcal{S}$ are disjoint and $\mathcal{S}$ is relatively open. We write

$$\mathrm{Var} := \{\underset{\sim}{u} \in C^1(\bar{\Omega}, \mathbb{R}^n) : \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \mathcal{D}\}.$$

3. The Constitutive Relation

For convenience we identify the body with the region $\bar{\Omega}$ that it occupies in a fixed reference configuration. A *deformation* $\underset{\sim}{f}$ of the body is a member of the space

$$\mathrm{Def} := \{\underset{\sim}{f} \in C^1(\bar{\Omega}, \mathbb{R}^n) : \det \nabla\underset{\sim}{f} > 0\}.$$

We assume that the body is hyperelastic with continuous response function $W: \bar{\Omega} \times \mathrm{Lin}^n \to \mathbb{R} \cup \{+\infty\}$. W gives the *stored energy*

$$W(\underset{\sim}{x}, \nabla\underset{\sim}{f}(\underset{\sim}{x}))$$

at any point $\underset{\sim}{x} \in \bar{\Omega}$ when the body is deformed by $\underset{\sim}{f} \in \mathrm{Def}$.

We further assume that W restricted to $\bar{\Omega} \times \mathrm{Lin}^n_+$ is C^2 and finite-valued. The derivatives

$$\underset{\sim}{S}(\underset{\sim}{x}, \underset{\sim}{F}) := \frac{\partial}{\partial \underset{\sim}{F}} W(\underset{\sim}{x}, \underset{\sim}{F}), \quad \underset{\sim}{C}(\underset{\sim}{x}, \underset{\sim}{F}) := \frac{\partial^2}{\partial \underset{\sim}{F}^2} W(\underset{\sim}{x}, \underset{\sim}{F})$$

are called the (Piola-Kirchhoff) *stress* and the *elasticity tensor*, respectively. We note that, as a consequence of the equality of the mixed partial derivatives of W,

$\mathbf{C}(\mathbf{x},\mathbf{F})\colon \mathrm{Lin}^n \to \mathrm{Lin}^n$ is *symmetric*, *i.e.*,

$$\mathbf{H} \cdot \mathbf{C}(\mathbf{x},\mathbf{F})[\mathbf{K}] = \mathbf{K} \cdot \mathbf{C}(\mathbf{x},\mathbf{F})[\mathbf{H}]$$

for all $\mathbf{x} \in \bar{\Omega}$, $\mathbf{F} \in \mathrm{Lin}^n_+$ and $\mathbf{H},\mathbf{K} \in \mathrm{Lin}^n$.

Given $\mathbf{x}_0 \in \bar{\Omega}$ and $\mathbf{f} \in \mathrm{Def}$ we write $\mathbf{C}_0 := \mathbf{C}(\mathbf{x}_0, \nabla\mathbf{f}(\mathbf{x}_0))$. We say that $\mathbf{C}_0$ satisfies the *Legendre-Hadamard condition* provided that

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathbf{C}_0[\mathbf{a} \otimes \mathbf{b}] \geq 0$$

whenever $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$.

Remark.[7] Consider the problem: Find $\mathbf{v}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ that satisfies

$$\mathrm{div}\, \mathbf{C}_0[\nabla\mathbf{v}] = \mathbf{v}_{tt} \text{ in } \mathbb{R}^n$$

where $\mathbf{v}$ has the form

$$\mathbf{v}(\mathbf{x},t) = \mathbf{a}\exp(\alpha t)\sin(\mathbf{b}\cdot\mathbf{x}),$$

$\alpha > 0$, and $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$. If such a solution were to exist one would probably call the rest state, $\mathbf{v} = \mathbf{0}$, unstable since there are initial data, arbitrarily close to $\mathbf{0}$ (in $C^0(\mathbb{R}^n)$), that give rise to a solution that grows exponentially in time (in $C^0(\mathbb{R}^n)$). However, the above equations imply that

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathbf{C}_0[\mathbf{a} \otimes \mathbf{b}] + \alpha^2|\mathbf{a}|^2 = 0$$

and hence that the Legendre-Hadamard condition is not satisfied.

---

[7]This observation is due to Hayes & Rivlin [12, Section 5]. See also Rivlin [23, pp. 409-425] and Truesdell & Noll [27, Section 73].

Thus one can view the Legendre-Hadamard condition as a dynamic stability requirement for linear elastodynamics.

Since we are interested in instances where the Legendre-Hadamard condition is an equality, for some vectors $\mathbf{a}$ and $\mathbf{b}$, we will need the following lemma.

Lemma 3.1. *Let $\mathbf{C}_0$ satisfy the Legendre-Hadamard condition and suppose that*

$$\mathbf{C}_0[\mathbf{e}_0 \otimes \mathbf{n}_0] = \mathbf{0} \tag{3.1}$$

for some $\mathbf{e}_0, \mathbf{n}_0 \in \mathbb{R}^n$. Then, for every $\mathbf{u}, \mathbf{t} \in \mathbb{R}^n$,

$$|\mathbf{u} \otimes \mathbf{n}_0 \cdot \mathbf{C}_0[\mathbf{e}_0 \otimes \mathbf{t}]|^2 \le (\mathbf{e}_0 \otimes \mathbf{t} \cdot \mathbf{C}_0[\mathbf{e}_0 \otimes \mathbf{t}])(\mathbf{u} \otimes \mathbf{n}_0 \cdot \mathbf{C}_0[\mathbf{u} \otimes \mathbf{n}_0]).$$

Proof. Let $\mathbf{u}, \mathbf{t} \in \mathbb{R}^n$ and consider the quartic polynomial

$$p(\alpha,\beta) := (\mathbf{e}_0+\alpha\beta\mathbf{u}) \otimes (\mathbf{n}_0+\beta\mathbf{t}) \cdot \mathbf{C}_0[(\mathbf{e}_0+\alpha\beta\mathbf{u}) \otimes (\mathbf{n}_0+\beta\mathbf{t})].$$

Then by the Legendre-Hadamard condition we find that $p(\alpha,\beta) \ge 0$ for every $\alpha, \beta \in \mathbb{R}$. However, since $\mathbf{C}_0$ is symmetric and satisfies (3.1) a simple computation shows that

$$p(\alpha,\beta) = \beta^2 q(\alpha,\beta),$$

where q is biquadratic. By continuity we conclude that, for every $\alpha \in \mathbb{R}$,

$$0 \le q(\alpha,0) = a\alpha^2 + b\alpha + c,$$

$$a := \mathbf{u} \otimes \mathbf{n}_0 \cdot \mathbf{C}_0[\mathbf{u} \otimes \mathbf{n}_0], \qquad c := \mathbf{e}_0 \otimes \mathbf{t} \cdot \mathbf{C}_0[\mathbf{e}_0 \otimes \mathbf{t}],$$

$$b := 2\mathbf{u} \otimes \mathbf{n}_0 \cdot \mathbf{C}_0[\mathbf{e}_0 \otimes \mathbf{t}].$$

Thus the discriminant of $q(\alpha,0)$ must be nonpositive, $b^2 - 4ac \le 0$, which is the desired result. ∎

For $\underset{\sim}{x}_0 \in \partial\Omega$ we write $\underset{\sim}{n}_0 = \underset{\sim}{n}(\underset{\sim}{x}_0)$ for the outward unit normal to $\partial\Omega$ and $\mathcal{H}$ for the half-space

$$\mathcal{H} := \{\underset{\sim}{y} \in \mathbb{R}^n: (\underset{\sim}{y}-\underset{\sim}{x}_0)\cdot\underset{\sim}{n}_0 < 0\}.$$

We consider the problem: Find $\underset{\sim}{w}: \bar{\mathcal{H}} \to \mathbb{C}^n$ that satisfies

$$\begin{aligned} \operatorname{div} \underset{\sim}{C}_0[\nabla\underset{\sim}{w}] &= \alpha^2 \underset{\sim}{w} \text{ in } \mathcal{H}, \\ \underset{\sim}{C}_0[\nabla\underset{\sim}{w}]\underset{\sim}{n}_0 &= \underset{\sim}{0} \text{ on } \partial\mathcal{H}, \end{aligned} \tag{3.2}$$

where $\alpha > 0$.

We are specifically interested in whether or not there are solutions of (3.2) that are <u>bounded exponentials</u>, that is, functions

$$\underset{\sim}{w}(\underset{\sim}{y}) = \underset{\sim}{z}(-(\underset{\sim}{y}-\underset{\sim}{x}_0)\cdot\underset{\sim}{n}_0)\exp(i(\underset{\sim}{y}-\underset{\sim}{x}_0)\cdot\underset{\sim}{t}) \tag{3.3}$$

for some $\underset{\sim}{t} \in \mathbb{R}^n$ that is <u>tangent</u> to $\partial\mathcal{H}$ ($\underset{\sim}{t}\cdot\underset{\sim}{n}_0 = 0$) and some $\underset{\sim}{z} \in C^2([0,\infty),\mathbb{C}^n)$ satisfying $\sup_s |\underset{\sim}{z}(s)| < \infty$.

We say that the pair $(\underset{\sim}{C}_0, \underset{\sim}{n}_0)$ satisfies *Agmon's condition*[8] provided that for every real $\alpha > 0$ the only bounded exponential solution of (3.2) is $\underset{\sim}{w} = \underset{\sim}{0}$.

<u>Remark</u>. Suppose that (3.2) has a bounded exponential solution $\underset{\sim}{w}_\alpha$ for some $\alpha > 0$. Then (3.2) has such a solution for <u>all</u> $\alpha > 0$. The required solution is given by

$$\underset{\sim}{w}_\beta(\underset{\sim}{x}) = \underset{\sim}{w}_\alpha(\frac{\beta}{\alpha}\underset{\sim}{x}).$$

---

[8] <u>Cf</u>. <u>Agmon</u> [1, p. 123].

Remark[9]. Consider the problem: Find $\mathbf{v}: \bar{\mathcal{H}} \times \mathbb{R} \to \mathbb{R}^n$ that satisfies

$$\operatorname{div} \mathbf{C}_0[\nabla \mathbf{v}] = \mathbf{v}_{tt} \text{ in } \mathcal{H},$$
$$\mathbf{C}_0[\nabla \mathbf{v}]\mathbf{n}_0 = \mathbf{0} \text{ on } \partial\mathcal{H}, \tag{3.4}$$

where $\mathbf{v}$ has the form

$$\mathbf{v}(\mathbf{x}, t) = \operatorname{Re}\{\mathbf{w}(\mathbf{x})\}\exp(\alpha t), \tag{3.5}$$

$\alpha > 0$, and $\mathbf{w}$ is a bounded exponential. If a solution to (3.4) of the form (3.5) were to exist one would probably call the rest state, $\mathbf{v} = \mathbf{0}$, unstable since there are initial data, arbitrarily close to $\mathbf{0}$ (in $C^0(\mathcal{H})$), that give rise to a solution growing exponentially in time (in $C^0(\mathcal{H})$). Thus one can view Agmon's condition as a dynamic stability requirement for linear elastodynamics.

We define $\mathbf{M}, \mathbf{N}_{\mathbf{t}}, \mathbf{P}_{\mathbf{t}}^{\alpha} \in \mathrm{Lin}^n$ by

$$\mathbf{M}\mathbf{a} := \mathbf{C}_0[\mathbf{a} \otimes \mathbf{n}_0]\mathbf{n}_0, \quad \mathbf{N}_{\mathbf{t}}\mathbf{a} := \mathbf{C}_0[\mathbf{a} \otimes \mathbf{t}]\mathbf{n}_0,$$
$$\mathbf{P}_{\mathbf{t}}^{\alpha}\mathbf{a} := \mathbf{C}_0[\mathbf{a} \otimes \mathbf{t}]\mathbf{t} + \alpha^2\mathbf{a}. \tag{3.6}$$

Then (3.2) and (3.3) reduce to a system of ordinary differential equations,

$$-\mathbf{M}\ddot{\mathbf{z}} + i(\mathbf{N}_{\mathbf{t}} + \mathbf{N}_{\mathbf{t}}^T)\dot{\mathbf{z}} + \mathbf{P}_{\mathbf{t}}^{\alpha}\mathbf{z} = \mathbf{0} \text{ on } (0, \infty), \tag{3.7}$$

---

[9] Cf. Simpson & Spector [25, p. 10].

with boundary conditions,

$$\mathbf{M}\dot{\mathbf{z}}(0) - i\mathbf{N}_{\mathbf{t}}\mathbf{z}(0) = \mathbf{0}. \tag{3.8}$$

We note that the existence of a bounded solution to (3.7) and (3.8) is completely determined by $\mathbf{M}$, $\mathbf{P}_{\mathbf{t}}^{0}$, and $\mathbf{N}_{\mathbf{t}}$. Thus Agmon's condition is an algebraic condition on the pair $(\mathbf{C}_0, \mathbf{n}_0)$ just as the Legendre-Hadamard condition is an algebraic condition on $\mathbf{C}_0$ (see section 8 for some simple examples).

Given $\alpha > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we define $\mathbf{L}: \mathbb{R} \to \mathrm{Sym}^n$ by

$$\mathbf{L}(\lambda) := \lambda^2\mathbf{M} + \lambda(\mathbf{N}_{\mathbf{t}} + \mathbf{N}_{\mathbf{t}}^T) + \mathbf{P}_{\mathbf{t}}^{\alpha}. \tag{3.9}$$

Then, by (3.6),

$$\mathbf{a}\cdot\mathbf{L}(\lambda)\mathbf{a} = \mathbf{a} \otimes (\lambda\mathbf{n}+\mathbf{t})\cdot\mathbf{C}_0[\mathbf{a} \otimes (\lambda\mathbf{n}+\mathbf{t})] + \alpha^2|\mathbf{a}| \tag{3.10}$$

and hence the Legendre-Hadamard condition implies that $\mathbf{L}(\lambda)$ is strictly positive definite for every $\lambda \in \mathbb{R}$. A related result is

Lemma 3.2. *Suppose that $\mathbf{L}(\lambda)$ is strictly positive definite for every $\lambda \in \mathbb{R}$. Let $\mathbf{z} \in C^2([0,\infty), \mathbb{C}^n)$ be a bounded solution of (3.7). Then $\mathbf{z} \in C^\infty([0,\infty),\mathbb{C}^n)$ and $\mathbf{z}$ and all its derivatives are contained in $L^2((0,\infty),\mathbb{C}^n)$.*

Proof. The standard existence and uniqueness theory[10] for systems of ordinary differential equations with constant coefficients implies that any solution of (3.7) must be a linear combination of solutions of the form

$$\mathbf{z}(s) = \mathbf{a}s^m\exp(-\eta s) \tag{3.11}$$

for some $\mathbf{a} \in \mathbb{C}^n$, $\eta \in \mathbb{C}$, and nonnegative integer m. Thus all solutions of (3.7) are contained in C^∞. Moreover, if (3.11) is a

[10] *Cf.*, *e.g.*, *Ince* [16, Chapter IV].

solution of (3.7) then so is

$$\underset{\sim}{z}(s) = \underset{\sim}{a}\exp(-\eta s). \qquad (3.12)$$

In order to show that any bounded solution has the property that it and all its derivatives are contained in $L^2((0,\infty),\mathbb{C}^n)$ we need only show that $\mathrm{Re}(\eta) \neq 0$ in (3.12). Therefore, if we let $\eta = i\lambda$ $(\lambda \in \mathbb{R})$ in (3.12) and substitute (3.12) into (3.7) we find that

$$[\lambda^2 \underset{\sim}{M} + \lambda(\underset{\sim}{N}_t + \underset{\sim}{N}_t^T) + \underset{\sim}{P}_t^\alpha]\underset{\sim}{a} = \underset{\sim}{L}(\lambda)\underset{\sim}{a} = \underset{\sim}{0},$$

which is not possible since $\underset{\sim}{L}(\lambda)$ is strictly positive definite.

