ON THE SYNTHETIC FACTORIZATION OF HOMOGENEOUS INVARIANTS

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Abstract. We prove that, after multiplication with a suitable monomial, every homogeneous bracket polynomial of rank $r \geq 3$ can be factored into a meet and join expression (synthetic construction) in the Grassmann (Cayley) algebra.

1. Introduction and statement of the result.

Every synthetic geometric construction in projective $(r-1)$-space can be written using meets and joins of points: $C(x_1, x_2, \ldots, x_m)$. This can be translated into the Grassmann algebra, using the operations of meet and join for extensors: $E(\wedge, \vee, x_1, x_2, \ldots, x_m)$. For an introduction to the Grassmann algebra (or *Cayley algebra* or *double algebra*) and the *bracket algebra* we refer the reader to [2],[4].

To express a theorem or elementary geometric property which results in the final collinearity of three points or concurrence of three lines we will obtain an equation $E(\wedge, \vee, x_1, x_2, \ldots, x_m) = 0$ in the Grassmann algebra. If this expression is expanded into the algebra of invariants, it becomes a polynomial in the basic invariants for projective $(r-1)$-space – the r-*brackets* $[x_{i_1} x_{i_2} \ldots x_{i_r}]$ representing the determinant of the homogeneous coordinates of r points. It is not difficult to see that this expanded bracket polynomial is homogeneous in the occurances of each point and has integer coefficients. This statement is a particular form of the metatheorem that every synthetic projective property can be written as an analytic projective property.

This suggests the natural question: can every bracket polynomial, with integer coefficients and homogeneous in the occurances of each point, be factored as a Grassmann algebra expression in meet and join (no additions or subtractions) – and hence interpreted as a synthetic construction ?

The simple answer is "no" as is seen in the example below. The problem of deciding whether a specific bracket algebra polynomial is Grassmann (Cayley) factorable is very difficult, in general. Recent progress in this direction has been made by N. White [4] who gave an algorithm based on invariant theory for Grassmann factorization in the multilinear case. (See also H. Crapo [1]). Let us remark that the factorization theorem proved in this paper is by no means competative to the White algorithm because it is much too general to

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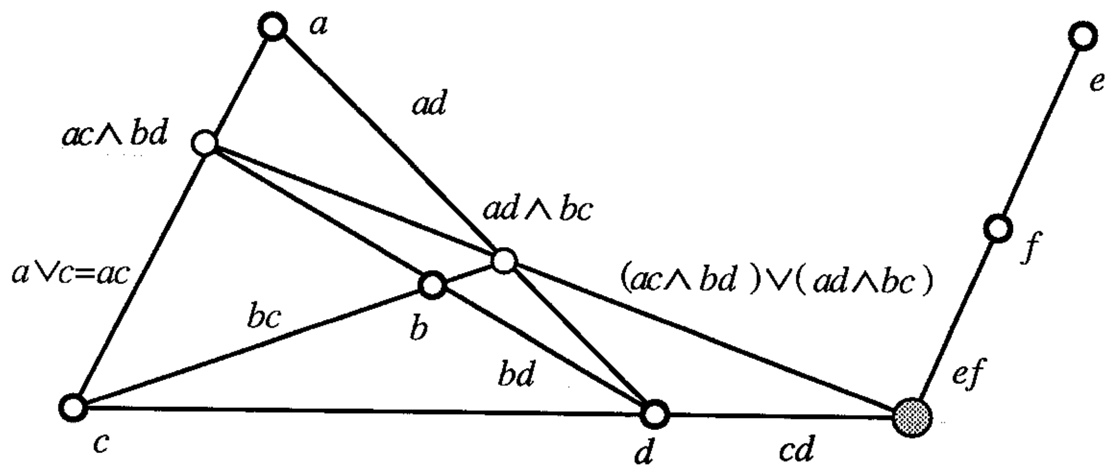

FIGURE 1. *The lines* $(ac \wedge bd) \vee (ad \wedge bc)$, cd *and* ef *are concurrent.*

be of any practical use. Their algorithm, however, suggests that Grassmann factorization might play a substantial role in future algorithms dealing with the important problem of geometrically interpreting algebraic expressions.

