PASSIVE QUASI-FREE STATES
OF THE NONINTERACTING FERMI GAS

BY

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<table>
<thead>
<tr>
<th>Preprint #</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>Workshop Summaries from the September 1982 workshop on Statistical Mechanics, Dynamical Systems and Turbulence</td>
</tr>
<tr>
<td>2</td>
<td>Raphael De la Llave</td>
<td>A Simple Proof of C. Siegel's Center Theorem</td>
</tr>
<tr>
<td>3</td>
<td>H. Simpson, S. Spector</td>
<td>On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials</td>
</tr>
<tr>
<td>4</td>
<td>George R. Sell</td>
<td>Vector Fields in the Vicinity of a Compact Invariant Manifold</td>
</tr>
<tr>
<td>5</td>
<td>Milan Miklavcic</td>
<td>Non-linear Stability of Asymptotic Suction</td>
</tr>
<tr>
<td>6</td>
<td>Hans Weinberger</td>
<td>A Simple System with a Continuum of Stable Inhomogeneous Steady States</td>
</tr>
<tr>
<td>7</td>
<td>Bau-Sen Du</td>
<td>Period 3 Bifurcation for the Logistic Mapping</td>
</tr>
<tr>
<td>8</td>
<td>Hans Weinberger</td>
<td>Optimal Numerical Approximation of a Linear Operator</td>
</tr>
<tr>
<td>9</td>
<td>L.R. Angel, D.F. Evans, B. Ninham</td>
<td>Three Component Ionic Microemulsions</td>
</tr>
<tr>
<td>10</td>
<td>D.F. Evans, D. Mitchell, S. Mukherjee, B. Ninham</td>
<td>Surfactant Diffusion; New Results and Interpretations</td>
</tr>
<tr>
<td>11</td>
<td>Leif Arkeryd</td>
<td>A Remark about the Final Aperiodic Regime for Maps on the Interval</td>
</tr>
<tr>
<td>12</td>
<td>Luis Magalhaes</td>
<td>Manifolds of Global Solutions of Functional Differential Equations</td>
</tr>
<tr>
<td>13</td>
<td>Kenneth Meyer</td>
<td>Tori in Resonance</td>
</tr>
<tr>
<td>14</td>
<td>C. Eugene Wayne</td>
<td>Surface Models with Nonlocal Potentials: Upper Bounds</td>
</tr>
<tr>
<td>16</td>
<td>George R. Sell</td>
<td>Smooth Linearization Near a Fixed Point</td>
</tr>
<tr>
<td>17</td>
<td>David Walklin</td>
<td>A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification</td>
</tr>
<tr>
<td>18</td>
<td>Pierre Collet</td>
<td>Local $C^r$ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z + z^2$</td>
</tr>
<tr>
<td>19</td>
<td>Henry C. Simpson, Scott J. Spector</td>
<td>On the Modified Bessel Functions of the First Kind On Barrelling for a Material in Finite Elasticity</td>
</tr>
<tr>
<td>20</td>
<td>George R. Sell</td>
<td>Linearization and Global Dynamics</td>
</tr>
<tr>
<td>21</td>
<td>P. Constantin, C. Foias</td>
<td>Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 3D Navier-Stokes Equations</td>
</tr>
<tr>
<td>22</td>
<td>Milan Miklavcic</td>
<td>Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator</td>
</tr>
<tr>
<td>23</td>
<td>P. Collet, H. Epstein, G. Gallavotti</td>
<td>Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties</td>
</tr>
<tr>
<td>24</td>
<td>J.E. Dunn, J. Serrin</td>
<td>On the Thermomechanics of Interstitial working</td>
</tr>
<tr>
<td>25</td>
<td>Scott J. Spector</td>
<td>On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension</td>
</tr>
<tr>
<td>26</td>
<td>W.A. Coppel</td>
<td>Maps on an Interval</td>
</tr>
<tr>
<td>27</td>
<td>James Kirkwood</td>
<td>Phase Transitions in the Ising Model with Traverse Field</td>
</tr>
<tr>
<td>28</td>
<td>Luis Magalhaes</td>
<td>The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different</td>
</tr>
<tr>
<td>29</td>
<td>Charles Tresser</td>
<td>Homoclinic Orbits for $\mathbb{R}^3$</td>
</tr>
<tr>
<td>30</td>
<td>Charles Tresser</td>
<td>About some Theorems by L.P. Sil'nikov</td>
</tr>
<tr>
<td>31</td>
<td>Michael Aizenmann</td>
<td>On the Renormalized Coupling Constant and the Susceptibility in $\phi^4$ Field Theory and the Ising Model in Four Dimensions</td>
</tr>
<tr>
<td>32</td>
<td>C. Eugene Wayne</td>
<td>The KAM Theory of Systems with Short Range Interactions I</td>
</tr>
</tbody>
</table>