∎

Remark. The above lemma shows that, given the Legendre-Hadamard condition, all bounded solutions of (3.7) are actually exponentially decaying. Thus, given the Legendre-Hadamard condition, Agmon's condition is equivalent to the requirement that there is no nontrivial solution to (3.2) of the form (3.3) where $\underset{\sim}{z}$ is exponentially decaying.

4. The Energy Minimizers

We suppose that $\underset{\sim}{b} \in C^0(\Omega,\mathbb{R}^n)$ is a given body force field, that $\underset{\sim}{s} \in C^0(\mathscr{S},\mathbb{R}^n)$ is a given surface force field, and let

$$E(\underset{\sim}{f}) := \int_\Omega [W(\underset{\sim}{x},\nabla\underset{\sim}{f}(\underset{\sim}{x})) - \underset{\sim}{b}(\underset{\sim}{x})\cdot\underset{\sim}{f}(\underset{\sim}{x})]d\underset{\sim}{x} - \int_{\mathscr{S}} \underset{\sim}{s}(\underset{\sim}{x})\cdot\underset{\sim}{f}(x)d\mathscr{S}_{\underset{\sim}{x}} \qquad (4.1)$$

denote the total energy when the body is deformed by $\underset{\sim}{f}$.

We are interested in a deformation $\underset{\sim}{f}$ that minimizes the total energy among certain classes of competing deformations, *i.e.*,

$$E(\underset{\sim}{f}) \leq E(\underset{\sim}{f} + \underset{\sim}{u}) \tag{4.2}$$

for certain variations $\underset{\sim}{u}$.

Definition. We say that $\underset{\sim}{f} \in C^1(\bar{\Omega}, \mathbb{R}^n)$ is a

(i) *global minimizer* provided that (4.2) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}$;

(ii) *strong local minimizer* provided that there is an $\epsilon > 0$ such that (4.2) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}$ such that

$$\sup_{\underset{\sim}{x}\in\Omega} |\underset{\sim}{u}(\underset{\sim}{x})| < \epsilon;$$

(iii) *weak local minimizer* provided that there is an $\epsilon > 0$ such that (4.2) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}$ such that

$$\sup_{\underset{\sim}{x}\in\Omega} |\underset{\sim}{u}(\underset{\sim}{x})| + \sup_{\underset{\sim}{x}\in\Omega} |\nabla\underset{\sim}{u}(\underset{\sim}{x})| < \epsilon.$$

Clearly, if $\underset{\sim}{f}$ is a global minimizer then it is also a strong local minimizer and if $\underset{\sim}{f}$ is a strong local minimizer then it is also a weak local minimizer. All of our results will be consequences of $\underset{\sim}{f}$ being a weak local minimizer. Thus the conditions we develop will also be necessary conditions for $\underset{\sim}{f}$ to be either a strong local minimizer or a global minimizer. Before proceeding we first recall some results concerning strong local minimizers.

Given any $\underset{\sim}{x}_0 \in \bar{\Omega}$, $\underset{\sim}{f} \in C^1(\bar{\Omega}, \mathbb{R}^n)$, and $r > 0$ we define
$\underset{\sim}{F}_0 := \nabla\underset{\sim}{f}(\underset{\sim}{x}_0)$,

$$\underset{\sim}{f}_0(\underset{\sim}{y}) := \underset{\sim}{F}_0\underset{\sim}{y}, \quad W_0(\underset{\sim}{G}) := W(\underset{\sim}{x}_0, \underset{\sim}{G})$$

for any $\underset{\sim}{G} \in \mathrm{Lin}^n$ and

$$\underset{\sim}{y} \in \mathcal{B} := \{\underset{\sim}{y} \in \mathbb{R}^n : |\underset{\sim}{y} - \underset{\sim}{x}_0| < r\}.$$

If, in addition, $\underset{\sim}{x}_0 \in \partial\Omega$ we write $\underset{\sim}{n}_0 := \underset{\sim}{n}(\underset{\sim}{x}_0)$, the outward unit normal to $\partial\Omega$, and

$$\mathcal{HB} := \{\underset{\sim}{y} \in \mathcal{B}: (\underset{\sim}{y}-\underset{\sim}{x}_0)\cdot\underset{\sim}{n}_0 < 0\}.$$

We interpret $\underset{\sim}{f}_0$ as a homogeneous deformation of a homogeneous body that occupies the region $\mathcal{B}$ or $\mathcal{HB}$ in a homogeneous reference configuration.

Proposition 4.1. *Let $\underset{\sim}{f} \in C^1(\bar{\Omega}, \mathbb{R}^n)$ be a strong local minimizer. Then, for every $\underset{\sim}{x}_0 \in \Omega$,*

$$\int_{\mathcal{B}} W_0(\underset{\sim}{F}_0)d\underset{\sim}{y} \le \int_{\mathcal{B}} W_0(\underset{\sim}{F}_0 + \nabla\underset{\sim}{u}(\underset{\sim}{y}))d\underset{\sim}{y} \qquad (4.3)^{11,12}$$

for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{B})$. Moreover, for every $\underset{\sim}{x}_0 \in \mathcal{S}$,

$$\int_{\mathcal{HB}} W_0(\underset{\sim}{F}_0)d\underset{\sim}{y} \le \int_{\mathcal{HB}} W_0(\underset{\sim}{F}_0 + \nabla\underset{\sim}{u}(\underset{\sim}{y}))d\underset{\sim}{y} \qquad (4.4)^{13}$$

for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$. Here

$$\mathrm{Var}(\mathcal{B}) := \{\underset{\sim}{u} \in C^1(\bar{\mathcal{B}}, \mathbb{R}^n): \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \partial\mathcal{B}\},$$

$$\mathrm{Var}(\mathcal{HB}) := \{\underset{\sim}{u} \in C^1(\overline{\mathcal{HB}}, \mathbb{R}^n): \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \partial\mathcal{B} \cap \partial(\mathcal{HB})\}. \qquad (4.5)$$

In other words, in order for $\underset{\sim}{f}$ to be a strong local minimizer each of the homogeneous deformations $\underset{\sim}{f}_0(\underset{\sim}{y}) = [\nabla\underset{\sim}{f}(\underset{\sim}{x}_0)]\underset{\sim}{y}$

---

[11] This result is essentially due to Meyers [17, pp. 128-131]. See Ball [3, Theorem 2.1] or Gurtin [10, pp. 9-10] for a direct proof.

[12] This inequality is Morrey's [18] quasiconvexity condition.

[13] This result is due to Ball & Marsden [5] who refer to the inequality as quasiconvexity at the boundary.

must be a *global* minimizer of the energy of a ball (half-ball) composed of the material at $\mathbf{x}_0$.

We now suppose that $\mathbf{f} \in \mathrm{Def}$ is a *weak* local minimizer. Then, for every $\mathbf{u} \in \mathrm{Var}$,

$$0 \leq \varphi(t) := E(\mathbf{f} + t\mathbf{u}) - E(\mathbf{f})$$

provided t is sufficiently small. Thus $\dot{\varphi}(0) = 0$ and $\ddot{\varphi}(0) \geq 0$. If f is C^2 one can then show that $\mathbf{f}$ satisfies the *equation of equilibrium* and the *natural boundary condition*, that is,

$$\operatorname{div} \mathbf{S}(\nabla\mathbf{f}) + \mathbf{b} = \mathbf{0} \text{ in } \Omega,$$

$$\mathbf{S}(\nabla\mathbf{f})\mathbf{n} = \mathbf{s} \text{ on } \mathcal{S}.$$

In addition (whether or not $\mathbf{f}$ is C^2) the *second variation* of E at $\mathbf{f}$ must be nonnegative, *i.e.*,

$$0 \leq \int_\Omega \nabla\mathbf{u} \cdot \mathbf{C}(\mathbf{x}, \nabla\mathbf{f}(\mathbf{x}))[\nabla\mathbf{u}(\mathbf{x})]d\mathbf{x} \tag{4.6}$$

for all $\mathbf{u} \in \mathrm{Var}$.

If we now define $\hat{W}(\mathbf{x}, \mathbf{U}) := \mathbf{U} \cdot \mathbf{C}(\mathbf{x}, \nabla\mathbf{f}(\mathbf{x}))[\mathbf{U}]$ and

$$\hat{E}(\mathbf{u}) := \int_\Omega \hat{W}(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))d\mathbf{x}$$

we find that (4.6) is equivalent to $\hat{E}(\mathbf{0}) \leq \hat{E}(\mathbf{u})$ for all $\mathbf{u} \in \mathrm{Var}$. Thus the function $\mathbf{0}$ is a strong local minimizer of $\hat{E}$ and hence we can apply Proposition 4.1 to get the following result.

Proposition 4.2. *Let* $\mathbf{f} \in \mathrm{Def}$ *be a weak local minimizer of* E. *Then, for every* $\mathbf{x}_0 \in \Omega$,

$$0 \leq \int_{\mathcal{B}} \nabla\mathbf{u}(\mathbf{y}) \cdot \mathbf{C}_0[\nabla\mathbf{u}(\mathbf{y})]d\mathbf{y} \tag{4.7}$$

for all $\mathbf{u} \in \mathrm{Var}(\mathcal{B})$. *Moreover, for every* $\mathbf{x}_0 \in \mathcal{S}$,

$$0 \leq \int_{\mathcal{HB}} \nabla \underset{\sim}{u}(\chi)\cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{u}(\chi)]d\chi$$

for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$. Here $\underset{\sim}{C}_0 := \underset{\sim}{C}(\underset{\sim}{x}_0, \nabla \underset{\sim}{f}(\underset{\sim}{x}_0))$.

Remark. A deformation that satisfies (4.6) is usually said to be infinitesimally stable.[14] Thus the above result states that in order for a deformation $\underset{\sim}{f}$ to be infinitesimally stable each of the homogeneous deformations $\underset{\sim}{f}_0(\chi) = [\nabla \underset{\sim}{f}(\underset{\sim}{x}_0)]\chi$ (of a ball or half-ball) must be infinitesimally stable.

Proposition 4.2 motivates one to consider the following problem: Find pointwise algebraic conditions on the elasticity tensor $\underset{\sim}{C}_0$ that are necessary and sufficient for (4.7) to be satisfied (for $\underset{\sim}{f}_0$ to be infinitesimally stable). The solution to this problem is well-known.

Proposition 4.3 (van Hove [29]). *A necessary and sufficient condition for (4.7) to be satisfied for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{B})$ is that $\underset{\sim}{C}_0$ satisfy the Legendre-Hadamard condition.*

The following result is then a direct consequence of Propositions 4.2 and 4.3.

Corollary 4.3.1. *Let $\underset{\sim}{f} \in$ Def be a weak local minimizer of E. Then $\underset{\sim}{C}_0 := \underset{\sim}{C}(\underset{\sim}{x}_0, \nabla \underset{\sim}{f}(\underset{\sim}{x}_0))$ satisfies the Legendre-Hadamard condition at every $\underset{\sim}{x}_0 \in \Omega$.*

---

[14] *Cf.*, *e.g.*, *Truesdell & Noll* [27, Section 68bis].

5. Main Results

We suppose that $\mathbf{f}_0$ is a homogeneous deformation of a homogeneous body composed of material with stored energy function W_0. Further, we suppose that the body occupies the region $\mathcal{HB}$ in a homogeneous reference configuration. We then consider the following problem: Find pointwise algebraic conditions on the elasticity tensor $\mathbf{C}_0$ that are necessary and sufficient for the nonnegativity of the second variation of the total energy, i.e.,

$$0 \le \int_{\mathcal{HB}} \nabla\mathbf{u}(\chi)\cdot\mathbf{C}_0[\nabla\mathbf{u}(\chi)]d\chi \tag{5.1}$$

for all $\mathbf{u} \in \mathrm{Var}(\mathcal{HB})$ (cf. (4.5)).

The following result solves this problem.

Theorem 1. *Necessary and sufficient conditions for (5.1) to be satisfied for all $\mathbf{u} \in \mathrm{Var}(\mathcal{HB})$ are that:*

(i) *$\mathbf{C}_0$ satisfies the Legendre-Hadamard condition;*

(ii) *$\mathbf{C}_0$ satisfies Agmon's condition;*

(iii) *If $\mathbf{a} \otimes \mathbf{n}_0\cdot\mathbf{C}_0[\mathbf{a} \otimes \mathbf{n}_0] = 0$ for some $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{C}_0[\mathbf{a} \otimes \mathbf{n}_0] = \mathbf{0}$.*

Here $\mathbf{n}_0$ is the outward unit normal to $\partial\mathcal{H}$.

Remark. $\mathcal{HB}$ need not be a half-ball in order for Theorem 1 to be valid. In particular if instead $\mathcal{HB}$ is contained in such a half-ball and its boundary is Lipschitz then (i), (ii), and (iii) will be sufficient for (5.1) to be satisfied for all $\mathbf{u}$ contained in

$$\mathrm{Var}(\mathcal{HB}) := \{\mathbf{u} \in C^1(\overline{\mathcal{HB}}, \mathbb{R}^n): \mathbf{u} = \mathbf{0} \text{ on } \overline{\partial(\mathcal{HB})\setminus\partial\mathcal{H}}\}.$$

Remark. Hestenes [13, pp. 328-337] has obtained an alternative algebraic characterization of (5.1). A slight modification of the argument used in [14] shows that a necessary and sufficient condition for (5.1) to be satisfied is that each of the linear transformations

$$\int_{-\infty}^{\infty} \mathbf{S}_t^{\alpha}(\tau)d\tau \qquad (\alpha > 0)$$

be positive semi-definite. Here (cf. (3.6))

$$\mathbf{S}_t^{\alpha}(\tau) := \mathbf{M} - (\tau\mathbf{M} + \mathbf{N}_t)\mathbf{L}^{-1}(\tau)(\tau\mathbf{M} + \mathbf{N}_t^T),$$

$$\mathbf{L}(\tau) := \tau^2\mathbf{M} + \tau(\mathbf{N}_t + \mathbf{N}_t^T) + \mathbf{P}_t^{\alpha} .$$

The following result, which is a direct consequence of Proposition 4.2 and Theorem 1, gives conditions that a general deformation of a general body must satisfy in order to be a local minimizer of the energy.