EXAMPLE 1. Given points a, b, c, d, e, f in the Euclidean plane, consider the triangle which is bounded by the lines ab, cd and ef. Up to a scalar factor, the oriented area of this triangle can be written as

$$ab \wedge cd \wedge ef \quad = \quad [abc][def] - [abd][cef].$$

On the other hand, the bracket polynomial $[abc][def] + [abd][cef]$ does not factor in the Grassmann algebra of rank 3. This can be seen easily as follows: The invariant $[abc][def] + [abd][cef]$ is antisymmetric in both (a, b) and (e, f) while it is symmetric in (c, d). This cannot be the case for any multilinear Grassmann algebra expression in a, b, c, d, e and f.

There is a more subtle answer. If we multiply this invariant by an appropriate product of brackets, then the resulting polynomial does factor as a synthetic construction.

EXAMPLE 1 (CONTINUED). The expression $[acd][bcd]\big([abc][def] + [abd][cef]\big)$ does factor, as follows.

$$\begin{aligned}
& \big\{ (ac \wedge bd) \vee (ad \wedge bc) \big\} \wedge ef \wedge cd \\
= \; & \big\{ ([acb]d - [acd]b) \vee ([adb]c - [adc]b) \big\} \wedge \{[efc]d - [efd]c\} \\
= \; & \big\{ ([acb][adb]dc - [acd][adb]bc - [acb][adc]db) \big\} \wedge \{[efc]d - [efd]c\} \\
= \; & -[acd][adb][efc][bcd] + [acb][adc][efd][dbc] \\
= \; & [acd][bcd]\big([abc][def] + [abd][cef]\big)
\end{aligned}$$

This synthetic construction is illustrated in Figure 1, and represents the construction of a point of intersection with cross ratio -1 to c, d and $ab \wedge cd$.

This raises the question whether every invariant polynomial, with integer coefficients, can be factored as a synthetic construction, if a suitable multiplier is chosen. It is the purpose of this note to prove that this is true for $r \geq 3$.

THEOREM 1. *Let $B([x_1 \dots x_r], \dots, [x_{m-r+1} \dots x_m])$ be a bracket algebra polynomial of rank $r \geq 3$, with integer coefficients and homogeneous in the occurances of each point x_i. Then there exists a Grassmann algebra expression in meet and join $E(\wedge, \vee, x_1, \dots, x_m)$, and a bracket monomial M such that*

$$E(\wedge, \vee, x_1, \dots, x_m) \quad = \quad M \cdot B([x_1 \dots x_r], \dots, [x_{m-r+1} \dots x_m]).$$

REMARK. For the 2-brackets, or the projective line, there are no significant synthetic constructions except the coincidence of points, and no useful factoring for the invariants. A simple expression like $[ab][cd] - [ac][bd]$ will never factor to a synthetic construction unless the projective line is embedded into some higher-dimensional projective space.

Theorem 1 is a natural extension of the standard coordinatization of the synthetic projective plane. The classical result says that synthetic constructions, with joins of points and intersections of lines, reproduce the algebra of the underlying field. We are answering a broader question: *Can all projective properties which are written in analytic geometry be reproduced in synthetic geometry ?*

If we take an analytic projective property, represented by a single polynomial equation, this equation must be invariant under the group of projective transformations of the space – the group of collineations which preserve lines, points and incidences. The First Fundamental Theorem of Invariant Theory (for any infinite field) says that such an equation is equivalent to a polynomial in the brackets, homogeneous in the entries for each point [6].

The polynomial equations of Theorem 1 have one additional restriction – the coefficients are integers. This leaves a residual question: do polynomial equations with incommensurable coefficients also represent projective properties ? This is clearly impossible for the field $\mathbb{Q}$ of rational numbers. If the underlying field is the complex numbers $\mathbb{C}$ (or any algebraically closed field) then the projective collineations include all automorphisms of the field. Invariance under these maps guarantees that a projective property is represented by a polynomial equation with integer coefficients [6]. If we restrict ourselves to the real projective plane, then because of the lack of automorphisms of $\mathbb{R}$, we have invariant equations such as

$$[abc][def] - \sqrt{2}\,[abd][cef] \quad = \quad 0$$

which cannot be checked with a synthetic construction. Synthetic geometry is more restrictive than analytic geometry !