(continued on back cover)
PASSIVE QUASI-FREE STATES
OF THE NONINTERACTING FERMI GAS

by

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Abstract

The passive quasi-free states of the noninteracting Fermi gas with continuous one-particle hamiltonian $H$ are computed. They turn out to be the well known Fermi-Dirac states, or limits thereof. This still holds true if the spectrum of $H$ has both a continuous and a discrete part, except for the appearance of a class of "ground state-like" states showing a local random excitation of the point spectrum in a neighborhood of the Fermi energy. When $H$ has only pure point spectrum, the requirement that a state be passive and quasi-free is no longer sufficient to characterize the Fermi-Dirac distributions.
1. **Introduction**

We shall be concerned with a system ("gas") composed of an arbitrary number of identical noninteracting fermions, any one of which is described by means of the (one-particle) Hilbert space $\mathcal{H}$ and the (one-particle) hamiltonian $H$. Specifically, the dynamics of a single particle is given by the strongly continuous one-parameter group $U_t = e^{-itH}(t \in \mathbb{R})$ of unitary operators on $\mathcal{H}$.

The equilibrium distributions of such a system are well known. Let $f$ be an eigenvector of unit length of the hamiltonian, i.e., $Hf = \lambda f$ for some $\lambda$ in $\mathbb{R}$; then the expected number $\langle n(f) \rangle_{\beta, \mu}$ of particles in the state $f$ at inverse temperature $\beta = (kT)^{-1}$ and chemical potential $\mu$ is given by the Fermi-Dirac distribution law

$$\langle n(f) \rangle_{\beta, \mu} = \left(1 + e^{\beta(\lambda - \mu)} \right)^{-1}.$$  \hspace{1cm} (1.1)

In the usual justifications of this formula (e.g., [12, chapter 9; 14, chapter 5]), based either on the maximum entropy principle or on one of the Gibbs ensembles, it is assumed that $\mathcal{H}$ contains a complete orthonormal set of eigenvectors of $H$. From the mathematical point of view this is of course a restrictive assumption.

One way of describing the content of this paper is to say that a derivation of the Fermi-Dirac law is presented that is valid for a hamiltonian with nonempty continuous spectrum (but in general not, it turns out, for a hamiltonian with pure point spectrum! Cf. Example 4.9). More specifically it is shown in Theorem 3.10 that, when $H$ has only continuous spectrum, then a state $\omega$ of the noninteracting Fermi gas corresponds to a Fermi-Dirac distribution, or to a limit of Fermi-Dirac distributions (ground state, Powers state, Fock or anti-Fock state) if (and only if)

(i) $\omega$ is quasi-free (Definition 2.1), i.e., all its truncated n-point functions vanish identically as soon as $n > 2$, and

(ii) $\omega$ is passive in the sense of Pusz and Woronowicz (Definition 2.2).
The first condition is of a purely statistical nature and independent of the dynamics: the state \( \omega \) does not admit any "proper" multiparticle correlations (this is analogous to the vanishing of the higher cumulants of the normal distribution). Condition (ii), as first argued in \([18]\), expresses a consequence of the second law of thermodynamics: the system is unable to perform work in a cyclic change of external conditions if it is initially in the state \( \omega \). We shall, however, give a somewhat different interpretation of (ii) later in this section.

If \( H \) has both point spectrum and continuous spectrum we only achieve a complete classification of the faithful passive quasi-free states, i.e., those states of the noninteracting Fermi gas for which \( \langle n(f) \rangle \neq 0,1 \) for all pure one-particle states \( f \) (Theorem 4.3). This is due to the existence of a somewhat intriguing class of passive quasi-free states that behave like ground states on the continuous part of the spectrum of \( H \), but still allow some random excitation of the point spectrum in a neighborhood of the "Fermi energy," provided that this energy does not lie in the continuous spectrum (Example 4.7). If \( H \) has no continuous spectrum at all, no statement of the sort made above is true in general (Example 4.9). This indicates the relevance of spectral considerations.

To give whatever "proof" of the well established Fermi–Dirac distribution law is not the main purpose of this work, however. Rather the aim is to draw attention to the passivity condition (ii). In combination with Fourier analysis it has proved an effective computational tool in the study of equilibrium states \([18, 8, 4, 5, 9]\). In general the definition of passivity applies to any invariant state of an arbitrary \( C^* \)-dynamical system, and its justification from the second law of thermodynamics \([18]\) is based on the perturbation theory for such systems. Fortunately in the case of a noninteracting Fermi gas the passivity condition has an elementary description and interpretation, which we now proceed to expose.

Let \( a^*(f) \) and \( a(f) \) \((f \in \mathbf{K})\) denote the creation, resp. annihilation operators acting on the antisymmetric Fock space \( \mathbf{F}^a_{a}(\mathbf{K}) \) built over the one-particle space \( \mathbf{K} \) (see, e.g., \([7, 13]\)). In particular we have the canonical anticommutation relations (CAR)
\[ a(f)a(g) + a(g)a(f) = 0 \]  \hspace{1cm} (1.2) \\
\[ a(f)a^*(g) + a^*(g)a(f) = (f, g) 1 \]

for all \( f \) and \( g \) in \( \mathcal{H} \), where \((\cdot, \cdot)\) denotes the scalar product in \( \mathcal{H} \) (linear in the second factor) and \( 1 \) is the identity operator on \( \mathcal{A}(\mathcal{H}) \). The CAR imply that \( \|a(f)\| = \|f\| \). Let \( \mathcal{A}_0(\mathcal{H}) \) be the algebra generated by \( 1 \) and all operators \( a(f), a^*(g) \) (the Clifford algebra over \( \mathcal{H} \)). A state \( \omega \) of \( \mathcal{A}_0(\mathcal{H}) \) is a positive (hence norm-bounded) linear functional on \( \mathcal{A}_0(\mathcal{H}) \) with \( \omega(1) = 1 \).