Theorem 2. Let $\mathbf{f} \in \mathrm{Def}$ be a weak local minimizer of

$$E(\mathbf{f}) = \int_{\Omega} [W(\mathbf{x}, \nabla\mathbf{f}(\mathbf{x})) - \mathbf{b}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]d\mathbf{x} - \int_{\mathscr{S}} \mathbf{s}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})d\mathscr{S}_{\mathbf{x}} .$$

Then, for every $\mathbf{x}_0 \in \mathscr{S}$,

(i) $\mathbf{C}_0$ satisfies the Legendre-Hadamard condition;

(ii) $\mathbf{C}_0$ satisfies Agmon's condition;

(iii) If $\mathbf{a} \otimes \mathbf{n}_0\cdot\mathbf{C}_0[\mathbf{a} \otimes \mathbf{n}_0] = 0$ for some $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{C}_0[\mathbf{a} \otimes \mathbf{n}_0] = \mathbf{0}$.

Here $\mathbf{C}_0 := \mathbf{C}(\mathbf{x}_0, \nabla\mathbf{f}(\mathbf{x}_0))$ $(\mathbf{C} := \partial^2 W/\partial(\nabla\mathbf{f})^2)$ and $\mathbf{n}_0 := \mathbf{n}(\mathbf{x}_0)$, the outward unit normal to $\partial\Omega$ at $\mathbf{x}_0$.

Remark. Theorem 2 is valid for a slightly more general class of loadings. In particular the body force can take the form

$$\underset{\sim}{b}(\underset{\sim}{x}) = \hat{\underset{\sim}{b}}(\underset{\sim}{x}, \underset{\sim}{f}(\underset{\sim}{x}))$$

where $\hat{\underset{\sim}{b}} \in C^2(\bar{\Omega}\times\mathbb{R}^n, \mathbb{R}^n)$.

Remark. We have not been able to obtain a direct physical interpretation to condition (iii). We conjecture that condition (iii) is needed to eliminate certain solutions to the linearized dynamic problem on a half space (see the remarks on p. 7 and p. 10).

Proof of Theorem 2. Proof of Theorem 1 (Necessity).[15] The necessity of (i) is well-known.[16] We show that (ii) and (iii) are direct consequences of (5.1).

(5.1) $\Rightarrow$ (ii). Let $b \in C^\infty(\mathbb{R},\mathbb{R})$ satisfy $b(s) = 1$ for $s \le r/3$ and $b(s) = 0$ for $s \ge 2r/3$. Define $\varphi(\underset{\sim}{x}) = b(|\underset{\sim}{x}-\underset{\sim}{x}_0|)$. Then given any $\underset{\sim}{w} \in C^2(\bar{\mathcal{H}}, \mathbb{C}^n)$, $\underset{\sim}{u} := \mathrm{Re}\{\varphi\underset{\sim}{w}\}$ and $\underset{\sim}{v} := \mathrm{Im}\{\varphi\underset{\sim}{w}\}$ are contained in $\mathrm{Var}(\mathcal{HB})$ and a straightforward computation shows that

$$\begin{aligned}\int_{\mathcal{HB}} (\nabla\underset{\sim}{u}\cdot\underset{\sim}{C}_0[\nabla\underset{\sim}{u}]+\nabla\underset{\sim}{v}\cdot\underset{\sim}{C}_0[\nabla\underset{\sim}{v}]) &= \int_{\mathcal{HB}} \nabla(\varphi\bar{\underset{\sim}{w}})\cdot\underset{\sim}{C}_0[\nabla(\varphi\underset{\sim}{w})] \\ &= \int_{\mathcal{HB}} (\bar{\underset{\sim}{w}} \otimes \nabla\varphi\cdot\underset{\sim}{C}_0[\underset{\sim}{w} \otimes \nabla\varphi] + \mathrm{Re}\{\nabla(\varphi^2\bar{\underset{\sim}{w}})\cdot\underset{\sim}{C}_0[\nabla\underset{\sim}{w}]\})\end{aligned} \tag{5.2}$$

---

[15] The sufficiency portion of the proof of Theorem 1 has been divided into three parts. The first is at the end of this section and the other two parts are at the end of sections 6 and 7.

[16] Cf., e.g., Truesdell & Noll [27, Section 68bis].

and, by the divergence theorem,

$$\int_{\mathcal{HB}} \nabla(\varphi^2 \bar{\mathbf{w}}) \cdot \mathbf{C}_0[\nabla \mathbf{w}] = -\int_{\mathcal{HB}} \varphi^2 \bar{\mathbf{w}} \cdot \operatorname{div} \mathbf{C}_0[\nabla \mathbf{w}] + \int_{\partial(\mathcal{HB})} \varphi^2 \bar{\mathbf{w}} \cdot \mathbf{C}_0[\nabla \mathbf{w}] \mathbf{n}_0 \quad (5.3)$$

We now suppose that Agmon's condition is not satisfied and we let $\mathbf{w}_1$ be a bounded exponential solution of (3.2) with $\alpha = 1$ (see the remark on page 9). Then $\mathbf{w}_\alpha(\mathbf{x}) := \mathbf{w}_1(\alpha \mathbf{x})$ will be a bounded exponential solution of (3.2) for any $\alpha > 0$. Moreover, $|\mathbf{w}_\alpha(\mathbf{x})|$ is uniformly bounded independent of α.

By (3.2), (5.2), and (5.4) we find that there is a constant $k > 0$ such that

$$\begin{aligned} \int_{\mathcal{HB}} \{\nabla \mathbf{u}_\alpha \cdot \mathbf{C}_0[\nabla \mathbf{u}_\alpha] + \nabla \mathbf{v}_\alpha \cdot \mathbf{C}_0[\nabla \mathbf{v}_\alpha]\} &\le k - \alpha^2 \int_{\mathcal{HB}} \varphi^2 |\mathbf{w}_\alpha|^2 \\ &\le k - \alpha^2 \int_{\mathcal{HB}_{1/3}} |\mathbf{w}_\alpha|^2 \end{aligned} \quad (5.4)$$

where $\mathcal{HB}_\rho$ is the intersection of $\mathcal{H}$ with the ball of radius ρ centered at $\mathbf{x}_0$.

We next note that $\mathbf{w}_\alpha$ is of the form (3.3) and hence $|\mathbf{w}_\alpha(s, \mathbf{y})| = |\mathbf{z}_\alpha(s)|$. Thus

$$\begin{aligned} \int_{\mathcal{HB}_{1/3}} |\mathbf{w}_\alpha(\mathbf{x})|^2 d\mathbf{x} &\ge (2/3n)^{n-1} \int_0^{1/3n} |\mathbf{z}_\alpha(s)|^2 ds \\ &= (2/3n)^{n-1} \int_0^{1/3n} |\mathbf{z}_1(\alpha s)|^2 ds \\ &= \alpha^{-1} (2/3n)^{n-1} \int_0^{\alpha/3n} |\mathbf{z}_1(\sigma)|^2 d\sigma \ . \end{aligned}$$

If we combine the last inequality with (5.4) we conclude that, for $\alpha \ge 3n$,

$$\int_{\mathcal{HB}} \{\nabla \mathbf{u}_\alpha \cdot \mathbf{C}_0[\nabla \mathbf{u}_\alpha] + \nabla \mathbf{v}_\alpha \cdot \mathbf{C}_0[\nabla \mathbf{v}_\alpha]\} \le k - \alpha (2/3n)^{n-1} \int_0^1 |\mathbf{z}_1(\sigma)|^2 d\sigma.$$

Therefore, by choosing α sufficiently large, we can find a

$\underset{\sim}{u}_\alpha \in \mathrm{Var}(\mathcal{HB})$ (or a $\underset{\sim}{v}_\alpha \in \mathrm{Var}(\mathcal{HB})$) for which (5.1) is not satisfied.

(5.1) $\Rightarrow$ (iii). Suppose that $\underset{\sim}{a} \in \mathbb{R}^n$ satisfies

$$\underset{\sim}{a}\cdot\underset{\sim}{M}\underset{\sim}{a} = \underset{\sim}{a} \otimes \underset{\sim}{n}_0\cdot\underset{\sim}{C}_0[\underset{\sim}{a} \otimes \underset{\sim}{n}_0] = 0.$$

Then, since $\underset{\sim}{C}_0$ is symmetric and satisfies the Legendre-Hadamard condition, $\underset{\sim}{M}$ is symmetric and positive semi-definite. Thus

$$\underset{\sim}{M}\underset{\sim}{a} = \underset{\sim}{C}_0[\underset{\sim}{a} \otimes \underset{\sim}{n}_0]\underset{\sim}{n}_0 = \underset{\sim}{0}. \qquad (5.5)$$

Next, let $\underset{\sim}{b}, \underset{\sim}{t}_0 \in \mathbb{R}^n$ with $\underset{\sim}{t}_0 \perp \underset{\sim}{n}_0$ and $|t_0| = 1$. Choose coordinates so that $\underset{\sim}{n}_0 = (-1,0,0,\ldots,0)$, $\underset{\sim}{t}_0 = (0,1,0,,0,\ldots,0)$ and write $\underset{\sim}{x} = (s,t,\underset{\sim}{z})$, where $s,t \in \mathbb{R}$ and $\underset{\sim}{z} \in \mathbb{R}^{n-2}$. Let $\varphi, \psi \in C^\infty(\mathbb{R},\mathbb{R})$ and $\rho \in C^\infty(\mathbb{R}^{n-2},\mathbb{R})$ satisfy

$$\begin{aligned} \int_{-1}^{1} |\dot\psi(t)|^2 dt &= 1, & \psi(t) &= 0 \quad \text{for} \quad |t| \ge 1, \\ \varphi(0) &= 1, & \varphi(s) &= 0 \quad \text{for} \quad |s| \ge 1, \\ \int_{|\underset{\sim}{z}|\le r} |\rho(\underset{\sim}{z})|^2 d\underset{\sim}{z} &= 1, & \rho(\underset{\sim}{z}) &= 0 \quad \text{for} \quad |\underset{\sim}{z}| \ge r/3. \end{aligned} \qquad (5.6)$$

For $\epsilon > 0$, $\delta > 0$, and $\beta > 0$ define

$$\underset{\sim}{u}^\beta_{\delta,\epsilon}(s,t,\underset{\sim}{z}) = \rho(\underset{\sim}{z})[\underset{\sim}{a}\delta^{-1}\varphi(\tfrac{s}{\epsilon\beta})\psi(\tfrac{t}{\beta}) + \underset{\sim}{b}\delta\varphi(\tfrac{s}{\beta})\dot\psi(\tfrac{t}{\beta})].$$

Then $\underset{\sim}{u}^\beta_{\delta,\epsilon} \in \mathrm{Var}(\mathcal{HB})$, provided ϵ and β are sufficiently small, and

$$\nabla\underset{\sim}{u}^\beta_{\delta,\epsilon} = \left[\begin{array}{l} \beta^{-1}\rho(\underset{\sim}{a}\otimes\underset{\sim}{n}_0\,\delta^{-1}\,\epsilon^{-1}\,\dot\varphi\,\psi + \underset{\sim}{b}\otimes\underset{\sim}{n}_0\,\delta\,\dot\varphi\,\dot\psi\,) \\ \quad +\beta^{-1}\rho(\underset{\sim}{a}\otimes\underset{\sim}{t}_0\,\delta^{-1}\,\varphi\,\dot\psi + \underset{\sim}{b}\otimes\underset{\sim}{t}_0\,\delta\,\varphi\,\ddot\psi\,) \\ \quad + (\underset{\sim}{a}\,\delta^{-1}\,\varphi\,\psi + \underset{\sim}{b}\,\delta\,\varphi\,\dot\psi)\otimes\nabla\rho \end{array}\right]. \qquad (5.7)$$

Now, by (5.1),

$$\int_{|\underset{\sim}{z}|\le r}\int_{-\beta}^{\beta}\int_0^{\beta} \nabla\underset{\sim}{u}^\beta_{\delta,\epsilon}\cdot\underset{\sim}{C}_0[\nabla\underset{\sim}{u}^\beta_{\delta,\epsilon}]ds\; dt\; d\underset{\sim}{z} \ge 0.$$

We make the change of variables $\sigma = s/\beta$ and $\tau = t/\beta$ (and hence $ds\,dt = \beta^2 d\sigma\, d\tau$) and then let $\beta \to 0^+$ in the last inequality to conclude, with the aid of (5.5), (5.6), and (5.7), that

$$\begin{aligned}
&\mathbf{b}\otimes\mathbf{n}_0\cdot\mathbf{C}_0[\mathbf{b}\otimes\mathbf{n}_0]\delta^2 \int_0^1 \dot{\varphi}^2 + \mathbf{a}\otimes\mathbf{t}_0\cdot\mathbf{C}_0[\mathbf{a}\otimes\mathbf{t}_0]\delta^{-2} \int_0^{\varepsilon} |\varphi(\tfrac{\sigma}{\varepsilon})|^2 d\sigma \\
&+ \mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0[\mathbf{b}\otimes\mathbf{t}_0]\delta^2 \int_0^1 \varphi^2 \int_{-1}^1 \ddot{\varphi}^2 + 2\mathbf{a}\otimes\mathbf{t}_0\cdot\mathbf{C}[\mathbf{b}\otimes\mathbf{n}_0]\int_0^{\varepsilon} \dot{\varphi}(\sigma)\varphi(\tfrac{\sigma}{\varepsilon})d\sigma \qquad (5.8) \\
&\qquad - 2\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0[\mathbf{a}\otimes\mathbf{n}_0]\varepsilon^{-1} \int_0^{\varepsilon} \dot{\varphi}(\tfrac{\sigma}{\varepsilon})\varphi(\sigma)d\sigma \ge 0.
\end{aligned}$$

In deriving the above equation we have used the identities

$$\int_{-1}^1 \varphi(\tau)\dot{\varphi}(\tau)d\tau = \int_{-1}^1 \dot{\varphi}(\tau)\ddot{\varphi}(\tau)d\tau = 0, \quad \int_{-1}^1 \varphi(\tau)\ddot{\varphi}(\tau)d\tau = -1,$$

which are consequences of $(5.6)_1$.