These observations give a metatheorem about the correspondences between analytic and synthetic projective properties by equations.

THEOREM 2.

a) *For projective geometry over $\mathbf{Q}$ or an algebraically closed field, every analytic property represented by a single polynomial equation corresponds to a synthetic construction with points and lines.*

b) *For projective geometry over the reals (or other non-rational, non-closed fields) there exist analytic projective properties represented by a single polynomial equation which do not correspond to a synthetic construction with points and lines.*

A general analytic projective property will be represented by a first order formula with polynomial equations. Every such property over an infinite field can be rewritten using only polynomial equations in the brackets, homogeneous in each point. It is unknown whether every such property over an algebraically closed field can be rewritten using only rational coefficients. If the answer is "yes", the translation is non-trivial. For example, the formula

$$[abc][def] - \sqrt{2}\,[abd][cef] = 0 \qquad \vee \qquad [abc][def] + \sqrt{2}\,[abd][cef] = 0$$

is projectively invariant over $\mathbf{C}$, since it is equivalent to

$$(\,[abc][def]\,)^2 \;-\; 2\,(\,[abd][cef]\,)^2 \;=\; 0.$$

Thus we remain one step away from the conjecture that synthetic geometry and analytic geometry are equivalent over $\mathbf{C}$. See the article of W. Whiteley [6] in this volume for details and further references.

2. Proof of the factorization theorem.

Our proof of the Theorem is based on the classical constructions of *projective addition*, *projective subtraction* and *projective multiplication*. With these techniques it is possible to encode arbitrary polynomial equations (with integer coefficients) into suitable synthetic constructions in the projective plane, see e.g. [3, Chapter 2], [5, Chapter 1]. For the purpose of the present paper these methods can be described as follows.

Let e_1, e_2, e_3, e_4 be in general position in the vector space $\mathbf{Q}^3$. Via homogeneous coordinates the vectors e_1, e_2, e_3, e_4 can be thought of as a projective basis of the projective plane. With every rational number τ or, more generally, with every rational function τ with rational coefficients we associate the vector $y(\tau) := e_1 + \tau e_2$. Again, $y(\tau)$ will be thought of as a point on the projective line $e_1 e_2$ with projective coordinate τ. We have the following lemma.

LEMMA. *Let $P \in \mathbf{Z}[\tau_1, \dots, \tau_n]$ be any homogeneous polynomial in n scalar variables τ_i. Then there exists a Grassmann algebra expression in join and meet (synthetic construction) $E_P(\wedge, \vee, e_1, \dots, e_4, y(\tau_1), \dots, y(\tau_n))$ and a bracket monomial $M_P = [e_1e_2e_3]^{\mu_4}[e_1e_2e_4]^{\mu_3}[e_1e_3e_4]^{\mu_2}[e_2e_3e_4]^{\mu_1}$ such that*

$$E_P(\wedge, \vee, e_1, \dots, e_4, y(\tau_1), \dots, y(\tau_n)) \quad = \quad P(\tau_1, \dots, \tau_n) \cdot M_P$$

Proof. We prove this lemma by induction on the structure of the polynomial P. Clearly, P can be written as an expression in the atoms $\tau_1, \dots, \tau_n$ and 1 using only the binary operations addition, subtraction and multiplication.

Step 1. Let $\omega := [e_1e_4e_3]/[e_4e_2e_3]$. We will think of the scalar ω as representing the unit "1". The vector $y(\omega)$ corresponding to the scalar ω is constructed as follows:

$$[e_4e_2e_3]\cdot y(\omega) \;=\; [e_4e_2e_3](e_1 + \omega\cdot e_2) \;=\; [e_4e_2e_3]e_1 + [e_1e_4e_3]e_2 \;=\; (e_1 \vee e_2) \wedge (e_3 \vee e_4).$$

Given two polynomials $P, Q \in \mathbf{Z}[\tau_1, \dots, \tau_n]$ we assume now that the vectors $y(P)$ and $y(Q)$ can be written (up to a bracket monomial multiplier) as a Grassmann algebra expression in the vectors $y(\omega), y(\tau_1), \dots, y(\tau_n), e_1, \dots, e_4$. Under this assumption we shall show that also the vectors $y(P+Q), y(P-Q)$ and $y(\omega\cdot P\cdot Q)$ possess such a representation.