Let \( P \) denote the spectral measure associated with the hamiltonian \( H \) (i.e., 
\[ H = \int \lambda P(d\lambda) \]). Suppose that \( f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_n \) are elements of \( \mathcal{H} \) such that

\[
\begin{align*}
  f_j &= P(\lambda_j, +\infty) f_j \quad (1 \leq j \leq m) \\
  g_k &= P(-\infty, \mu_k) g_k \quad (1 \leq k \leq n)
\end{align*}
\]  \hspace{1cm} (1.3)

and

\[ \sum_{k=1}^n \mu_k \leq \sum_{j=1}^m \lambda_j \]  \hspace{1cm} (1.4)

Let \( x \) be the product of the operators \( a(f_1), a(f_2), \ldots, a(f_m), a^*(g_1), a^*(g_2), \ldots, a^*(g_n) \) (in any order). The passivity condition requires that

\[ \omega(x^*x) \leq \omega(xx^*) \]  \hspace{1cm} (1.5)

for all such \( x \).

To see what this means let us assume that all the \( f \)'s and \( g \)'s above are normalized and pairwise orthogonal. Using the CAR (1.2) we find

\[
\begin{align*}
  x^*x &= \prod_{j=1}^m a^*(f_j)a(f_j) \prod_{k=1}^n a(g_k)a^*(g_k) \\
  xx^* &= \prod_{j=1}^m a(f_j)a^*(f_j) \prod_{k=1}^n a^*(g_k)a(g_k)
\end{align*}
\]
where the factors in the products commute pairwise. Moreover \( a^*(f)a(f) \) is a projection operator, corresponding to the observation "some particle of the system occupies the state \( f \)," whereas \( a(f)a^*(f) = 1 - a^*(f)a(f) \) corresponds to the opposite statement. Hence \( \omega(x^*x) \) is the probability that one observes a particle in each of the states \( f_1, f_2, \ldots, f_m \), but no particle in any of the states \( g_1, g_2, \ldots, g_n \).

According to (1.5) this probability cannot exceed the probability that each of the states \( g_1, g_2, \ldots, g_n \) and none of the states \( f_1, f_2, \ldots, f_m \) is occupied. Since a particle in the state \( f_j \) carries an amount of energy of at least \( \lambda_j \) by (1.3), while in the state \( g_k \) the energy is at most \( \mu_k \), the inequalities (1.4) and (1.5) express the fact that the "finite dimensional distributions" of \( \omega \) favor the lower energy configurations. In other words, passivity appears as a minimal energy principle.

We close this discussion with two remarks. If in (1.3) we choose \( f_j = P[\lambda_j, +\infty)f_j \) and \( g_k = P(-\infty, \mu_k]g_k \), then we obtain a condition that has virtually the same physical interpretation as passivity, but that is nevertheless mathematically strictly stronger (unless \( H \) has no point spectrum). We refer to this condition as \underline{strong passivity} (cf. Remarks 2.3, 3.7, and 4.4). On the other hand we shall usually restrict our attention to the gauge-invariant part of \( \mathcal{H}_0(x) \). In that case we will require (1.5) to hold only for those \( x \) that are products of an equal number of creation and annihilation operators, i.e., \( m = n \) in (1.3), (1.4).

The organization of the paper is as follows. In Section 2 the definition of a quasi-free state is recalled, and the link between the condition (1.3) - (1.5) and earlier work on passivity is established. Section 3 is devoted to the study of the passive quasi-free states of the noninteracting Fermi gas with continuous one-particle hamiltonian. In Section 4 a partial extension of the results to more general hamiltonians is given, together with counterexamples.
2. Quasi-free states and passivity

Let \( \mathcal{A} (x) \) denote the norm-closure of \( \mathcal{U}_0 (x) \). We shall formulate our results in terms of \( \mathcal{A} (x) \) rather than \( \mathcal{U}_0 (x) \), in order to be able to relate them to the general theory of C*-dynamical systems. (In particular we shall make numerous references to the KMS condition [10; 7, Section 5.3].) This is only a matter of presentation: it is important to notice that all proofs are algebraic in nature, in that they really take place in the dense subalgebra \( \mathcal{U}_0 (x) \) of \( \mathcal{A} (x) \).

In fact, the analytic arguments are made at the level of the one-particle space \( \mathcal{K} \), not the Fock space \( \mathcal{F}_a (x) \).