We note that the change of variables $y = \sigma/\varepsilon$ yields

$$\varepsilon^{-1} \int_0^{\varepsilon} \dot{\varphi}(\tfrac{\sigma}{\varepsilon})\varphi(\sigma)d\sigma = \int_0^1 \dot{\varphi}(y)\varphi(\varepsilon y)dy$$

and hence by the bounded convergence theorem and $(5.6)_2$ we find that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1} \int_0^{\varepsilon} \dot{\varphi}(\tfrac{\sigma}{\varepsilon})\varphi(\sigma)d\sigma = \varphi(0) \int_0^1 \dot{\varphi}(y)dy = -1. \qquad (5.9)$$

Finally, we note that φ is bounded and let $\varepsilon \to 0^+$ followed by $\delta \to 0^+$ in (5.8) to conclude, with the aid of (5.9), that

$$0 \le 2\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0[\mathbf{a}\otimes\mathbf{n}_0] = 2\mathbf{b}\cdot\mathbf{C}_0[\mathbf{a}\otimes\mathbf{n}_0]\mathbf{t}_0 .$$

Since $\mathbf{b}$ is arbitrary we find that

$$\mathbf{C}_0[\mathbf{a}\otimes\mathbf{n}_0]\mathbf{t} = \mathbf{0} \quad \text{for any} \quad \mathbf{t} \perp \mathbf{n}_0 . \qquad (5.10)$$

The desired result now follows from (5.5) and (5.10). ∎

The first step in proving the sufficiency portion of Theorem 1 is to define a Fourier transform in the tangential variables.

With this in mind we choose coordinates so that $\underset{\sim}{x}_0 = \underset{\sim}{0}$, $\underset{\sim}{n} := \underset{\sim}{n}_0 = (-1,0,0,0,\dots,0)$ and write $\underset{\sim}{x} = (s,\underset{\sim}{y})$ where $s \in \mathbb{R}$ and $\underset{\sim}{y} \in \mathbb{R}^{n-1}$.

For any $\underset{\sim}{\xi} \in \mathbb{R}^{n-1}$, $\alpha \in \mathbb{R}$, and $\underset{\sim}{z} \in C_0^\infty([0,\infty),\mathbb{C}^n)$ we write

$$Q_\alpha^{\underset{\sim}{\xi}}(\underset{\sim}{z}) := \int_0^\infty (\overline{\underset{\sim}{H}}(\underset{\sim}{z}(s),\underset{\sim}{\xi})\cdot \underset{\sim}{C}_0[\underset{\sim}{H}(\underset{\sim}{z}(s),\underset{\sim}{\xi})]+\alpha^2|\underset{\sim}{z}(s)|^2)ds,$$
$$\underset{\sim}{H}(\underset{\sim}{z}(s),\underset{\sim}{\xi}) := -\dot{\underset{\sim}{z}}(s) \otimes \underset{\sim}{n} + i\underset{\sim}{z}(s) \otimes (0,\underset{\sim}{\xi}). \tag{5.11}$$

The following result then utilizes van Hove's [29] technique of reducing an analytic inequality to an algebraic inequality in the transform variables. It is essentially due[17] to Agmon [1, Theorem 5.1] and De Figueiredo [8, Theorem 3.1].

Lemma 5.3. *A necessary and sufficient condition for* (5.1) *to be satisfied for all* $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$ *is that*

$$Q_\alpha^{\underset{\sim}{\xi}}(\underset{\sim}{z}) \geq 0 \tag{5.12}$$

for every $\underset{\sim}{z} \in C_0^1([0,\infty),\mathbb{C}^n)$, $\alpha > 0$, *and* $\underset{\sim}{\xi} \in \mathbb{R}^{n-1}$.

Remark. Since C_0^1 is dense in H^1 and Q is continuous on H^1 inequality (5.12) is satisfied by all $\underset{\sim}{z} \in H^1((0,\infty),\mathbb{C}^n)$ and hence, in particular, by all $\underset{\sim}{z} \in C^\infty([0,\infty),\mathbb{C}^n)$ with $\underset{\sim}{z}$ and all its derivatives contained in $L^2((0,\infty),\mathbb{C}^n)$.

Proof of Lemma 5.3. *Sufficiency.* Given $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$ we extend the domain of $\underset{\sim}{u}$ to $\overline{\mathcal{H}}$ by setting $\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{0}$ for $\underset{\sim}{x} \notin \mathcal{HB}$ and define

$$\hat{\underset{\sim}{u}}(s,\underset{\sim}{\xi}) := (2\pi)^{-(n-1)/2}\int_{\mathbb{R}^{n-1}} \underset{\sim}{u}(s,\underset{\sim}{y})\exp(-i\underset{\sim}{\xi}\cdot\underset{\sim}{y})d\underset{\sim}{y}.$$

Then

[17] See also Chen [7, Proposition 1].

$$\int_{\mathcal{HB}} \nabla \underset{\sim}{u} \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{u}] d\underset{\sim}{x} = \int_{\mathbb{R}^{n-1}} \int_0^\infty \nabla \bar{\underset{\sim}{u}} \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{u}] d\chi \, ds$$

and hence by Plancherel's Theorem and (5.11)

$$\int_{\mathcal{HB}} \nabla \underset{\sim}{u} \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{u}] d\underset{\sim}{x} = \int_{\mathbb{R}^{n-1}} Q_0^{\underset{\sim}{\xi}}(\hat{\underset{\sim}{u}}(\cdot, \underset{\sim}{\xi})) d\underset{\sim}{\xi} \, .$$

Now, if $Q_\alpha^{\underset{\sim}{\xi}} \geq 0$ for every $\alpha > 0$ and $\underset{\sim}{\xi} \in \mathbb{R}^{n-1}$ it follows by continuity that $Q_0^{\underset{\sim}{\xi}} \geq 0$ for every $\underset{\sim}{\xi} \in \mathbb{R}^{n-1}$ and hence by the last equation we conclude that (5.1) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$.

<u>Necessity</u>. We write $\mathcal{B}^{n-1}$ for the unit ball in $\mathbb{R}^{n-1}$ and let $\varphi \in C_0^\infty(\mathcal{B}^{n-1}, \mathbb{R})$, $\underset{\sim}{\xi}^0 \in \mathbb{R}^{n-1}$, and $\underset{\sim}{z} \in C_0^1([0,\infty), \mathbb{C}^n)$. For $\epsilon > 0$ we define

$$\underset{\sim}{w}^\epsilon(s,\chi) := \epsilon\, \varphi(\chi) \underset{\sim}{z}(s/\epsilon) \exp(i \underset{\sim}{\xi}^0 \cdot \chi/\epsilon)$$

and note that, for ϵ sufficiently small, both $\underset{\sim}{u}^\epsilon := \mathrm{Re}\{\underset{\sim}{w}^\epsilon\}$ and $\underset{\sim}{v}^\epsilon := \mathrm{Im}\,\{\underset{\sim}{w}^\epsilon\}$ are contained in $\mathrm{Var}(\mathcal{HB})$.

Now $\hat{\underset{\sim}{w}}^\epsilon(s, \underset{\sim}{\xi}) = \epsilon\, \hat{\varphi}(\underset{\sim}{\xi} - \underset{\sim}{\xi}^0/\epsilon) \underset{\sim}{z}(s/\epsilon)$ and thus (5.1), (5.11), and Plancherel's Theorem imply that

$$\begin{aligned} 0 &\leq \int_{\mathcal{HB}} (\nabla \underset{\sim}{u}^\epsilon \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{u}^\epsilon] + \nabla \underset{\sim}{v}^\epsilon \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{v}^\epsilon]) d\underset{\sim}{x} \\ &= \int_{\mathcal{HB}} \nabla \bar{\underset{\sim}{w}}^\epsilon \cdot \underset{\sim}{C}_0[\nabla \underset{\sim}{w}^\epsilon] d\underset{\sim}{x} = \int_{\mathbb{R}^{n-1}} Q_0^{\epsilon \underset{\sim}{\xi}}(\underset{\sim}{z}(s/\epsilon)) |\hat{\varphi}(\underset{\sim}{\xi} - \underset{\sim}{\xi}^0/\epsilon)|^2 d\underset{\sim}{\xi} \, . \end{aligned}$$

If we make the change of variables $\underset{\sim}{\eta} = \underset{\sim}{\xi} - \underset{\sim}{\xi}^0/\epsilon$ and $\sigma = s/\epsilon$ we find that, for every (sufficiently small) $\epsilon > 0$,

$$0 \leq \int_{\mathbb{R}^{n-1}} \int_0^\infty \bar{\underset{\sim}{H}}(\underset{\sim}{z}(\sigma), \epsilon\underset{\sim}{\eta} + \underset{\sim}{\xi}^0) \cdot \underset{\sim}{C}_0[H(\underset{\sim}{z}(\sigma), \epsilon\underset{\sim}{\eta} + \underset{\sim}{\xi}^0)] |\hat{\varphi}(\underset{\sim}{\eta})|^2 ds \, d\underset{\sim}{\eta} \, ,$$

$$\underset{\sim}{H}(\underset{\sim}{z}(\sigma), \epsilon\underset{\sim}{\eta} + \underset{\sim}{\xi}^0) = -\dot{\underset{\sim}{z}}(\sigma) \otimes \underset{\sim}{n} + i\, \underset{\sim}{z}(\sigma) \otimes (0, \epsilon\underset{\sim}{\eta} + \underset{\sim}{\xi}^0) \, .$$

Thus, since the last expression is continuous in ϵ, we let $\epsilon \to 0^+$ to conclude that

$$0 \le Q_{0}^{\underset{\sim}{\xi}^{0}}(\underset{\sim}{z}) \int_{\mathbb{R}^{n-1}} |\hat{\varphi}(\underset{\sim}{\eta})|^2 d\underset{\sim}{\eta}$$

and hence that

$$0 \le Q_{0}^{\underset{\sim}{\xi}^{0}}(\underset{\sim}{z}) \le Q_{\alpha}^{\underset{\sim}{\xi}^{0}}(\underset{\sim}{z})$$

for all $\underset{\sim}{z} \in C_0^1([0,\infty), \mathbb{C}^n)$. ∎

Proof of Theorem 1 (Sufficiency). Case I. $\underset{\sim}{M} = \underset{\sim}{0}$. Then condition (iii) and (3.6) imply that $\underset{\sim}{N}_{\underset{\sim}{t}} = \underset{\sim}{0}$. Thus (3.6) and the Legendre-Hadamard condition imply that

$$Q_{\alpha}^{\underset{\sim}{\xi}}(\underset{\sim}{z}) = \int_0^\infty \bar{\underset{\sim}{z}} \cdot P_{\underset{\sim}{t}}^{\alpha} \underset{\sim}{z} ds \ge \alpha^2 \int_0^\infty |\underset{\sim}{z}(s)|^2 ds,$$

where $\underset{\sim}{t} = (0, \underset{\sim}{\xi})$. Therefore $Q_{\alpha}^{\underset{\sim}{\xi}}$ is nonnegative and hence by Lemma 5.3 we find that (5.1) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{B})$. ∎

6. Quadratic Forms on a Half-line I. Regular Forms

In order to complete the sufficiency portion of the proof of Theorem 1 we will need to establish some auxiliary results concerning quadratic forms on a half-line. Motivated by (3.5) and (5.11) we let $\underset{\sim}{A}:(0,\infty) \to \mathrm{Sym}^m$, $\underset{\sim}{B}:(0,\infty) \to \mathrm{Sym}^m$, and $\underset{\sim}{F}:(0,\infty) \to \mathrm{Lin}^m$ be continuous and consider the quadratic functional

$$q_\alpha(\underset{\sim}{f}) := \int_0^\infty (\dot{\bar{\underset{\sim}{f}}} \cdot \underset{\sim}{A}(\alpha)\dot{\underset{\sim}{f}} + \bar{\underset{\sim}{f}} \cdot \underset{\sim}{B}(\alpha)\underset{\sim}{f} - 2\,\mathrm{Re}(i\dot{\bar{\underset{\sim}{f}}} \cdot \underset{\sim}{F}(\alpha)\underset{\sim}{f}))ds.$$

For fixed $\alpha > 0$ we associate with q_α the tensor field $\underset{\sim}{R}:\mathbb{R} \to \mathrm{Sym}^m$ defined by

$$\underset{\sim}{R}(\lambda) := \lambda^2 \underset{\sim}{A}(\alpha) + \lambda(\underset{\sim}{F}(\alpha) + \underset{\sim}{F}^T(\alpha)) + \underset{\sim}{B}(\alpha), \tag{6.1}$$

the system of ordinary differential equations (the Euler equations corresponding to q_α)

$$-\underset{\sim}{A}(\alpha)\ddot{\underset{\sim}{f}} + i(\underset{\sim}{F}(\alpha)+\underset{\sim}{F}^T(\alpha))\dot{\underset{\sim}{f}} + \underset{\sim}{B}(\alpha)\underset{\sim}{f} = \underset{\sim}{0} \text{ on } (0,\infty), \tag{6.2}$$

and the natural boundary condition

$$\underset{\sim}{A}(\alpha)\dot{\underset{\sim}{f}}(0) - i\,\underset{\sim}{F}(\alpha)\underset{\sim}{f}(0) = \underset{\sim}{0}. \tag{6.3}$$

Definition. Fix $\alpha > 0$. We say that q_α is:

(i) regular provided that there are constants $c_1 = c_1(\alpha) > 0$ and $c_2 = c_2(\alpha) > 0$ such that, for every $\underset{\sim}{e} \in \mathbb{R}^m$,

$$\underset{\sim}{e}\cdot\underset{\sim}{R}(\lambda)\underset{\sim}{e} \geq (c_1\lambda^2+c_2)|\underset{\sim}{e}|^2;$$

(ii) coercive provided that there is a constant $k = k(\alpha) > 0$ such that (cf. (2.1))

$$q_\alpha(\underset{\sim}{f}) \geq k\|\underset{\sim}{f}\|_1^2$$

for all $\underset{\sim}{f} \in C_0^1([0,\infty),\mathbb{C}^m)$.

We note that if q_α is regular and $\underset{\sim}{z}$ is any bounded solution to (6.2) then, by Lemma 3.2, $\underset{\sim}{z}$ is C^∞ and $\underset{\sim}{z}$ and all its derivatives are contained in L^2. The main tool we use is the following result of De Figueiredo [8, Theorem 1.2], a proof of which can be found in the Appendix.

Proposition 6.1. *Let $\alpha > 0$ be given. Then necessary and sufficient conditions for q_α to be coercive are that q_α be regular and that $q_\alpha(\underset{\sim}{f})$ be strictly positive whenever $\underset{\sim}{f}$ is a nontrivial bounded solution of (6.2).*

Remark. The quadratic form Q_0^{ξ} is regular at every $\xi \in \mathbb{R}^{n-1}$ if and only if C_0 is strongly elliptic, i.e.,

$$a \otimes b \cdot C_0[a \otimes b] > 0$$

whenever $a, b \in \mathbb{R}^n$ with $a \neq 0$, $b \neq 0$. Thus, given that C_0 is strongly elliptic (which precludes condition (iii) of Theorem 1) a necessary and sufficient condition for the nonnegativity of the second variation (eq. (5.1)) is either Agmon's condition or equivalently De Figueiredo's condition, that is, for every $\alpha > 0$ and $\xi \in \mathbb{R}^{n-1}$ Q_α^{ξ} is strictly positive on the finite dimensional space of solutions to (3.7).

A direct consequence of Proposition 6.1 is the following result.