Step 2. (Projective addition)

$$\begin{aligned}
& \Big\{ \big(\{(e_2e_4 \wedge e_1e_3) \vee \underline{y(P)}\} \wedge e_2e_3 \big) \vee \big(e_2e_4 \wedge \underline{y(Q)}e_3\big) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \big(\{(-[e_2e_4e_1]e_3 + [e_2e_4e_3]e_1) \vee (e_1 + Pe_2)\} \wedge e_2e_3 \big) \vee \big(e_2e_4 \wedge y(Q)e_3\big) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \big(\{ -[e_2e_4e_1]e_3e_1 - P[e_2e_4e_1]e_3e_2 + P[e_2e_4e_3]e_1e_2 \} \wedge e_2e_3 \big) \vee \big(e_2e_4 \wedge y(Q)e_3\big) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \big([e_2e_4e_1][e_3e_1e_2]e_3 + P[e_2e_4e_3][e_1e_2e_3]e_2\big) \vee \big(-[e_2e_4e_1]e_3 + [e_2e_4e_3]e_1 + Q[e_2e_4e_3]e_2\big) \Big\} \\
& \hfill \wedge\; e_1e_2 \\
= \;& \Big\{ +[e_2e_4e_1][e_3e_1e_2][e_2e_4e_3]e_3e_1 + Q[e_2e_4e_1][e_3e_1e_2][e_2e_4e_3]e_3e_2 \\
& \quad - P[e_2e_4e_3][e_1e_2e_3][e_2e_4e_1]e_2e_3 + P[e_2e_4e_3][e_1e_2e_3][e_2e_4e_3]e_2e_1 \Big\} \wedge e_1e_2 \\
= \;& \quad [e_1e_2e_4][e_1e_2e_3][e_2e_4e_3][e_1e_2e_3] \cdot (\, e_1 + Pe_2 + Qe_2 \,) \\
= \;& \quad [e_1e_2e_4][e_1e_2e_3][e_2e_4e_3][e_1e_2e_3] \cdot \underline{y(P+Q)}
\end{aligned}$$

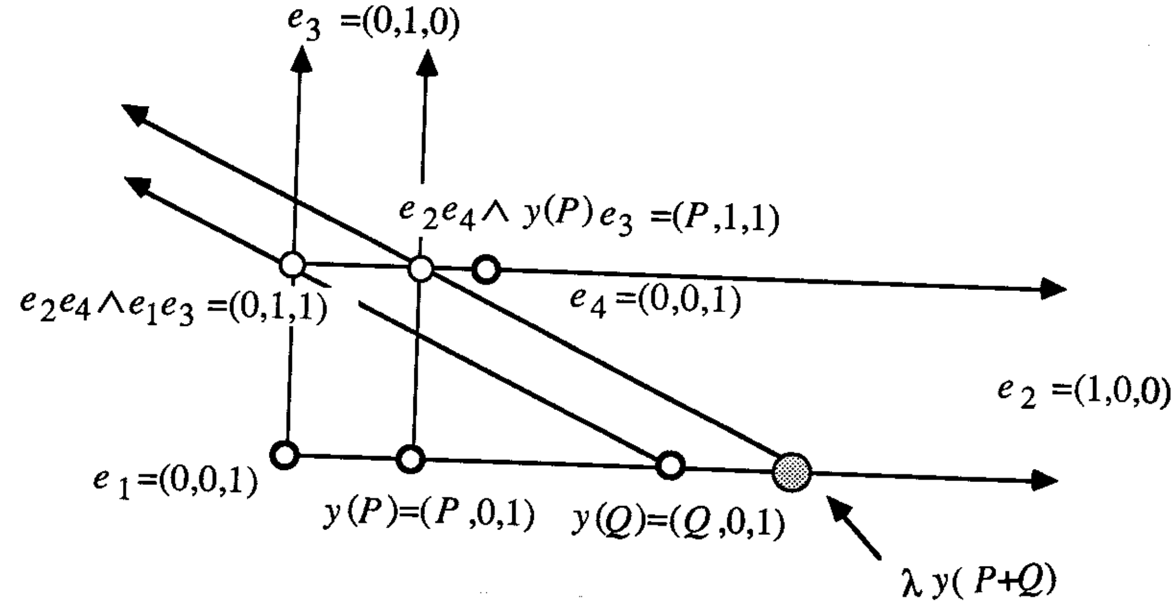

FIGURE 2. *Synthetic construction of $y(P+Q)$ from $y(P)$ and $y(Q)$.*

Step 3. (Projective subtraction)