The real line \( \mathbb{R} \) acts periodically as a gauge group on \( \mathcal{A} (x) \) by automorphisms \( \gamma_\theta \) satisfying

\[
\gamma_\theta (a(f)) = e^{i \theta} a(f)
\]

(\( \theta \in \mathbb{R}, f \in \mathcal{K} \)). For each integer \( d \), we define spectral subspaces

\[
\mathcal{A} (x)^{(d)} = \{ x \in \mathcal{A} (x) \mid \gamma_\theta (x) = e^{id\theta} x \text{ for all real } \theta \}.
\]

The fixed point algebra \( \mathcal{A} (x)^{(0)} \) is often called the observable algebra (as opposed to the field algebra \( \mathcal{A} (x) \)), sometimes more specifically the current algebra. Its structure has been thoroughly investigated in [6].

Every state \( \omega \) of \( \mathcal{A} (x)^{(0)} \) has a unique extension to a gauge-invariant state of \( \mathcal{A} (x) \). Most often we shall not distinguish between \( \omega \) and its gauge-invariant extension.

**Definition 2.1.** A (not necessarily gauge-invariant) state \( \omega \) of \( \mathcal{A} (x) \) is called quasi-free if it is even (i.e., it vanishes on all odd degree monomials in the \( a(f) \)'s and \( a^*(g) \)'s) and if, for all choices of \( n \geq 2 \) and \( f_1, f_2, \ldots, f_{2n} \) in \( \mathcal{K} \) one has

\[
\omega (a_{f_1} a_{f_2} \cdots a_{f_{2n}}) = \sum_{\tau} \sum_{j=1}^{n} \epsilon (\tau) \prod_{j=1}^{n} \omega (a_{\tau(2j-1)} f_{\tau(2j-1)} a_{\tau(2j)} f_{\tau(2j)})
\]

(2.1)
where \( a_j(f) \) stands for either \( a(f) \) or \( a^*(f) \), and where the sum is taken over all permutations \( \tau \) of the set \( \{1, 2, \ldots, 2n\} \) such that \( \tau(1) < \tau(3) < \ldots < \tau(2n-1) \) and \( \tau(2j-1) < \tau(2j) \) \( (1 \leq j \leq n) \). As usual \( \epsilon(\tau) = 1 \) if \( \tau \) is even, \( \epsilon(\tau) = 0 \) if \( \tau \) is odd. \( \Box \)

Formula (2.1) says that every correlation among creation and (or) creation operators can be expressed in terms of correlations between pairs of such operators.

If \( \omega \) is moreover gauge-invariant, then by (2.1) it is completely determined by the operator \( Q \) on \( \mathcal{H} \) given by

\[
\omega(a(\xi)ag) = (f, Qg)
\]

(\( f, g \in \mathcal{H} \)). The operator \( Q \) (which can be defined for any state of \( \mathcal{U}(\mathcal{H}) \)) will be called the one-particle density of \( \omega \). Clearly \( 0 \leq Q \leq 1 \). It can be shown that, conversely, every operator \( Q \) on \( \mathcal{H} \) satisfying \( 0 \leq Q \leq 1 \) defines a gauge-invariant quasi-free state of \( \mathcal{U}(\mathcal{H}) \) through (2.1) and (2.2). Furthermore, \( \omega \) is faithful if and only if both \( Q \) and \( 1 - Q \) are injective (to prove the "if" statement, one can show that \( \omega \) is KMS with respect to some time evolution and use the simplicity of \( \mathcal{U}(\mathcal{H}) \). I am indebted to M. Fannes for pointing out this argument to me). All the facts about quasi-free states we need can be found, e.g., in [7, 13].

Next we turn to the general definition of passivity. Let \( (\mathcal{U}, \alpha) \) be a \( C^\ast \)-dynamical system, i.e., \( \mathcal{U} \) is a \( C^\ast \)-algebra and \( t \mapsto \alpha_t \) is a strongly continuous one-parameter group of \( \ast \)-automorphisms of \( \mathcal{U} \). To each open subset \( \mathcal{O} \) of \( \mathbb{R} \) there corresponds a spectral subspace \( R(\mathcal{O}) \) of \( \mathcal{U} \), by definition the closed linear span of the set of elements of the form \( \int f(t)\alpha_t(x)dt \), where \( x \in \mathcal{U} \), \( f \in L^1(\mathbb{R}) \) and the support of the Fourier cotransform of \( f \) is compact and contained in \( \mathcal{O} \) (for the theory of spectral subspaces, see [3; 16; 7, Section 3.2.3]).

**Definition 2.2** [8, Definition 1.2]. A state \( \omega \) of \( \mathcal{U} \) is spectrally passive with respect to \( \alpha \) if it is \( \alpha \)-invariant and

\[
\omega(x^\ast x) \leq \omega(xx^\ast)
\]

for all \( x \) in \( R(\infty, 0) \). \( \Box \)
This condition is actually formally weaker than passivity as defined by Pusz and Woronowicz (although no example of a spectrally passive, nonpassive state seems to be known at present, and the conditions are equivalent in special cases [4, Corollary 7]). The relationship between passivity and spectral passivity is discussed in [8].