Proposition 6.2. *Let q_α be regular for all $\alpha > 0$ and nonnegative for sufficiently large α. Suppose that there is an $\alpha_* > 0$ such that q_{α_*} is not nonnegative. Then there is an $\alpha_0 > 0$ and a nontrivial $f_p \in C^\infty([0,\infty),\mathbb{C}^m)$, with f_p and all its derivatives contained in $L^2((0,\infty),\mathbb{C}^m)$, such that*

(a) $q_{\alpha_0}(f_p) = 0$,

(b) $q_{\alpha_0}(f) \geq 0$ *for all* $f \in C_0^1([0,\infty),\mathbb{C}^m)$.

Moreover, f_p satisfies (6.2) *and* (6.3) *with* $\alpha = \alpha_0$.

Proof. Define

$$\alpha_0 := \inf\{\alpha > 0\colon\ q_\beta \text{ is nonnegative for all } \alpha > \beta\}. \tag{6.4}$$

Then by hypothesis $0 < \alpha_0 < +\infty$. Also the continuity of the maps $\alpha \mapsto \underset{\sim}{A}(\alpha)$, $\alpha \mapsto \underset{\sim}{B}(\alpha)$, and $\alpha \mapsto \underset{\sim}{F}(\alpha)$ implies that (b) is satisfied.

Suppose, for the sake of contradiction, that $q_\alpha(\underset{\sim}{f})$ is strictly positive whenever $\underset{\sim}{f} \in C^\infty((0,\infty),\mathbb{C}^m)\setminus\{0\}$ with $\underset{\sim}{f}$ and all its derivatives contained in $L^2((0,\infty),\mathbb{C}^m)$. Since q_α is regular all bounded solutions of (6.2) are contained in C^∞ (see Lemma 3.2) and hence q_α is strictly positive on solutions of (6.2). Therefore, by Proposition 6.1, q_{α_0} is coercive. However, the continuity of the maps $\alpha \mapsto \underset{\sim}{A}(\alpha)$, $\alpha \mapsto \underset{\sim}{B}(\alpha)$, and $\alpha \mapsto \underset{\sim}{F}(\alpha)$ then implies that q_α is coercive for all α in some neighborhood of α_0. This contradicts (6.4). Thus there must exist a nontrivial $\underset{\sim}{f}_p \in C^\infty([0,\infty),\mathbb{C}^m)$, with $\underset{\sim}{f}_p$ and all its derivatives contained in $L^2((0,\infty),\mathbb{C}^m)$, that satisfies (a).

Finally, (a) and (b) imply that the first variation of q_{α_0} at $\underset{\sim}{f}_p$ is zero and hence that $\underset{\sim}{f}_p$ is a weak solution of the corresponding Euler equation (6.2) and that $\underset{\sim}{f}_p$ satisfies the natural boundary condition (6.3). ∎

Our next result gives a sufficient condition for the quadratic form $Q_\alpha^{\underset{\sim}{\xi}}$ (as given by (5.11)) to be regular.

Lemma 6.3. *Let $\underset{\sim}{C}_0$ satisfy the Legendre-Hadamard condition. Suppose, in addition, that $\underset{\sim}{M}$ is strictly positive definite. Then for every $\alpha > 0$ and $\underset{\sim}{t} \in \mathbb{R}^n$ there are constants $c_1 = c_1(\alpha,\underset{\sim}{t}) > 0$ and $c_2 = c_2(\alpha,\underset{\sim}{t}) > 0$ such that* (cf. (3.9))

$$\underset{\sim}{a}\cdot\underset{\sim}{L}(\lambda)\underset{\sim}{a} = \underset{\sim}{a}\cdot[\lambda^2\underset{\sim}{M}+\lambda(\underset{\sim}{N}_{\underset{\sim}{t}}+\underset{\sim}{N}_{\underset{\sim}{t}}^T)+\underset{\sim}{P}_{\underset{\sim}{t}}^\alpha]\underset{\sim}{a} \geq (c_1\lambda^2+c_2)|\underset{\sim}{a}|^2$$

for every $\lambda \in \mathbb{R}$ and $\underset{\sim}{a} \in \mathbb{R}^n$. Here $\underset{\sim}{M}$, $\underset{\sim}{N}_{\underset{\sim}{t}}$ and $\underset{\sim}{P}_{\underset{\sim}{t}}^\alpha$ are given by (3.6).

Proof. Given $\alpha > 0$ and $\underset{\sim}{t} \in \mathbb{R}^n$, (3.10) and the Legendre-Hadamard condition imply that, for every $\lambda \in \mathbb{R}$ and $\underset{\sim}{a} \in \mathbb{R}^n$,

$$\underset{\sim}{a} \cdot \underset{\sim}{L}(\lambda)\underset{\sim}{a} \ge \alpha^2 |\underset{\sim}{a}|^2. \tag{6.5}$$

Suppose, for the sake of contradiction, that for every positive integer i there is a $\lambda_i \in \mathbb{R}$ and an $\underset{\sim}{a}^i \in \mathbb{R}^n$, with $|\underset{\sim}{a}^i| = 1$, such that

$$\underset{\sim}{a}^i \cdot \underset{\sim}{L}(\lambda_i)\underset{\sim}{a}^i < (\tfrac{1}{i}\lambda_i^2 + \tfrac{1}{2}\alpha^2) |\underset{\sim}{a}^i|^2. \tag{6.6}$$

Since the unit ball is compact in $\mathbb{R}^n$ there is an $\underset{\sim}{a} \in \mathbb{R}^n$, with $|\underset{\sim}{a}| = 1$, and a subsequence (also denoted $\underset{\sim}{a}^i$) such that $\underset{\sim}{a}^i \to \underset{\sim}{a}$ as $i \to \infty$. In addition, there is a sub-subsequence (also denoted λ_i) for which either $|\lambda_i| \to +\infty$ or $\lambda_i \to \lambda$ (for some $\lambda \in \mathbb{R}$) as $i \to \infty$.

Case I. $\lambda_i \to \lambda$ as $i \to \infty$. We note that $\underset{\sim}{L}$ is continuous in both λ and $\underset{\sim}{a}$ and let $i \to \infty$ in (6.6) to conclude that

$$\underset{\sim}{a} \cdot \underset{\sim}{L}(\lambda)\underset{\sim}{a} \le \tfrac{1}{2}\alpha^2 |\underset{\sim}{a}|^2.$$

This contradicts (6.5).

Case II. $|\lambda_i| \to +\infty$ as $i \to +\infty$. We first divide (6.6) by λ_i^2 and then let $i \to +\infty$ to conclude, with the aid of (3.9), that

$$\underset{\sim}{a} \cdot \underset{\sim}{M}\underset{\sim}{a} \le 0.$$

This contradicts the assumption that $\underset{\sim}{M}$ is strictly positive definite. ∎

Next, we show that the assumptions of the above lemma imply that $Q_\alpha^{\underset{\sim}{\xi}}$ also satisfies the other hypothesis needed to invoke Proposition 6.2.

Lemma 6.4. *Let C_0 satisfy the Legendre-Hadamard condition. Suppose, in addition, that M is strictly positive definite. Then for every $\xi \in \mathbb{R}^{n-1}$ there is an $\alpha_1 > 0$ such that*

$$Q_{\alpha}^{\xi}(z) \geq 0 \quad \text{for all} \quad z \in C_0^1([0,\infty), \mathbb{C}^m)$$

whenever $\alpha \geq \alpha_1$.

Proof. By (3.6) we can rewrite (5.11) in the form

$$Q_{\alpha}^{\xi}(z) = \int_0^{\infty}(\dot{\bar{z}} \cdot M\dot{z} + \bar{z} \cdot P_t^{\alpha} z - 2\,\mathrm{Re}(i\dot{\bar{z}} \cdot N_t z))ds$$

where $t = (0, \xi)$. However, M is strictly positive definite and C_0 satisfies the Legendre-Hadamard condition. Thus, for every $t \in \mathbb{R}^n$, there are constants $k > 0$ and $K > 0$ such that

$$k|a|^2 \leq \bar{a} \cdot Ma, \quad |\bar{a} \cdot N_t b| \leq K|a||b|$$

$$\alpha^2 |b|^2 \leq \bar{b} \cdot P_t^{\alpha} b\ ,$$

for every $a, b \in \mathbb{C}^n$ and $\alpha > 0$. Therefore the quadratic form

$$\bar{a} \cdot Ma + \bar{b} \cdot P_t^{\alpha} b - 2\,\mathrm{Re}(i\bar{a} \cdot N_t b)$$

will be strictly positive definite for α sufficiently large, which implies the desired result. ∎

Proof of Theorem 1 (Sufficiency). Case II. M is strictly positive definite. Fix $\xi \in \mathbb{R}^{n-1}$. Then the Legendre-Hadamard condition, Lemma 6.3, and Lemma 6.4 imply that Q_{α}^{ξ} is regular for all $\alpha > 0$ and nonnegative for sufficiently large α.

Suppose, for the sake of contradiction, that there is an $\alpha_* > 0$ such that $Q_{\alpha_*}^{\xi}$ is not nonnegative. Then by Proposition 6.2

there is an $\alpha_0 > 0$ and a nontrivial $\underset{\sim}{f}_p \in C^\infty([0,\infty),\mathbb{C}^n)$, with $\underset{\sim}{f}_p$ and all its derivatives contained in $L^2([0,\infty),\mathbb{C}^m)$, such that $\underset{\sim}{f}_p$ satisfies (6.2) and (6.3) (with $\alpha = \alpha_0$) where $\underset{\sim}{A}(\alpha) = \underset{\sim}{M}$, $\underset{\sim}{B}(\alpha) = \underset{\sim}{P}^\alpha_{\underset{\sim}{t}}$, and $\underset{\sim}{F}(\alpha) = \underset{\sim}{N}_{\underset{\sim}{t}}$ ($\underset{\sim}{t} = (0,\underset{\sim}{\xi})$). Therefore (cf. (3.7)-(3.8)) Agmon's condition is not satisfied, a contradiction. Thus $Q^{\underset{\sim}{\xi}}_\alpha$ is nonnegative for every $\alpha > 0$ and $\underset{\sim}{\xi} \in \mathbb{R}^{n-1}$ and hence by Lemma 5.3 we find that (5.1) is satisfied for all $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{H}\mathcal{B})$. ∎

7. Quadratic Forms on a Half-line II. Singular Forms

Thus far we have proven the sufficiency portion of Theorem 1 under the additional hypothesis that either $\underset{\sim}{M} = \underset{\sim}{O}$ or $\underset{\sim}{M}$ is strictly positive definite. In this section we consider the final case where $\underset{\sim}{M} \neq \underset{\sim}{O}$ has a nontrivial kernel.

The Legendre-Hadamard condition implies that $\underset{\sim}{M}$ is positive semi-definite and that $\underset{\sim}{P}^\alpha_{\underset{\sim}{t}}$ is strictly positive definite for $\alpha > 0$. Since $\underset{\sim}{M}$ and $\underset{\sim}{P}^\alpha_{\underset{\sim}{t}}$ are also symmetric we apply the spectral theorem and write

$$\underset{\sim}{M} = \left[\begin{array}{c|c} \underset{\sim}{A}_* & \underset{\sim}{O} \\ \hline \underset{\sim}{O} & \underset{\sim}{O} \end{array}\right], \qquad \underset{\sim}{P}^\alpha_{\underset{\sim}{t}} = \left[\begin{array}{c|c} \underset{\sim}{B}^*_\alpha & \underset{\sim}{E}^T \\ \hline \underset{\sim}{E} & \underset{\sim}{D}_\alpha \end{array}\right], \tag{7.1}$$

where $\underset{\sim}{A}_* \in \mathrm{Sym}^m$, $\underset{\sim}{B}^*_\alpha$ $(= \underset{\sim}{B}^*_0 + \alpha^2\underset{\sim}{I}) \in \mathrm{Sym}^m$ and $\underset{\sim}{D}_\alpha$ $(= \underset{\sim}{D}_0 + \alpha^2\underset{\sim}{I}) \in \mathrm{Sym}^{n-m}$ are strictly positive definite for all $\alpha > 0$ and some positive integer $m < n$. In addition, condition (iii) of Theorem 1 implies that certain blocks in such a representation for $\underset{\sim}{N}_{\underset{\sim}{t}}$

must be zero. In particular

$$\underset{\sim}{N}_t = \left[\begin{array}{c|c} \underset{\sim}{F}_* & \underset{\sim}{G}^T \\ \hline \underset{\sim}{0} & \underset{\sim}{0} \end{array} \right]. \tag{7.2}$$

Thus, if we write

$$\underset{\sim}{z}(s) = \left[\frac{\underset{\sim}{f}(s)}{\underset{\sim}{z}_0(s)} \right] \tag{7.3}$$

then (3.6), (5.11), and (7.1)-(7.3) imply that

$$\begin{aligned} Q_\alpha^{\tilde{\xi}}(\underset{\sim}{z}) &= \int_0^\infty (P(\dot{\underset{\sim}{f}}, \underset{\sim}{f}) + r(\dot{\underset{\sim}{f}}, \underset{\sim}{f}, \underset{\sim}{z}_0))ds \\ P(\underset{\sim}{a}, \underset{\sim}{b}) &:= \bar{\underset{\sim}{a}} \cdot \underset{\sim}{A}_* \underset{\sim}{a} + \bar{\underset{\sim}{b}} \cdot \underset{\sim}{B}_\alpha^* \underset{\sim}{b} - 2\,\mathrm{Re}\{i\bar{\underset{\sim}{a}} \cdot \underset{\sim}{F}_* \underset{\sim}{b}\} \\ r(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) &:= \bar{\underset{\sim}{c}} \cdot \underset{\sim}{D}_\alpha \underset{\sim}{c} + 2\,\mathrm{Re}\{\bar{\underset{\sim}{c}} \cdot \underset{\sim}{E} \underset{\sim}{b} - i\bar{\underset{\sim}{a}} \cdot \underset{\sim}{G}^T \underset{\sim}{c}\}. \end{aligned} \tag{7.4}$$

If $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{C}^m$ are fixed then (7.4) is quadratic in the vector $\underset{\sim}{c}$. Since $\underset{\sim}{D}_\alpha$ is strictly positive definite this quadratic has a minimum that is achieved by a unique vector $\underset{\sim}{c}_M$. If we take the first variation of (7.4) with respect to $\underset{\sim}{c}$ we find that, for every $\underset{\sim}{h} \in \mathbb{C}^{n-m}$,

$$\underset{\sim}{h} \cdot \underset{\sim}{D}_\alpha \bar{\underset{\sim}{c}}_M + \bar{\underset{\sim}{h}} \cdot \underset{\sim}{D}_\alpha \underset{\sim}{c}_M + 2\,\mathrm{Re}\{\underset{\sim}{h} \cdot (\underset{\sim}{E}\bar{\underset{\sim}{b}} - i\underset{\sim}{G}\bar{\underset{\sim}{a}})\} = 0$$

and hence that

$$\begin{aligned} \underset{\sim}{D}_\alpha \bar{\underset{\sim}{c}}_M + \underset{\sim}{D}_\alpha \underset{\sim}{c}_M + 2\,\mathrm{Re}\{\underset{\sim}{E}\bar{\underset{\sim}{b}} - i\underset{\sim}{G}\bar{\underset{\sim}{a}}\} &= \underset{\sim}{0}, \\ \underset{\sim}{D}_\alpha \bar{\underset{\sim}{c}}_M - \underset{\sim}{D}_\alpha \underset{\sim}{c}_M - 2i\,\mathrm{Im}\{\underset{\sim}{E}\bar{\underset{\sim}{b}} - i\underset{\sim}{G}\bar{\underset{\sim}{a}}\} &= \underset{\sim}{0}. \end{aligned} \tag{7.5}$$

If we add $(7.5)_1$ to $(7.5)_2$ and solve for $\underset{\sim}{c}_M$ we find that

$$\underset{\sim}{c}_M = -\underset{\sim}{D}_\alpha^{-1}(\underset{\sim}{E}\underset{\sim}{b} + i\underset{\sim}{G}\underset{\sim}{a})$$

and hence

$$r(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}_M) = -(\overline{\underset{\sim}{E}\underset{\sim}{b} + i\underset{\sim}{G}\underset{\sim}{a}}) \cdot \underset{\sim}{D}_\alpha^{-1}(\underset{\sim}{E}\underset{\sim}{b} + i\underset{\sim}{G}\underset{\sim}{a}).$$

Since $\underset{\sim}{c}_M$ minimizes (7.4) we arrive at the following result.