$$
\begin{aligned}
& \Big\{ \Big(\{(e_2e_4 \wedge \underline{y(P)}e_3) \vee \underline{y(Q)}\} \wedge e_2e_3 \Big) \vee (e_2e_4 \wedge e_1e_3) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \Big(\{((e_2e_4 \wedge e_1e_3) + (e_2e_4 \wedge Pe_2e_3)) \vee y(Q)\} \wedge e_2e_3 \Big) \vee (e_2e_4 \wedge e_1e_3) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \Big(\{(-[e_2e_4e_1]e_3 + [e_2e_4e_3]e_1 - P[e_2e_3e_4]e_2) \vee (e_1 + Qe_2)\} \wedge e_2e_3 \Big) \vee (e_2e_4 \wedge e_1e_3) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \Big(\{ -[e_2e_4e_1]e_3e_1 - (Q-P)[e_2e_3e_4]e_1e_2 - Q[e_1e_2e_4]e_3e_2 \} \wedge e_2e_3 \Big) \vee (e_2e_4 \wedge e_1e_3) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ \Big([e_2e_4e_1][e_3e_1e_2]e_3 - (Q-P)[e_2e_3e_4][e_1e_2e_3]e_2 \Big) \vee (-[e_2e_4e_1]e_3 + [e_2e_4e_3]e_1) \Big\} \wedge e_1e_2 \\
= \;& \Big\{ -(P-Q)[e_2e_3e_4][e_1e_2e_3][e_2e_4e_1]e_2e_3 + [e_2e_4e_1][e_3e_1e_2][e_2e_4e_3]e_3e_1 \\
& \qquad\qquad - (Q-P)[e_2e_3e_4][e_1e_2e_3][e_2e_4e_3]e_2e_1 \Big\} \wedge e_1e_2 \\
= \;& (P-Q)[e_1e_2e_3]^2[e_1e_2e_4][e_2e_3e_4]e_2 + [e_1e_2e_3]^2[e_1e_2e_4][e_2e_4e_3]e_1 \\
= \;& [e_1e_2e_3]^2\,[e_1e_2e_4][e_2e_4e_3] \cdot \underline{y(P-Q)}
\end{aligned}
$$