In the case of the noninteracting Fermi gas the time evolution is given by the one-parameter group \( t \mapsto \alpha_t \) of \( \ast \)-automorphisms determined by

\[
\alpha_t(a(f)) = a(U_{-t}f)
\]

(\( f \in \mathcal{H} \), \( t \in \mathbb{R} \)). This prescription corresponds to "second quantization" of \( \mathcal{H} \). Clearly \( \mathcal{U}(\mathcal{H})\langle 0 \rangle \) is globally \( \alpha \)-invariant. We shall use \( R(\Theta) \) to denote the spectral subspaces of \( \mathcal{U}(\mathcal{H}) \); then the corresponding subspace of \( \mathcal{U}(\mathcal{H})\langle 0 \rangle \) (associated with the restriction of \( \alpha \) to \( \mathcal{U}(\mathcal{H})\langle 0 \rangle \)) is simply \( R(\Theta) \cap \mathcal{U}(\mathcal{H})\langle 0 \rangle \).

Now it is easy to show that the condition (1.3)–(1.5) discussed in the Introduction is a consequence of spectral passivity. For if

\[
f_j = P(\lambda_j, +\infty)f_j \quad (j = 1, 2, \ldots, m) \quad \text{and} \quad g_k = P(-\infty, \mu_k)g_k \quad (k = 1, 2, \ldots, n)
\]

then \( a(f_j) \in R(-\infty, -\lambda_j) \) and \( a(\ast g_k) \in R(-\infty, \mu_k) \), because the map \( a^\ast \) is an isometric embedding of \( \mathcal{H} \) into \( \mathcal{U}(\mathcal{H}) \) intertwining \( U_t \) and \( \alpha_t \). Let \( x \) be a product of the operators \( a(f_1), a(f_2), \ldots, a(f_m), a(\ast g_1), a(\ast g_2), \ldots, a(\ast g_n) \); then

\[
x \in R \left( -\infty, \sum_{k=1}^{n} \mu_k - \sum_{j=1}^{n} \lambda_j \right) \subseteq R(-\infty, 0)
\]

and hence the spectral passivity of \( \omega \) requires exactly

\[
\omega(x^\ast x) \leq \omega(xx^\ast) \quad . \tag{1.5}
\]

Remark 2.3. For a general \( C^\ast \)-dynamical system \((\mathcal{U}, \alpha)\) one can construct spectral subspaces \( M(\mathcal{F}) \) corresponding to the closed subsets \( \mathcal{F} \) of \( \mathcal{H} \) \([3, 16, 7]\). In \([9]\) an \( \alpha \)-invariant state \( \omega \) of \( \mathcal{U} \) was defined to be strongly spectrally passive if

\[
x \in M(-\infty, 0] \implies \omega(x^\ast x) \leq \omega(xx^\ast) \quad .
\]
Unlike spectral passivity this condition implies that \( \omega \) is a trace on the subalgebra of \( \alpha \)-fixed elements. For the noninteracting Fermi gas it leads to a stronger form of the minimum energy principle, as mentioned at the end of Section 1. \( \Box \)

Finally we introduce the Fermi-Dirac distribution in the present setting. Let \( \beta \) (resp. \( \mu \)) be a nonzero positive (resp. a real) number. The assumption that the equilibrium state \( \omega_{\beta, \mu} \) of the noninteracting Fermi gas at inverse temperature \( \beta \) and chemical potential \( \mu \) is completely determined by the mean occupation numbers

\[
\langle n(f) \rangle_{\beta, \mu} = \omega_{\beta, \mu} (a^\dagger f a(f)), \quad \|f\| = 1
\]

(as one tacitly assumes in the classical textbooks) amounts to requiring that \( \omega_{\beta, \mu} \) be (gauge-invariant) quasi-free. Comparing with (1.1), we define the Fermi-Dirac state \( \omega_{\beta, \mu} \) to be the gauge-invariant quasi-free state corresponding to the one-particle density

\[
Q_{\beta, \mu} = (1 + e^{\beta(H - \mu)})^{-1}
\]

on \( \mathcal{K} \). In fact \( \omega_{\beta, \mu} \) is the unique \( \beta \)-KMS state of \( \mathcal{A}(\mathcal{K}) \) with respect to the automorphism group \( t \mapsto \alpha_t \gamma_{-\mu t} \) \( [7, \text{Prop. 5.2.23}] \) (for general information on the KMS condition we refer to \( [7, \text{Section 5.3}] \)).

We shall also encounter the limiting cases \( \beta = +\infty, 0 \) and \( \mu = +\infty. \) First we define

\[
Q_{+\infty, \mu} = P(-\infty, \mu) + \frac{1}{2} P(\{\mu\})
\]

for real \( \mu \). Each of these one-particle densities defines a ground state \( [7, \text{Def. 5.3.18}] \) \( \omega_{+\infty, \mu} \) for the evolution \( t \mapsto \alpha_t \gamma_{-\mu t} \), though not necessarily the only one. We shall use the term ground state with Fermi energy \( \mu \) for every gauge-invariant quasi-free state whose one-particle density \( Q \) coincides with \( P(-\infty, \mu) \) on \((P(\{\mu\})\mathcal{K})^\perp\) (the Fermi-energy \( \mu \) may not be uniquely defined, cf. Section 4).
Corresponding to the densities 0 and 1 one has (respectively) the Fock state \( \omega_0 \) and the anti-Fock state \( \omega_1 \). Neither the ground states nor the Fock and anti-Fock states are faithful (except, for the former, in the trivial case \( H = \mu 1 \)).