Proposition 7.1. *Let C_0 satisfy the Legendre-Hadamard condition and condition (iii) of Theorem 1. Suppose that $M \neq 0$ is singular. Then for every $z = [f|z_0]^T \in C_0^1([0,\infty),\mathbb{C}^n)$*

$$Q_\alpha^{\xi}(z) \geq q_\alpha(f)$$

with equality if and only if

$$z_0(s) = -D_\alpha^{-1}(Ef(s) + iG\dot{f}(s))$$

for every $s \in [0,\infty)$. Here

$$\begin{gathered} q_\alpha(f) := \int_0^\infty (\dot{\bar{f}}\cdot A(\alpha)\dot{f} + \bar{f}\cdot B(\alpha)f - 2\,\mathrm{Re}(i\dot{\bar{f}}\cdot F(\alpha)f))ds \\ A(\alpha) := A_* - G^T D_\alpha^{-1} G, \quad B(\alpha) := B_\alpha^* - E^T D_\alpha^{-1} E, \\ F(\alpha) := F_* - G^T D_\alpha^{-1} E \end{gathered} \tag{7.6}$$

In order to apply Proposition 6.2 we show that q_α is regular.

Lemma 7.2. *Let C_0 satisfy the Legendre-Hadamard condition and condition (iii) of Theorem 1. Suppose that $M \neq 0$ is singular. Then for every $\alpha > 0$ and $t \in \mathbb{R}^n$ there are constants $c_1 = c_1(\alpha,t) > 0$ and $c_2 = c_2(\alpha,t) > 0$ such that* (cf. (6.1)), *for every $\lambda \in \mathbb{R}$ and $e \in \mathbb{R}^m$,*

$$e\cdot R(\lambda)e = e\cdot[\lambda^2 A(\alpha) + \lambda(F(\alpha)+F(\alpha)^T) + B(\alpha)]e \geq (c_1\lambda^2 + c_2)|e|^2,$$

where $A(\alpha)$, $B(\alpha)$, and, $F(\alpha)$ are given by (7.6).

Proof. Given $\alpha > 0$ and $t \in \mathbb{R}^n$, (3.10) and the Legendre-Hadamard condition imply that, for every $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}^n$

$$a\cdot L(\lambda)a \geq \alpha^2|a|^2.$$

If we let $a = [e|c]^T$ then by (7.1), (7.2), and (6.1) we find that

$$\begin{bmatrix} \lambda^2 \underset{\sim}{e}\cdot \underset{\sim}{A}_* \underset{\sim}{e} + \lambda \underset{\sim}{e}\cdot(\underset{\sim}{E}_* + \underset{\sim}{E}_*^T)\underset{\sim}{e} + \underset{\sim}{e}\cdot \underset{\sim}{B}_\alpha^* \underset{\sim}{e} \\ + \underset{\sim}{c}\cdot \underset{\sim}{D}_\alpha \underset{\sim}{c} + 2\underset{\sim}{c}\cdot(\underset{\sim}{E}\underset{\sim}{e} + \lambda \underset{\sim}{G}\underset{\sim}{e}) \end{bmatrix} \geq \begin{bmatrix} \alpha^2 |\underset{\sim}{e}|^2 \\ + \\ \alpha^2 |\underset{\sim}{c}|^2 \end{bmatrix}.$$

In particular we let $\underset{\sim}{c} = -\underset{\sim}{D}_\alpha^{-1}(\underset{\sim}{E}\underset{\sim}{e} + \lambda\underset{\sim}{G}\underset{\sim}{e})$ to conclude, with the aid of (7.6), that

$$\underset{\sim}{e}\cdot \underset{\sim}{R}(\lambda)\underset{\sim}{e} \geq \alpha^2 |\underset{\sim}{e}|^2 \tag{7.7}$$

for every $\underset{\sim}{e} \in \mathbb{R}^m$.

The proof of Lemma 6.3 now shows that (7.7) implies the desired result <u>provided</u> $\underset{\sim}{A}(\alpha)$ is strictly positive definite for every $\alpha > 0$. In order to prove this we let $\underset{\sim}{e}_0 \in \mathbb{R}^n$ and $\underset{\sim}{u} \in \mathbb{R}^n$ be of the form

$$\underset{\sim}{e}_0 = \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{u}_0 \end{array}\right], \quad \underset{\sim}{u} = \left[\begin{array}{c} \underset{\sim}{u}_* \\ \hline \underset{\sim}{0} \end{array}\right].$$

Then (7.1), (7.2), and Lemma 3.1 imply that

$$|\underset{\sim}{u}_*\cdot \underset{\sim}{G}^T \underset{\sim}{u}_0|^2 \leq (\underset{\sim}{u}_0\cdot \underset{\sim}{D}_0 \underset{\sim}{u}_0)(\underset{\sim}{u}_*\cdot \underset{\sim}{A}_* \underset{\sim}{u}_*). \tag{7.8}$$

Define

$$\underset{\sim}{T}_\alpha := \left[\begin{array}{c|c} \underset{\sim}{A}_* & \underset{\sim}{G}^T \\ \hline \underset{\sim}{G} & \underset{\sim}{D}_\alpha \end{array}\right].$$

Then $\underset{\sim}{T}_\alpha$ is symmetric and

$$\left[\begin{array}{c} x\,\underset{\sim}{u}_* \\ \hline y\,\underset{\sim}{u}_0 \end{array}\right]\cdot \underset{\sim}{T}_\alpha \left[\begin{array}{c} x\,\underset{\sim}{u}_* \\ \hline y\,\underset{\sim}{u}_0 \end{array}\right] = \begin{array}{l} x^2(\underset{\sim}{u}_*\cdot \underset{\sim}{A}_* \underset{\sim}{u}_*) + y^2(\underset{\sim}{u}_0\cdot \underset{\sim}{D}_0 u_0) \\ + 2xy(\underset{\sim}{u}_*\cdot \underset{\sim}{G}^T \underset{\sim}{u}_0) + y^2(\alpha^2 |\underset{\sim}{u}_0|^2). \end{array} \tag{7.9}$$

Since $\underset{\sim}{A}_*$ is strictly positive definite and $\underset{\sim}{D}_0$ is positive semi-definite (by the Legendre-Hadamard condition), equations (7.8) and (7.9) imply that $\underset{\sim}{T}_\alpha$ is strictly positive definite for every $\alpha > 0$.

Finally, if we let $\underset{\sim}{u} \in \mathbb{R}^n$ be of the form

$$\underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}_* \\ -\underset{\sim}{D}_\alpha^{-1}\underset{\sim}{G}\underset{\sim}{u}_* \end{bmatrix}$$

we find that

$$\underset{\sim}{u}\cdot\underset{\sim}{T}_\alpha\underset{\sim}{u} = \underset{\sim}{u}_*\cdot(\underset{\sim}{A}_* - \underset{\sim}{G}^T\underset{\sim}{D}_\alpha^{-1}\underset{\sim}{G})\underset{\sim}{u}_* .$$

Therefore $\underset{\sim}{A}(\alpha) = \underset{\sim}{A}_* - \underset{\sim}{G}^T\underset{\sim}{D}_\alpha^{-1}\underset{\sim}{G}$ is strictly positive definite. ∎

Next, we show that the assumptions of the above lemma imply that q_α also satisfies the other hypothesis needed to invoke Proposition 6.2.

Lemma 7.3. Let $\underset{\sim}{C}_0$ satisfy the Legendre-Hadamard condition and condition (iii) of Theorem 1. Suppose that $\underset{\sim}{M} \neq \underset{\sim}{0}$ is singular. Then for every $\underset{\sim}{t} \in \mathbb{R}^n$ there is an $\alpha_1 > 0$ such that

$$q_\alpha(\underset{\sim}{f}) \geq 0 \quad \text{for all} \quad \underset{\sim}{f} \in C_0^1([0,\infty),\mathbb{C}^m)$$

whenever $\alpha \geq \alpha_1$. Here $\underset{\sim}{A}(\alpha)$, $\underset{\sim}{B}(\alpha)$, and $\underset{\sim}{F}(\alpha)$ are given by (7.1), (7.2), and (7.6).

Proof. We note that $\underset{\sim}{D}_\alpha = \underset{\sim}{D}_0 + \alpha^2\underset{\sim}{I}$ and hence for every $\varepsilon > 0$ there is an $\alpha_2 > 0$ such that

$$|\underset{\sim}{D}_\alpha^{-1}| \leq \varepsilon \quad \text{whenever} \quad \alpha > \alpha_2 .$$

Thus by (7.6) and the fact that $\underset{\sim}{A}(\alpha)$ is positive definite (cf. the proof of Lemma 7.2) there is an $\alpha_3 > \alpha_2$ and constants $k > 0$ and $K > 0$ such that

$$k|\underset{\sim}{a}|^2 \leq \bar{\underset{\sim}{a}}\cdot\underset{\sim}{A}(\alpha)\underset{\sim}{a}, \quad |\bar{\underset{\sim}{a}}\cdot\underset{\sim}{F}(\alpha)\underset{\sim}{b}| \leq K|\underset{\sim}{a}||\underset{\sim}{b}|$$
$$(\alpha^2-1)|\underset{\sim}{b}|^2 \leq \bar{\underset{\sim}{b}}\cdot\underset{\sim}{B}(\alpha)\underset{\sim}{b}$$

for every $\alpha > \alpha_3$ and every $\underset{\sim}{a},\underset{\sim}{b} \in \mathbb{C}^m$. Therefore the quadratic form

$$\bar{a}\cdot A(\alpha)a + \bar{b}\cdot B(\alpha)b - 2\,\mathrm{Re}(i\bar{a}\cdot F(\alpha)b)$$

will be strictly positive definite for α sufficiently large, which implies the desired result. ∎

Proof of Theorem 1 (Sufficiency). Final Case. $M \neq 0$ is singular. Fix $\xi \in \mathbb{R}^{n-1}$. Then the Legendre-Hadamard condition, condition (iii) of Theorem 1, Lemma 7.2, and Lemma 7.3 imply that q_α is regular for all $\alpha > 0$ and nonnegative for sufficiently large α.

Suppose, for the sake of contradiction that there is an $\alpha_* > 0$ such that $Q^{\xi}_{\alpha_*}$ is not nonnegative. Then by Proposition 7.1 q_{α_*} is not nonnegative. Thus, by Proposition 6.2 there is an $\alpha_0 > 0$ and a nontrivial $f_p \in C^\infty([0,\infty),\mathbb{C}^m)$, with f_p and all its derivatives contained in $L^2([0,\infty),\mathbb{C}^m)$, such that f_p satisfies (6.2) and (6.3), with $\alpha = \alpha_0$.

Define

$$z_p = \begin{bmatrix} f_p \\ -D_\alpha^{-1}(Ef_p + iG\dot{f}_p) \end{bmatrix}.$$

Then a straightforward computation shows that z_p satisfies (3.7) and (3.8). Therefore Agmon's condition is not satisfied, a contradiction. Thus Q^{ξ}_{α} is nonnegative for every $\alpha > 0$ and $\xi \in \mathbb{R}^{n-1}$ and hence by Lemma 5.3 we find that (5.1) is satisfied for all $u \in \mathrm{Var}(\mathcal{HB})$. ∎

8. Some Examples

Our first example shows that condition (iii) of Theorem 1 is not a consequence of the Legendre-Hadamard condition and Agmon's condition.[18] In particular we let

$$W(\mathbf{F}) = h(\det \mathbf{F}) \tag{8.1}$$

for every $\mathbf{x} \in \bar{\Omega}$ and $\mathbf{F} \in \mathrm{Lin}^n_+$ where $h \in C^2((0,\infty),\mathbb{R})$. Then the Piola-Kirchhoff stress $\mathbf{S}$ and the elasticity tensor $\mathbf{C}$ are given by

$$\begin{aligned} \mathbf{S}(\mathbf{F}) &= th'(t)\mathbf{F}^{-T}, \quad t := \det \mathbf{F} \\ \mathbf{C}(\mathbf{F})[\mathbf{H}] &= -th'(t)\mathbf{F}^{-T}\mathbf{H}^T\mathbf{F}^{-T} + (t^2h''(t)+th'(t))(\mathbf{F}^{-T}\cdot\mathbf{H})\mathbf{F}^{-T}. \end{aligned} \tag{8.2}$$

We now fix $\mathbf{F} \in \mathrm{Lin}^n_+$ and define $\mathbf{A}\colon \mathrm{Lin}^n \to \mathrm{Lin}^n$ by

$$\begin{aligned} \mathbf{A}[\mathbf{K}] &:= \mathbf{F}^T\mathbf{C}(\mathbf{F})[\mathbf{F}\mathbf{K}] = a\mathbf{K}^T + b(\mathbf{K}\cdot\mathbf{I})\mathbf{I}, \\ a &:= -th'(t), \quad b := t^2h''(t) + th'(t). \end{aligned} \tag{8.3}$$

Since $\mathbf{F}$ is invertible it is then clear that

$$\int_\Omega \nabla\mathbf{u}\cdot\mathbf{C}(\mathbf{F})[\nabla\mathbf{u}] \geq 0 \quad \text{for all} \quad \mathbf{u} \in \mathrm{Var}$$

if and only if

$$\int_\Omega \nabla\mathbf{v}\cdot\mathbf{A}[\nabla\mathbf{v}] \geq 0 \quad \text{for all} \quad \mathbf{v} \in \mathrm{Var}.$$

We will prove

Proposition 8.1. *Let $\mathbf{A}$ be given by $(8.3)_1$ and $\mathbf{n} \in \mathbb{R}^n$. Then*

(i) *$\mathbf{A}$ satisfies the Legendre-Hadamard condition if and only if* $a+b \geq 0$;

[18] It is known (see, *e.g.*, *Simpson & Spector* [25, Section 8]) that Agmon's condition is not a consequence of the Legendre-Hadamard condition and condition (iii) of Theorem 1. It is not clear whether or not the Legendre-Hadamard condition is a consequence of the other two conditions.