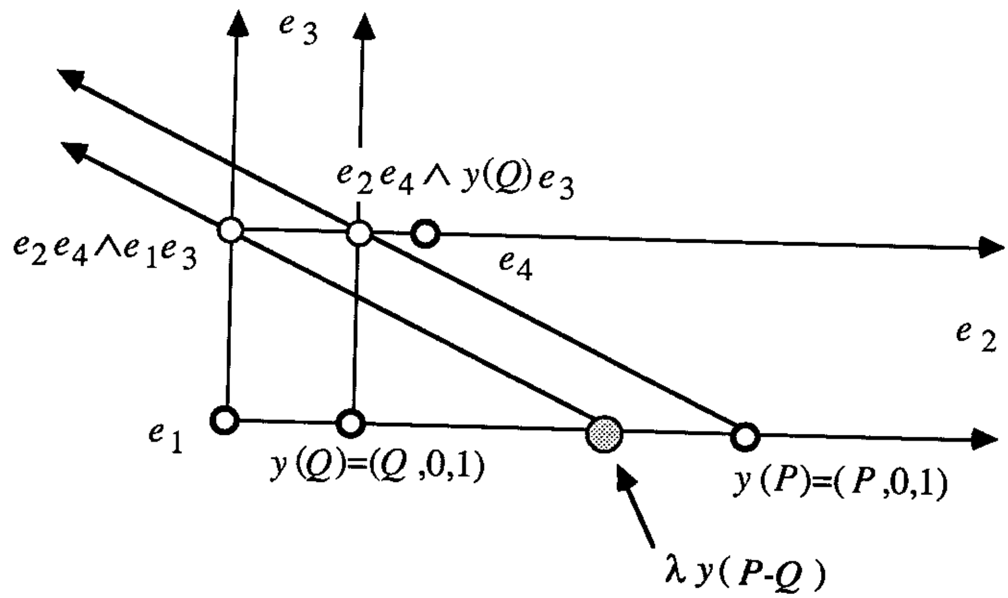

FIGURE 3. *Synthetic construction of $y(P-Q)$ from $y(P)$ and $y(Q)$.*

Step 4. (Projective multiplication)

$$
\begin{aligned}
&\Big\{\big(\{(e_2e_4 \wedge e_1e_3) \vee \underline{y(P)}\} \wedge e_2e_3\big) \\
&\qquad\qquad \vee \Big(\big\{\big(\{(e_2e_4 \wedge e_1e_3) \vee (e_3e_4 \wedge e_2e_1)\} \wedge e_2e_3\big) \vee \underline{y(Q)}\big\} \wedge e_3e_1\Big)\Big\} \wedge e_2e_1 \\
= \;& -[e_1e_2e_3][e_2e_3e_4]\cdot \Big\{\big(\{(e_2e_4 \wedge e_1e_3) \vee y(P)\} \wedge e_2e_3\big) \\
&\qquad\qquad\qquad \vee \Big(\Big\{\big([e_1e_2e_4]e_3 + [e_1e_3e_4]e_2\big) \vee (e_1 + Qe_2)\Big\} \wedge e_3e_1\Big)\Big\} \wedge e_2e_1 \\
= \;& -[e_1e_2e_3]^2[e_2e_3e_4]\cdot \Big\{\big([e_2e_4e_1]e_3 + P[e_2e_4e_3]e_2\big) \\
&\qquad\qquad\qquad \vee \Big(\Big\{\big([e_1e_2e_4]e_3 + [e_1e_3e_4]e_2\big) \vee (e_1 + Qe_2)\Big\} \wedge e_3e_1\Big)\Big\} \wedge e_2e_1 \\
= \;& -[e_1e_2e_3]^3[e_2e_3e_4]\Big\{\big([e_2e_4e_1]e_3 + P[e_2e_4e_3]e_2\big) \vee \big([e_1e_3e_4]e_1 - Q[e_1e_2e_4]e_3\big)\Big\} \wedge e_2e_1 \\
= \;& [e_1e_2e_3]^4[e_1e_2e_4][e_2e_3e_4]\Big\{[e_1e_3e_4]e_1 + PQ[e_2e_4e_3]e_2\Big\} \\
= \;& \quad [e_1e_2e_3]^4[e_1e_2e_4][e_1e_3e_4][e_2e_3e_4]\cdot \underline{y(\omega^{-1}\cdot P\cdot Q)}
\end{aligned}
$$

