

SYMBOLIC COMPUTATIONS IN GEOMETRY

By

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Preface

The papers collected in this volume were presented at a concentrated week on *Symbolic Computations in Projective Geometry*, held at the Institute for Mathematics and its Applications, Minneapolis, October 12–16, 1987. This week was intended to study the use of coordinate-free algebraic methods in computational geometry. These methods include the Cayley or Grassmann algebra, the bracket algebra of invariants of the general linear group, and the Cayley-Menger algebra which is generated by Euclidean invariants. These structures, because of their coordinate-free (invariant) nature, lie closer to synthetic geometry than the usual algebra of coordinates versus a fixed frame of reference.

Papers I, V, and VI discuss Cayley factorization, which is the crucial step in interpreting algebraic expressions in projective geometry. Paper II examines some algebraic issues regarding the computer-aided coordinatization of combinatorial configurations, and in paper VII these methods are applied to a large class of previously unrealized configurations. Paper III deals with the logical underpinnings of invariant methods, and paper IV applies the algebra of Cayley-Menger determinants to computational problems in Euclidean geometry.

The volume is partially expository, but also contains many interesting new results, which will hopefully stimulate further research in this area. We wish to thank the I.M.A. for its support during our week.

Minneapolis, January 1988

Bernd Sturmfels and Neil White

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