(ii) $(\mathbf{A}, \mathbf{n})$ _satisfies Agmon's condition if and only if_ $a+b \geq 0$;

(iii) $(\mathbf{A}, \mathbf{n})$ _satisfies condition_ (iii) _of Theorem_ 1 _if and only if_ $a = 0$.

An immediate consequence of Proposition 8.1 and Theorem 1 is

Corollary 8.1.1. _Let_ $\mathbf{A}$ _be given by_ $(8.3)_1$ _and_ $\mathbf{n} \in \mathbb{R}^n$. _Then necessary and sufficient conditions for_

$$0 \leq \int_{\mathcal{HB}} \nabla \mathbf{u} \cdot \mathbf{A}[\nabla \mathbf{u}] \quad \text{for all} \quad \mathbf{u} \in \mathrm{Var}(\mathcal{HB})$$

are that $a = 0$ _and_ $b \geq 0$.

Remark. The above corollary implies that, for the special material (8.1), $\mathbf{A}$ must be positive semi-definite in order for (5.1) to be satisfied. To see this choose $\mathbf{K} = \mathbf{W} = -\mathbf{W}^T$ and $\mathbf{K} = \mathbf{E}_0 = \mathbf{E}_0^T$ with $\mathrm{trace}(\mathbf{E}_0) = 0$ to conclude, with the aid of $(8.3)_1$, that

$$\mathbf{W} \cdot \mathbf{A}[\mathbf{W}] = -a|\mathbf{W}|^2,$$

$$\mathbf{E}_0 \cdot \mathbf{A}[\mathbf{E}_0] = a|\mathbf{E}_0|^2.$$

Thus $a = 0$ is necessary for $\mathbf{A}$ to be positive semi-definite. When $a = 0$ we find that

$$\mathbf{K} \cdot \mathbf{A}[\mathbf{K}] = b(\mathbf{K} \cdot \mathbf{I})^2.$$

Therefore $b \geq 0$ is necessary for $\mathbf{A}$ to be positive semi-definite. Clearly $a = 0$ and $b \geq 0$ are also sufficient.

Proof of Proposition 8.1. (i). Let $\underset{\sim}{e}, \underset{\sim}{n} \in \mathbb{R}^n$. Then, by $(8.3)_1$,

$$\underset{\sim}{e} \otimes \underset{\sim}{n} \cdot \underset{\sim}{A}[\underset{\sim}{e} \otimes \underset{\sim}{n}] = (a+b)(\underset{\sim}{e} \cdot \underset{\sim}{n})^2 \tag{8.4}$$

and the result is clear.

(ii). Without loss of generality let $\underset{\sim}{n} = (-1,0,0,0,\ldots,0)$ and $\underset{\sim}{t} = \tau(0,1,0,0,0,\ldots,0)$. Then, by (3.6) and $(8.3)_1$,

$$\underset{\sim}{M}\underset{\sim}{e} = -(a+b)e_1\underset{\sim}{n}, \quad \underset{\sim}{N}_{\underset{\sim}{t}}\underset{\sim}{e} = -ae_1\underset{\sim}{t} + b\tau e_2\underset{\sim}{n},$$

$$\underset{\sim}{P}^{\alpha}_{\underset{\sim}{t}}\underset{\sim}{e} = (a+b)\tau e_2\underset{\sim}{t} + \alpha^2\underset{\sim}{e},$$

and hence (3.7) and (3.8) reduce to

$$\left.\begin{aligned} -(a+b)\ddot{z}_1 - i\tau(a+b)\dot{z}_2 + \alpha^2 z_1 &= 0 \\ -i\tau(a+b)\dot{z}_1 + [\tau^2(a+b)+\alpha^2]z_2 &= 0 \end{aligned}\right\} \text{ on } (0,\infty), \tag{8.5}$$

$$\begin{aligned} (a+b)\dot{z}_1(0) + i\tau b\, z_2(0) &= 0, \\ i\tau a\, z_1(0) &= 0. \end{aligned} \tag{8.6}$$

Case I. $a+b = 0$. Then (8.5) implies that $z_1 = z_2 = 0$ and thus Agmon's condition is satisfied.

Case II. $a+b < 0$. Let $\tau = 0$. Then it is straightforward to show that $z_2 = 0$ and $z_1(s) = \cos(\alpha s/(-a-b)^{1/2})$ will satisfy (8.5) and (8.6). Thus $a+b \geq 0$ is necessary for Agmon's condition to be satisfied.

Case III. $a+b > 0$. If we differentiate $(8.5)_2$, solve for $\dot{z}_2$, and substitute the result into $(8.5)_1$ we find that (8.5) is equivalent to

$$\ddot{z}_1(s) - c^2 z_1(s) = 0, \quad z_2(s) = i\tau\dot{z}_1(s)/c^2, \tag{8.7}$$

where

$$c := \left[\frac{\alpha^2+(a+b)\tau^2}{a+b}\right]^{1/2}$$

is real and strictly positive. In addition $(8.7)_2$ implies that (8.6) is equivalent to

$$(\alpha^2+a\tau^2)\dot{z}_1(0) = 0, \qquad a\tau z_1(0) = 0. \tag{8.8}$$

Since the only nontrivial bounded solution to $(8.7)_1$ is $z_1(s) = \exp(-cs)$ we find that in order for (8.8) to be satisfied we must set $a\tau = 0$ and $\alpha^2+a\tau^2 = 0$. However the last two equations imply that $\alpha = 0$. Therefore Agmon's condition is satisfied.

(iii). Let $\underset{\sim}{e}, \underset{\sim}{n} \in \mathbb{R}^n$. Then, by (8.4), necessary and sufficient conditions for

$$\underset{\sim}{e} \otimes \underset{\sim}{n} \cdot \underset{\sim}{A}[\underset{\sim}{e} \otimes \underset{\sim}{n}] = 0 \tag{8.9}$$

are either $\underset{\sim}{e} \cdot \underset{\sim}{n} = 0$ or $a+b = 0$.

<u>Case I</u>. $a+b \neq 0$. Then $\underset{\sim}{e} \cdot \underset{\sim}{n} = 0$ and hence

$$\underset{\sim}{A}[\underset{\sim}{e} \otimes \underset{\sim}{n}] = a\underset{\sim}{n} \otimes \underset{\sim}{e}.$$

Thus a necessary and sufficient condition for condition (iii) of Theorem 1 to be satisfied is $a = 0$.

<u>Case II</u>. $a+b = 0$. Then given $\underset{\sim}{n} \in \mathbb{R}^n$ (8.9) is satisfied for every $\underset{\sim}{e} \in \mathbb{R}^n$. In particular choose $\underset{\sim}{e} \perp \underset{\sim}{n}$. This reduces the problem to Case I. ∎

The above example is not completely satisfactory from a physical viewpoint since the constitutive relation (8.1) is that of an elastic fluid. We therefore give a second example that shows that condition (iii) of Theorem 1 is not a consequence of the other two conditions. Let $\mathbf{e} \in \mathbb{R}^n$, with $|\mathbf{e}| = 1$, and

$$W(\mathbf{F}) = \frac{\mu}{2}\,\mathbf{F}\cdot\mathbf{F} + h(\det \mathbf{F}) + \frac{\omega}{2}\,|\mathbf{F}\mathbf{e}|^2 + \frac{\omega}{2}\,|\mathbf{F}^{-T}\mathbf{e}|^2.$$

Then

$$\mathbf{S}(\mathbf{F}) = \left[\begin{array}{c} \mu\mathbf{F} + (\det \mathbf{F})h'(\det \mathbf{F})\mathbf{F}^{-T} \\ + \omega\mathbf{F}\mathbf{e}\otimes\mathbf{e} - \omega\mathbf{F}^{-T}\mathbf{e}\otimes\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{e} \end{array}\right],$$

$$\mathbf{C}(\mathbf{I})[\mathbf{H}] = 2\mu\mathbf{E} + \lambda(\mathbf{E}\cdot\mathbf{I})\mathbf{I} + 2\omega\mathbf{E}\mathbf{e}\otimes\mathbf{e} + 2\omega\mathbf{e}\otimes\mathbf{E}\mathbf{e},$$

where $\lambda = h''(1)-\mu$ and $\mathbf{E} = \frac{1}{2}(\mathbf{H}+\mathbf{H}^T)$. We note that the choice $\mu = -h'(1)$ insures that the reference configuration is a natural state.

We now let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and define

$$\alpha := \mathbf{a}\cdot\mathbf{e}, \qquad \mathbf{A} := \mathbf{a} - \alpha\mathbf{e},$$

$$\beta := \mathbf{b}\cdot\mathbf{e}, \qquad \mathbf{B} := \mathbf{b} - \beta\mathbf{e}.$$

Then

$$\mathbf{C}(\mathbf{I})[\mathbf{a}\otimes\mathbf{b}] = \left[\begin{array}{c} \mu(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}) + \lambda(\mathbf{a}\cdot\mathbf{b})\mathbf{I} \\ +\omega\beta(\mathbf{a}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{a}) + \omega\alpha(\mathbf{b}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{b}) \end{array}\right] \tag{8.10}$$

and hence

$$\mathbf{a}\otimes\mathbf{b}\cdot\mathbf{C}[\mathbf{a}\otimes\mathbf{b}] = \left[\begin{array}{l} \mu|\mathbf{A}|^2|\mathbf{B}|^2 + (\mu+\lambda)(\mathbf{A}\cdot\mathbf{B})^2 + (2\mu+\lambda+4\omega)\alpha^2\beta^2 \\ +2(\mu+\lambda+\omega)\alpha\beta\mathbf{A}\cdot\mathbf{B} + (\mu+\omega)(\alpha^2|\mathbf{B}|^2 + \beta^2|\mathbf{A}|^2) \end{array}\right]$$

where we have written $\mathbf{C}$ for $\mathbf{C}(\mathbf{I})$. We let

$$2\mu + \lambda + 4\omega = 0 \tag{8.11}$$

and apply the arithmetic-geometric mean inequality to conclude that

$$\underset{\sim}{a}\otimes\underset{\sim}{b}\cdot\underset{\sim}{C}[\underset{\sim}{a}\otimes\underset{\sim}{b}] \geq \mu|\underset{\sim}{A}|^2|\underset{\sim}{B}|^2 + (\mu+\lambda)(\underset{\sim}{A}\cdot\underset{\sim}{B})^2 - \lambda(\alpha^2|\underset{\sim}{B}|^2+\beta^2|\underset{\sim}{A}|^2).$$

Thus, given (8.11), the Legendre-Hadamard condition will be satisfied if

$$\mu \geq 0,\ 2\mu + \lambda \geq 0,\ \text{and}\ \lambda < 0. \tag{8.12}$$

Next, we combine (8.10) and (8.11) to conclude that

$$\underset{\sim}{C}[\underset{\sim}{e}\otimes\underset{\sim}{e}] = \lambda[\underset{\sim}{I}-\underset{\sim}{e}\otimes\underset{\sim}{e}]$$

and hence that

$$\underset{\sim}{e}\otimes\underset{\sim}{e}\cdot\underset{\sim}{C}[\underset{\sim}{e}\otimes\underset{\sim}{e}] = 0$$

Thus, given (8.11) and (8.12), condition (iii) of Theorem 1 will not be satisfied.

Finally, in order to verify that Agmon's condition is satisfied, we let $\underset{\sim}{n} = \underset{\sim}{e}$ and, without loss of generality, we choose coordinates so that $\underset{\sim}{n} = (-1,0,0,\ldots,0)$ and $\underset{\sim}{t} = (0,\tau,0,0,\ldots,0)$. Then by (3.6), (8.10), and (8.11)

$$\underset{\sim}{M}\underset{\sim}{a} = (\mu+\omega)[\underset{\sim}{a} - (\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{n}],$$

$$\underset{\sim}{N}_{\underset{\sim}{t}}\underset{\sim}{a} = (\mu+\omega)(\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{t} + \lambda(\underset{\sim}{a}\cdot\underset{\sim}{t})\underset{\sim}{n},$$

$$\underset{\sim}{P}^{\alpha}_{\underset{\sim}{t}}\underset{\sim}{a} = (\mu\tau^2+\alpha^2)\underset{\sim}{a} + (\mu+\lambda)(\underset{\sim}{a}\cdot\underset{\sim}{t})\underset{\sim}{t} + \omega\tau^2(\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{n}.$$

and hence (3.7) and (3.8) reduce to

$$\left.\begin{aligned} -i\tau(\mu+\lambda+\omega)\dot{z}_2 + [(\mu+\omega)\tau^2+\alpha^2]z_1 &= 0 \\ -(\mu+\omega)\ddot{z}_2 - i\tau(\mu+\lambda+\omega)\dot{z}_1 + [(2\mu+\lambda)\tau^2+\alpha^2]z_2 &= 0 \\ -(\mu+\omega)\ddot{z}_j + (\mu\tau^2+\alpha^2)z_j &= 0 \end{aligned}\right\}\ \text{on}\ (0,\infty) \tag{8.13}$$

and

$$\left.\begin{aligned} \lambda\tau z_2(0) = 0, \quad (\mu+\omega)\dot{z}_j(0) = 0 \\ (\mu+\omega)(\dot{z}_2(0) - i\tau z_1(0)) = 0 \end{aligned}\right\} \tag{8.14}$$

for $j = 3,4,\ldots,n$.

If we assume that μ and λ satisfy (8.12) then, by (8.11),

$$\mu + \omega > 0 \tag{8.15}$$

and hence $(8.13)_3$, (8.14), and the fact that $|\underset{\sim}{z}|$ is bounded imply that $z_j \equiv 0$ for $j = 3,4,\ldots,n$.

If we differentiate $(8.13)_1$ and substitute the result into $(8.13)_2$ we find, with the aid of (8.12), $(8.13)_1$, (8.14), and (8.15), that

$$-\left[(\mu+\omega) - \frac{\tau^2(\mu+\lambda+\omega)^2}{(\mu+\omega)\tau^2+\alpha^2}\right]\ddot{z}_2 + [(2\mu+\lambda)\tau^2+\alpha^2]z_2 = 0 \text{ on } (0,\infty),$$

$$\tau z_2(0) = 0, \quad \left[1 + \frac{\tau^2(\mu+\lambda+\omega)}{(\mu+\omega)\tau^2+\alpha^2}\right]\dot{z}_2(0) = 0.$$

It is now easy to show that (8.12) implies that the coefficients in the above differential equation have opposite signs for every $\alpha \neq 0$ and hence that $z_2 \equiv 0$ is the only bounded solution to this initial value problem. It then follows from $(8.13)_1$ that $z_1 \equiv 0$ and hence that $\underset{\sim}{z} \equiv \underset{\sim}{0}$ is the only bounded solution to (8.13) and (8.14).