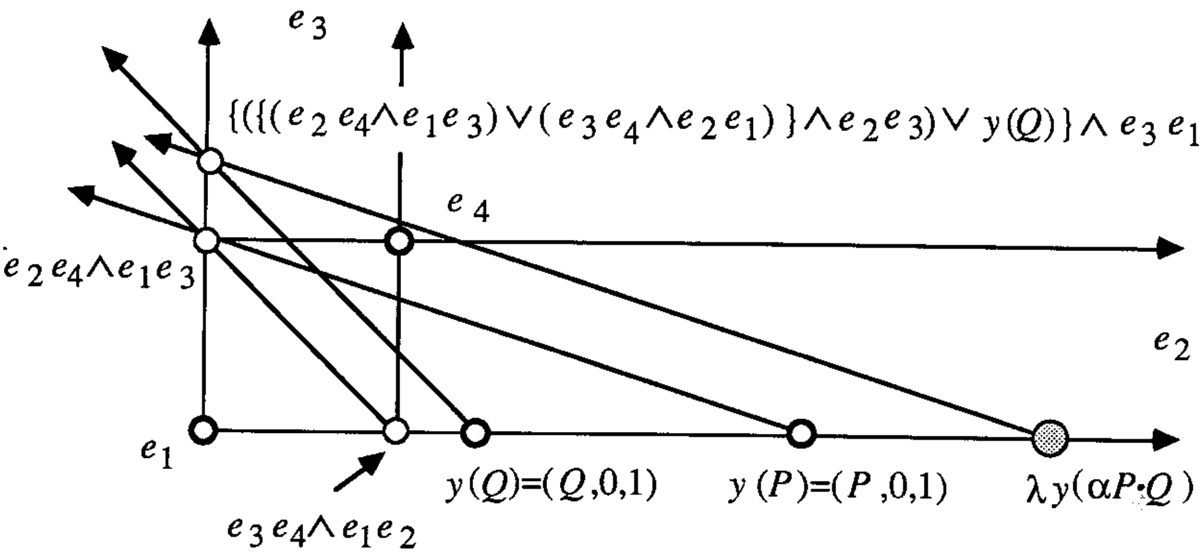

FIGURE 4. *Synthetic construction of* $y(\omega^{-1} \cdot P \cdot Q)$ *from* $y(P)$ *and* $y(Q)$.

Now assume that $P \in \mathbf{Z}[\tau_1, \ldots, \tau_n]$ is an arbitrary homogeneous polynomial of degree k. Combining steps 1–4 while treating ω as the unit, we obtain a Grassmann algebra expression in $y(\tau_1), \ldots, y(\tau_n), e_1, \ldots, e_4$ which is equal (up to a monomial multiplier) to the vector $y(\omega^{-k} \cdot P(\tau_1, \ldots, \tau_n)) = e_1 + \omega^{-k} \cdot P(\tau_1, \ldots, \tau_n) \cdot e_2$. Taking the meet of this expression with $e_1 \vee e_3$ and clearing denominators, we obtain

$$E_P(\wedge, \vee, e_1, \ldots, e_4, y(\tau_1), \ldots, y(\tau_n)) \quad = \quad P(\tau_1, \ldots, \tau_n) \cdot M_P,$$

as desired. This completes the proof of the Lemma. $\square$

Now we are ready to derive our factorization theorem. The proof will be given only for 3-brackets or the projective plane. For invariants in the r-brackets, $r > 3$, one can simply work with a larger projective basis, and project the coordinates down into a plane spanned by three suitable basis points.

Proof of Theorem 1. Let $B([x_1 x_2 x_3], \ldots, [x_i x_j x_k], \ldots, [x_{m-2} x_{m-1} x_m])$ be an invariant with integer coefficients of degree k in the 3-brackets on m points $x_1, \ldots, x_m$, and let, as before, (e_1, e_2, e_3, e_4) be a projective basis. We multiply B by $[e_1 e_2 e_3]^{2k}$, and we use the superexchange identity

$$[e_1 e_2 e_3]^2 [x_i x_j x_k] \quad = \quad \det \begin{pmatrix} [x_1 e_2 e_3] & [e_1 x_1 e_3] & [e_1 e_2 x_1] \\ [x_2 e_2 e_3] & [e_1 x_2 e_3] & [e_1 e_2 x_2] \\ [x_3 e_2 e_3] & [e_1 x_3 e_3] & [e_1 e_2 x_3] \end{pmatrix}$$

to reduce all summands of B to expressions in the homogeneous coordinates $([x_i e_2 e_3], [e_1 x_i e_3], [e_1 e_2 x_i])$ of the points x_i versus the basis (e_1, e_2, e_3). We rewrite this coordinate vector as $[x_i e_2 e_3] \cdot (1, x_i^1, x_i^2)$ where $x_i^1 := [e_1 x_i e_3]/[x_i e_2 e_3]$, $x_i^2 := [e_1 e_2 x_i]/[x_i e_2 e_3]$.

We can express the bracket polynomial B in terms of the projective coordinates x_i^j as

$$(1)\qquad B(\ldots,[x_ie_2e_3],[e_1x_ie_3],[e_1e_2x_i],\ldots) \;=\; P(x_1^1,x_1^2,\ldots,x_m^1,x_m^2)\cdot\prod_{i=1}^{m}[x_ie_2e_3]^{\nu_i}$$

where $P \in \mathbf{Z}(x_1^1,x_1^2,\ldots,x_m^1,x_m^2)$ is homogeneous.