Notice that \( \omega_0 \) and \( \omega_1 \), when restricted to \( \mathfrak{A}(x) \( ^{(0)} \)), are multiplicative linear functionals \([5, \text{Section 5}]\), hence they are \( \beta \)-KMS with respect to \( \alpha \) for arbitrary \( \beta \).

Finally we consider the gauge-invariant quasi-free states \( \omega_c \ (0 < c < 1) \) with one-particle density \( c \). We call these the Powers states because they were used by R. Powers to construct an uncountable family of nonisomorphic type III factors \([17] \). As a state on \( \mathfrak{A}(x) \), \( \omega_c \) is \( \beta \)-KMS with respect to \( \gamma \) for \( \beta = \log(c^{-1} - 1) \) (not necessarily positive). Restricted to \( \mathfrak{A}(x) \( ^{(0)} \)), it is the weak \( \ast \)-limit of the \( \omega_{\beta, \mu} \) when \( \beta \to 0, \ \mu \to \pm \infty, \ \beta \mu = -\log(c^{-1} - 1) \), and hence it is an \( \alpha \)-invariant trace \([7, \text{Prop. 5.3.23}] \). The field algebra \( \mathfrak{A}(x) \) has only one trace, namely \( \omega_{1/2} \).

Remark 2.4. Replacing \( H \) by \( H - \mu 1 \) has the effect of replacing \( \alpha_t \) by \( \alpha_t \gamma \mu t \).

However, such a "shift of the energy scale" does not affect the restriction of \( \alpha \) to \( \mathfrak{A}(x) \( ^{(0)} \)). In the sequel we shall repeatedly make use of the freedom to choose a convenient zero point for the energy. \( \square \)
3. The continuous case

In this section (except in Remark 3.7 and Lemma 3.16) we make the standing assumption that the one-particle Hamiltonian $H$ has no point spectrum, i.e., $P(\lambda) = 0$ for all $\lambda$ in $\mathbb{R}$. The spectrum of $H$ will be denoted by $\sigma(H)$. We want to show that every quasi-free state of $\mathcal{U}(\mathcal{H})^{(0)}$ which is passive with respect to $\alpha$ is equal to one of the states $\omega^\beta_\mu, \omega^\mu_\omega, \omega^\mu_0, \omega^\mu_1$ or $\omega_c$.

3.1 Functional dependence between state and Hamiltonian.

With every state $\omega$ of $\mathcal{U}(\mathcal{H})^{(0)}$ (or $\mathcal{U}(\mathcal{K})$) we can associate a one-particle density operator $Q$ defined via equation (2.2). Our first aim is to show that, if $\omega$ is passive with respect to $\alpha$ (not necessarily quasi-free), then $Q = \phi(H)$ where $\phi$ is some decreasing function defined on $\mathbb{R}$.

**Definition 3.1.** For an arbitrary state $\omega$ of $\mathcal{U}(\mathcal{K})^{(0)}$ (or $\mathcal{U}(\mathcal{K})$) we define real functions $\phi$ and $\psi$ on $\mathbb{R}$ by

$$
\phi(\lambda) = \inf \left\{ \omega \left( a^*(f) a(f) \right) \left\| f \right\|^{-2} \mid f = P(-\infty, \lambda)f \neq 0 \right\}
$$

$$
\psi(\lambda) = \sup \left\{ \omega \left( a^*(f) a(f) \right) \left\| f \right\|^{-2} \mid f = P(\lambda, +\infty)f \neq 0 \right\}
$$

where by convention $\inf \phi = 1$ and $\sup \phi = 0$.

**Lemma 3.2.** The functions $\phi$ and $\psi$ have the following properties:

(i) $0 \leq \phi(\lambda) \leq 1$, $0 \leq \psi(\lambda) \leq 1$ for all $\lambda$ in $\mathbb{R}$.

(ii) $\phi$ and $\psi$ are decreasing.

(iii) $\phi$ (resp. $\psi$) is everywhere continuous from the left (resp. the right).

(iv) If $\lambda \in \sigma(H)$ then $\phi(\lambda+) \leq \psi(\lambda-)$. (Here and in the sequel we use the notation $\phi(\lambda+) = \lim_{\mu \nearrow \lambda} \phi(\mu)$, $\psi(\lambda) = \lim_{\mu \searrow \lambda} \psi(\mu)$.)