Our third example shows that, unlike Agmon's condition, the complementing condition[19] can fail at the unique global minimizer of the energy. Following Ball & James [4, Section 7b] we let

---

[19]The complementing condition is equivalent to the requirement that $\underset{\sim}{w} = \underset{\sim}{0}$ be the only bounded exponential solution to (3.2) when $\alpha = 0$.

$n = 2$ and

$$W(\underset{\sim}{F}) = \frac{1}{2}\underset{\sim}{F}\cdot\underset{\sim}{F} + h(\det \underset{\sim}{F}) \tag{8.16}$$

where $h \in C^2(\mathbb{R}^+,\mathbb{R})$ and satisfies

$$\begin{gathered} \lim_{t\to 0^+} h(t) = \lim_{t\to\infty} h(t) = +\infty, \\ h''(t) > 0 \quad \text{for} \quad t \in (0,1) \cup (1,\infty), \\ h'(1) = -1, \quad h''(1) = 0. \end{gathered} \tag{8.17}$$

Then, by (8.16) and $(8.17)_3$,

$$\underset{\sim}{S}(\underset{\sim}{F}) = \underset{\sim}{F} + (\det \underset{\sim}{F})h'(\det \underset{\sim}{F})\underset{\sim}{F}^{-T}, \tag{8.18}$$

$$\underset{\sim}{C}(\underset{\sim}{I})[\underset{\sim}{H}] = \underset{\sim}{H} + \underset{\sim}{H}^T - (\underset{\sim}{H}\cdot\underset{\sim}{I})\underset{\sim}{I} .$$

Thus (cf., e.g., Simpson & Spector [24, 25]) the complementing condition fails at $(\underset{\sim}{C}(\underset{\sim}{I}),\underset{\sim}{n})$ for any $\underset{\sim}{n} \in \mathbb{R}^2$.

We now consider the problem of minimizing the energy of a half-ball composed of material with constitutive relation (8.10). We claim that

$$\int_{\mathcal{HB}} W(\underset{\sim}{I})d\underset{\sim}{x} < \int_{\mathcal{HB}} W(\underset{\sim}{I} + \nabla\underset{\sim}{u}(\underset{\sim}{x}))d\underset{\sim}{x}$$

for every nontrivial $\underset{\sim}{u} \in \mathrm{Var}(\mathcal{HB})$ and hence that $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{x}$ is the unique global minimizer of the energy.

To verify the above claim we first minimize (8.16). By $(8.17)_1$ we find that a global minimizer to (8.16) must exist and satisfy $\underset{\sim}{S} = \underset{\sim}{0}$. By (8.18) we then conclude that this minimizer must satisfy

$$\underset{\sim}{F}\,\underset{\sim}{F}^T = k\underset{\sim}{I}, \quad k = (\det \underset{\sim}{F})h'(\det \underset{\sim}{F}) \tag{8.19}$$

and hence

$$W_{min} = k + h(k).$$

If we minimize the last expression with respect to k we find that $h'(k) = -1$ and therefore we conclude, with the aid of $(8.17)_2$ and $(8.17)_3$ that $k = 1$. Thus the global minimizers of (8.16) are the orthogonal linear transformations with positive determinant.

By the previous paragraph we find that any global minimizer of the energy must satisfy

$$\nabla \underset{\sim}{f}(\underset{\sim}{x}) \nabla \underset{\sim}{f}(\underset{\sim}{x})^T = \underset{\sim}{I}.$$

However a standard result[20] then implies that $\nabla \underset{\sim}{f}(\underset{\sim}{x}) \equiv \underset{\sim}{Q}_0$, a constant. Since $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{x}$ on $\partial(\mathcal{H}\mathcal{B})/\partial\mathcal{H}$ this proves our assertion.

Remark. Ball & James [4, Section 7b] consider a family of constitutive relations that include (8.16) with $(8.17)_1$, $(8.17)_3$, and

$$h''(t) = 0 \quad \text{for} \quad t \in (1-a, 1+b),$$
$$h''(t) > 0 \quad \text{for} \quad t \in (0, 1-a) \cup (1+b, +\infty).$$

(Thus any global minimizer must satisfy (8.19) with $k \in (1-a, 1+b)$.) They show that there are an infinite number of linearly independent global minimizers to this problem. (In particular any analytic function whose derivative is uniformly close to 1 on $\mathcal{H}\mathcal{B}$ will minimize the energy.) They use this example to illustrate the occurence of minimizers that contain arbitrarily small oscillations at the boundary.

9. Appendix

We herein sketch a proof of the sufficiency portion of Proposition 6.1. All of the following results are taken from De Figueiredo [8].

---

[20] Cf., e.g., Gurtin [9, p. 49].

Proposition 9.1. Let q_α be regular at a given $\alpha > 0$. Then there is a constant $k_1 = k_1(\alpha) > 0$ such that (cf. (2.1))

$$q_\alpha(\underset{\sim}{f}) \geq k_1 \|\underset{\sim}{f}\|_1^2$$

for all $\underset{\sim}{f} \in C_0^1([0,\infty),\mathbb{C}^m)$ that satisfy $\underset{\sim}{f}(0) = \underset{\sim}{0}$.

Proof. Given $\underset{\sim}{f} \in C_0^1([0,\infty),\mathbb{C}^m)$ that satisfies $\underset{\sim}{f}(0) = \underset{\sim}{0}$ extend the domain of $\underset{\sim}{f}$ to $\mathbb{R}$ by setting $\underset{\sim}{f}(s) = \underset{\sim}{0}$ for $s \leq 0$. Define

$$\hat{\underset{\sim}{f}}(\tau) := (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \underset{\sim}{f}(s)\exp(-i\tau s)ds.$$

Then, by Plancherel's Theorem,

$$q_\alpha(\underset{\sim}{f}) = \int_{-\infty}^{+\infty} (\tau^2\overline{\hat{\underset{\sim}{f}}}\cdot\underset{\approx}{A}\hat{\underset{\sim}{f}}+\overline{\hat{\underset{\sim}{f}}}\cdot\underset{\approx}{B}\hat{\underset{\sim}{f}}+\tau\overline{\hat{\underset{\sim}{f}}}\cdot(\underset{\sim}{F}+\underset{\sim}{F}^T)\hat{\underset{\sim}{f}})d\tau.$$

Therefore, since q_α is regular, there are constants $c_1 = c_1(\alpha) > 0$ and $c_2 = c_2(\alpha) > 0$ such that

$$q_\alpha(\underset{\sim}{f}) \geq \int_{-\infty}^{\infty} (c_1\tau^2+c_2)\,|\hat{\underset{\sim}{f}}(\tau)|^2 d\tau.$$

The desired result now follows from Plancherel's Theorem. ∎

The following is then a standard corollary.[21]

Corollary 9.1.1. Let q_α be regular at a given $\alpha > 0$. Then for any $\underset{\sim}{e} \in \mathbb{C}^m$ there exists a unique $\underset{\sim}{f} \in C^\infty([0,\infty),\mathbb{C}^m)$, with $\underset{\sim}{f}$ and all its derivatives contained in $L^2((0,\infty),\mathbb{C}^m)$, such that $\underset{\sim}{f}$ satisfies (6.2) and $\underset{\sim}{f}(0) = \underset{\sim}{e}$.

---

[21] Cf., e.g., De Figueiredo [8, Section 3] or Agmon, Douglis & Nirenberg [2, Theorem 3.2].

Lemma 9.2. *Let* q_α *be regular at a given* $\alpha > 0$. *Suppose in addition that* $q_\alpha(\underset{\sim}{f})$ *is strictly positive whenever* $\underset{\sim}{f}$ *is a nontrivial solution of* (6.2). *Then there is a constant* $k_2 = k_2(\alpha) > 0$ *such that*

$$q_\alpha(\underset{\sim}{f}) \geq k_2 \| \underset{\sim}{f} \|_1^2$$

whenever $\underset{\sim}{f}$ *is a solution of* (6.2).

Proof. The set of functions that satisfy (6.2) is a finite dimensional vector space. It is not difficult to show that both $q_\alpha(\cdot)$ and $\|\cdot\|_1$ are norms on this space. Since any two norms are equivalent on a finite dimensional vector space, the result follows. ∎

We now define a bilinear form associated with q_α. Let

$$b_\alpha(\underset{\sim}{f},\underset{\sim}{g}) := \int_0^\infty (\dot{\bar{\underset{\sim}{f}}}\cdot \underset{\sim}{A}\dot{\underset{\sim}{g}} + \bar{\underset{\sim}{f}}\cdot \underset{\sim}{B}\underset{\sim}{g} - 2\,\mathrm{Re}\{i\dot{\bar{\underset{\sim}{f}}}\cdot \underset{\sim}{F}\underset{\sim}{g}\})ds.$$

Proof of Proposition 6.1 (*Sufficiency*). Suppose that q_α is regular and strictly positive on bounded solutions of (6.2). Let $\underset{\sim}{g} \in C_0^1([0,\infty),\mathbb{C}^m)$. Define $\underset{\sim}{f} \in C^\infty([0,\infty),\mathbb{C}^m)$ to be the unique bounded solution of (6.2) with $\underset{\sim}{f}(0) = \underset{\sim}{g}(0)$ and let $\underset{\sim}{u} := \underset{\sim}{g}-\underset{\sim}{f}$. Then

$$q_\alpha(\underset{\sim}{g}) = q_\alpha(\underset{\sim}{f}) + b_\alpha(\underset{\sim}{u},\underset{\sim}{f}) + b_\alpha(\underset{\sim}{f},\underset{\sim}{u}) + q_\alpha(\underset{\sim}{u}).$$

Now the middle two terms on the right-hand side of the above equation are zero since $\underset{\sim}{f}$ satisfies (6.2) and $\underset{\sim}{u}(0) = \underset{\sim}{0}$. Therefore, by Proposition 9.1 and Lemma 9.2 we find that

$$\begin{aligned} q_\alpha(\underset{\sim}{g}) &\geq k_2(\alpha)\|\underset{\sim}{f}\|_1^2 + k_1(\alpha)\|\underset{\sim}{u}\|_1^2 \\ &\geq k\|\underset{\sim}{f}+\underset{\sim}{u}\|_1^2 = k\|\underset{\sim}{g}\|_1^2 \end{aligned}$$

where $k = \min(k_1,k_2)/2$. This is the desired result. ∎

Acknowledgement. We thank Professors J. L. Ericksen, R. D. James, and K. A. Pericak-Spector for their many useful discussions and suggestions. This work was supported in part by the Institute for Mathematics and its Applications and Heriot-Watt University. It was also supported by the National Science Foundation (DMS 86-00281 and DMS 86-02201) and the Air Force Office of Scientific Research (USAFOSR 86-0184).

REFERENCES

1. Agmon, S., On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math. 15 (1962), 119-147.

2. Agmon, S., A. Douglis, & L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Comm. Pure Appl. Math. 17 (1964), 35-92.

3. Ball, J. M., Constitutive inequalities and existence theorems in nonlinear elastostatics. Nonlinear Analysis and Mechanics Vol. I (R. J. Knops, ed.). London: Pitman, 1977.

4. Ball, J. M., & R. D. James, Fine phase mixtures as minimizers of energy. To appear: Arch. Rational Mech. Anal.

5. Ball, J. M., & J. E. Marsden, Quasiconvexity at the boundary, positivity of the second variation, and elastic stability. Arch. Rational Mech. Anal. 86 (1984), 251-277.

6. Biot, M. A., Mechanics of Incremental Deformations. New York: Wiley, 1965.

7. Chen, Y.-c., Stability of pure homogeneous deformations of an elastic plate with fixed edges. To appear: Q. J. Mech. Appl. Math.

8. De Figueiredo, D. G., The coerciveness problem for forms over vector valued functions. Comm. Pure Appl. Math. 16 (1963), 63-94.

9. Gurtin, M. E., An Introduction to Continuum Mechanics. New York: Academic Press, 1981.

10. Gurtin, M. E., Two-phase deformations of elastic solids. Arch. Rational Mech. Anal. **84** (1983), 1-29.

11. Hadamard, J., Lecons sur la Propagation des Ondes et les Equations de l'Hydrodynamique. Paris: Hermann, 1903.

12. Hayes, M., & R. S. Rivlin, Propagation of a plane wave in an isotropic elastic material subjected to pure homogeneous deformation. Arch. Rational Mech. Anal. **8** (1961), 15-22.

13. Hestenes, M. R., Quadratic variational theory and linear elliptic partial differential equations. Trans. Amer. Math. Soc. **101** (1961), 306-350.

14. Hestenes, M. R., & E. J. McShane, A theorem on quadratic forms and its application in the calculus of variations. Trans. Amer. Math. Soc. **47** (1940), 501-512.

15. Hutchinson, J. W., & V. Tvergaard, Surface instabilities on statically strained plastic solids. Int. J. Mech. Sci. **22** (1980), 339-354.

16. Ince, E. L., Ordinary Differential Equations. New York: Dover, 1956.

17. Meyers, N. G., Quasi-convexity and lower semi-continuity of multiple variational integrals of any order. Trans. Amer. Math. Soc. **119** (1965), 125-149.

18. Morrey, C. B., Quasi-convexity and the lower semicontinuity of multiple integrals. Pacific J. Math. **2** (1952), 25-53.

19. Nowinski, J. L., Surface instability of a half-space under high two-dimensional compression. J. Franklin Inst. **288** (1969), 367-376.

20. Nowinski, J. L., On the surface instability of an isotropic highly elastic half-space. Indian J. Math. Mech. **18** (1969), 1-10.

21. Reddy, B. D., Surface instabilities on an equibiaxially stretched elastic half-space. Math. Proc. Camb. Phil. Soc. **91** (1982), 491-501.

22. Reddy, B. D., The occurence of surface instabilities and shear bands in plane-strain deformation of an elastic half-space. Q. J. Mech. Appl. Math. **36** (1983), 337-350.

23. Rivlin, R. S., Some reflections on material stability. Mechanics Today, vol. 5, Oxford: Pergamon, 1980.

24. Simpson, H. C., & S. J. Spector, On failure of the complementing condition and nonuniqueness in linear elastostatics. J. Elasticity **15** (1985), 229-331.

25. Simpson, H. C., & S. J. Spector, On the positivity of the second variation in finite elasticity. Arch. Rational Mech. Anal. **98** (1987), 1-30.

26. Terpstra, F. J., Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. Math. Ann. **116** (1939), 166-180.

27. Truesdell, C., & W. Noll, The Non-linear Field Theories of Mechanics. Handbuch der Physik III/3 (S. Flügge, ed.). Berlin: Springer-Verlag, 1965.

28. Usmani, S. A., & M. F. Beatty, On the surface instability of a highly elastic half-space. J. Elasticity **4** (1974), 249-263.

29. van Hove, L., Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs fonctions inconnues. Proc. Koninklijke Nederlandsche Akademie Van Wetenschappen **50** #1 (1947), 18-23.