By the Lemma, there exists a synthetic construction for the polynomial P. More precisely, there is a Grassmann algebra expression E_P and a bracket monomial $M_P = [e_1e_2e_3]^{\mu_4}[e_1e_2e_4]^{\mu_3}[e_1e_3e_4]^{\mu_2}[e_2e_3e_4]^{\mu_1}$ such that

$$(2)\qquad E_P(\wedge,\vee,e_1,\ldots,e_4,\,y(x_1^1),y(x_1^2),\ldots,y(x_m^2)) \;=\; P(x_1^1,x_1^2,\ldots x_m^1,x_m^2)\cdot M_P.$$

¿From the equations (1) and (2) we get a representation

$$(3)\qquad E'(\wedge,\vee,e_1,\ldots,e_4,\,y(x_1^1),y(x_1^2),\ldots,y(x_m^1),y(x_m^2)) \;=\; M'\cdot B,$$

where B is the given bracket polynomial, M' a suitable bracket monomial, and E' a Grassmann algebra expression. Note that in the derivation from of (3) we are making use of the fact that an additional bracket monomial is allowed as a (synthetic) factor in the Grassmann algebra expression E'.

Finally, we need to express the vectors $y(x_i^1)$ and $y(x_i^2)$, which represent the scalars x_i^1, x_i^2 in terms of suitable synthetic constructions (up to a bracket factor). We assume at this point that $e_4 = e_1+e_2+e_3$, and we get

$$(4)\qquad e_1 + x_i^1\cdot e_2 \;=\; [x_ie_2e_3]\,y(x_i^1) \;=\; (x_i\vee e_3)\wedge(e_1\vee e_2).$$

We obtain by expansion

$$(5)\qquad \begin{aligned} &\big(\{((e_1e_2\wedge e_3e_4)\vee(e_2e_4\wedge e_1e_3))\wedge e_2e_3\}\vee(e_2x_i\wedge e_1e_3)\big)\wedge e_1e_2 \\ =\;& [e_1e_2e_3]^2[e_2e_3e_4]\big([e_2e_1e_4][x_ie_2e_3]e_1 + [e_1e_3e_4][e_1e_2x_i]e_2\big) \\ =\;& [e_1e_2e_3]^2[e_2e_3e_4][e_1e_3e_4][x_ie_2e_3]\cdot y(x_i^2). \end{aligned}$$

We solve the equations (4) and (5) for $y(x_i^1)$ and $y(x_i^2)$ respectively, and we plug in the resulting (fractional) Grassmann algebra expressions into the equation (3). Clearing denominators yields

$$(6)\qquad E''(\wedge,\vee,e_1,\ldots,e_4,\,x_1,x_2,\ldots,x_m) \;=\; M''\cdot B,$$

where M'' is a bracket monomial and E'' a Grassmann algebra expression.

The expressions on both sides are projectively invariant, and therefore the assumption $e_4 = e_1+e_2+e_3$ can be dropped at this point. Equation (6) remains valid if (e_1,e_2,e_3,e_4)

is replaced by any projective basis. In particular, we can replace the e_i by suitable x_j as long as the non-degeneracy assumptions

$$[e_1e_2e_3] \neq 0, \ [e_4e_2e_3] \neq 0, \ [e_1e_4e_3] \neq 0, \ [e_1e_2e_4] \neq 0, \ \text{and} \ [x_ie_2e_3] \neq 0$$

are satisfied. These conditions, which are needed in the proof of the Lemma, are trivially satisfied when the points x_j are in general position in the projective plane.

With this substitution equation (6) can be rewritten as

$$E(\wedge, \vee, x_1, \dots, x_m) \quad = \quad M \cdot B\big([x_1x_2x_3], \dots, [x_{m-2}x_{m-1}x_m]\big). \tag{7}$$

This completes the proof of Theorem 1. $\square$

REMARK. *Suppose that the original polynomial B has degree k in the 3-brackets and that it consists of l summands with coefficients ± 1. Tracing carefully all constructions, it can be seen that the degree of the multiplier monomial M is bounded by $105\,k\,l$.*

Of course this bound could be improved, but our construction is only of theoretical, not practical, interest.

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