**Proof:** (i) follows from $\|a(f)\| = \|f\|$. (ii) is obvious. To prove (iii), for every $f$ in $\mathcal{K}$ define a function $\phi_f$ on $\mathbb{R}$ by

$$
\phi_f(\lambda) = \omega \left( a^* \left( P(-\infty, \lambda)f \right) a \left( P(-\infty, \lambda)f \right) \right) \left\| P(-\infty, \lambda)f \right\|^{-2}
$$
if \( P(-\infty, \lambda)f \neq 0 \), and \( \phi_f(\lambda) = 1 \) otherwise. Using well known properties of spectral projections it is easy to see that \( \phi_f \) is everywhere continuous from the left. Since

\[
\phi(\lambda) = \inf\{\phi_f(\lambda) \mid f \in \mathcal{H}\}
\]

for all \( \lambda \) one concludes that \( \phi \) is upper semicontinuous from the left. As \( \phi \) decreases it is also lower semicontinuous from the left.

Finally, if \( \lambda \in \sigma(H) \) and \( \epsilon > 0 \) one can find \( f = P(\lambda - \epsilon, \lambda + \epsilon)f \neq 0 \). Hence

\[
\phi(\lambda + \epsilon) \leq \omega(a^\ast(f)a(f)) \|f\|^{-2} \leq \psi(\lambda - \epsilon) .
\]

This proves (iv). \( \Box \)

**Lemma 3.3.** If \( \omega \) is spectrally passive with respect to \( \alpha \) (as a state of \( \mathfrak{A}(\mathfrak{X})^{(0)} \)) then \( \phi(\lambda) \geq \psi(\lambda) \) for all \( \lambda \) in \( \mathbb{R} \).

**Proof:** We can suppose \( \phi(\lambda) < 1 \) and \( \psi(\lambda) > 0 \), for otherwise the inequality in the statement holds trivially (Lemma 3.2(i)). If \( f = P(\lambda, +\infty)f \) and \( g = P(-\infty, \lambda)g \), then the passivity assumption implies (equations (1.3) - (1.5)):

\[
\omega(a^\ast(f)a(g)a^\ast(g)a(f)) \leq \omega(a^\ast(g)a(f)a^\ast(f)a(g)) .
\]

Using the CAR (1.2) one obtains

\[
\|g\|^2 \omega(a^\ast(f)a(f)) - \omega(a^\ast(f)a^\ast(g)a(g)a(f)) \\
\leq \|f\|^2 \omega(a^\ast(g)a(g)) - \omega(a^\ast(g)a^\ast(f)a(f)a(g)) .
\]

The second terms in both sides cancel, again by (1.2), and it follows that

\[
\omega(a^\ast(f)a(f)) \|f\|^{-2} \leq \omega(a^\ast(g)a(g)) \|g\|^{-2} ,
\]

provided \( f \neq 0 \) and \( g \neq 0 \). This implies the lemma. \( \Box \)

**Remark 3.4.** If \( \omega \) is spectrally passive as a state of the field algebra \( \mathfrak{A}(\mathfrak{X}) \), then \( \phi(0) > 1/2 > \psi(0) \). This follows from the inequality
\[ \omega(a^\varphi(f)a(h)) \geq \omega(a(f)a^\varphi(h)) = \|f\|^2 - \omega(a^\varphi(h)a(f)) \]

which is valid for all \( f \) in \( P(-\infty,0)\). \( \Box \)

Next we investigate the consequences of our assumption that \( H \) has a continuous spectrum.

Proposition 3.5. The one-particle density \( Q \) is a decreasing function of \( H \). More precisely \( Q = \phi(H) = \psi(H) \).

Proof: Fix \( \epsilon > 0 \). Since \( \phi \) is decreasing and has bounded range, it is possible to find \( \lambda_1 < \lambda_2 < \ldots < \lambda_n \) in \( \mathbb{R} \) such that

\[ \phi(\lambda_k^+) - \phi(\lambda_{k+1}^-) < \epsilon \quad (3.1) \]

for all \( k = 0,1,\ldots, n \), where \( \lambda_0 = -\infty \) and \( \lambda_{n+1} = +\infty \). (Moreover \( \phi(\lambda_{k+1}^-) = \phi(\lambda_{k+1}^+) \) by continuity, but this is not important for the proof.) Since \( H \) has no point spectrum, every \( f \) in \( \mathfrak{H} \) can be written as \( f = \sum_{k=0}^{n} f_k \), where \( f_k = P(\lambda_k, \lambda_{k+1})f \). By definition of \( \phi(H) \) one has

\[ \phi(\lambda_{k+1}^-) \|f_k\|^2 \leq (f_k, \phi(H)f_k) \leq \phi(\lambda_k^+) \|f_k\|^2 \quad (3.2) \]

\( (0 \leq k \leq n) \).

On the other hand, if \( 1 \leq k \leq n-1 \), \( 0 < \delta < (\lambda_{k+1} - \lambda_k)/2 \) then

\[ \phi(\lambda_{k+1}^- - \delta) \|P(\lambda_{k+1} + \delta, \lambda_{k+1} + \delta)f\|^2 \leq \left( P(\lambda_{k+1} + \delta, \lambda_{k+1} + \delta)f, QP(\lambda_{k+1} + \delta, \lambda_{k+1} + \delta)f \right) \]

\[ \leq \psi(\lambda_k + \delta) \|P(\lambda_{k+1} + \delta, \lambda_{k+1} + \delta)f\|^2 \]

by definition of \( \phi, \psi \) and \( Q \). By Lemma 3.4 one has \( \psi(\lambda_k + \delta) \leq \phi(\lambda_k + \delta) \), hence taking the limit as \( \delta \to 0 \) one obtains the inequalities

\[ \phi(\lambda_{k+1}^-) \|f_k\|^2 \leq (f_k, Qf_k) \leq \phi(\lambda_k^+) \|f_k\|^2 \quad (3.3) \]
which are valid not only if $1 \leq k \leq n - 1$, but also for $k = 0, n$, as shown by an obvious variation of the above argument. Subtracting (3.3) from (3.2), adding the $(n + 1)$ resulting inequalities and taking (3.1) into account, one concludes that

$$-\epsilon \sum_{k=0}^{n} \|f_k\|^2 \leq \sum_{k=0}^{n} (f_k, (\phi(H) - Q)f_k) \leq \epsilon \sum_{k=0}^{n} \|f_k\|^2.$$

Now $\sum_{k=0}^{n} (f, Qf_k) = (f, Qf)$ because, as $\omega$ is $\alpha$-invariant, $Q$ commutes with the spectral projections of $H$. Hence

$$-\epsilon \|f\|^2 \leq (f, (\phi(H) - Q)f) \leq \epsilon \|f\|^2,$$

or $\phi(H) = Q$, since both $\epsilon$ and $f$ were arbitrary.

Finally, by comparison of the Lemmas 3.2(iv) and 3.3, it is clear that $\phi(\lambda) = \psi(\lambda)$ for all $\lambda$ in $\sigma(H)$ where both $\phi$ and $\psi$ are continuous. But the set of points where either $\phi$ or $\psi$ is discontinuous is at most countable (since $\phi$ and $\psi$ are decreasing, Lemma 3.2(ii)), hence has $P$-measure zero. Consequently $\phi(H) = \psi(H)$. \[\square\]

It will be useful in the sequel to have a name, $D$, say, for the set of points of $\sigma(H)$ where at least one of the functions $\phi$ or $\psi$ is discontinuous. Furthermore, let $L$ (resp. $R$) be the set of points of $\sigma(H)$ that are left (resp. right) isolated in $\sigma(H)$. Since these three sets are countable, the spectral measure $P$ is carried by $S = \sigma(H) \setminus (D \cup L \cup R)$. Notice also that $\phi = \psi$ on $S$. Using the fact that $\sigma(H)$ is a perfect set, one easily shows the following: If $\lambda \in \sigma(H) \setminus L$ (resp. $\lambda \in \sigma(H) \setminus R$) then for every $\epsilon > 0$ the set $\sigma(H) \cap (\lambda - \epsilon, \lambda)$ (resp. $\sigma(H) \cap (\lambda, \lambda + \epsilon)$) is uncountable.

As a corollary to Proposition 3.5 we can give a useful "local" definition of $\phi$ and $\psi$.

**Lemma 3.6.** (i) Suppose $\lambda \in \sigma(H) \setminus L$ and $\epsilon > 0$. Then

$$\phi(\lambda) = \inf \left\{ \omega(\lambda, \lambda) \|f\|^2 \mid f = P(\lambda - \epsilon, \lambda) f \neq 0 \right\}.$$
(ii) Suppose $\lambda \in \sigma(H) \setminus R$ and $\epsilon > 0$. Then

$$\psi(\lambda) = \sup \left\{ \omega \left( a^* f a(f) \right) \| f \|^{-2} \bigg| f \in P(\lambda, \lambda + \epsilon) f \neq 0 \right\} .$$

Proof: We only prove (i). Let us write $\tilde{\phi}(\lambda)$ for the right hand side of the equation in the statement. Obviously $\phi(\lambda) \leq \tilde{\phi}(\lambda)$. On the other hand, for every $\delta > 0$ there exists $\epsilon' > 0$ such that $\phi(\lambda - \epsilon') \leq \phi(\lambda) + \delta$ (Lemma 3.2). Since $\lambda$ is not left isolated in $\sigma(H)$, one can find a nonzero $f$ in $P(\lambda - \min(\epsilon, \epsilon'), \lambda)$. Finally

$$\tilde{\phi}(\lambda) \leq \omega \left( a^* f a(f) \right) \| f \|^{-2} = (f, Q_0 f) \| f \|^{-2} = (f, \phi(H) f) \| f \|^{-2} \leq \phi(\lambda - \epsilon') \leq \phi(\lambda) + \delta ,$$

where we used Proposition 3.5. This concludes the proof, as $\delta$ was arbitrary. $\square$

Remark 3.7. If one assumes that $\omega$ is strongly spectrally passive (Remark 2.3), then Proposition 3.5 and an appropriate modification of Lemma 3.6 are valid for arbitrary $H$. One only needs to redefine $\phi$ according to

$$\phi(\lambda) = \inf \left\{ \omega \left( a^* f a(f) \right) \| f \|^{-2} \bigg| f = P(-\infty, \lambda] f \neq 0 \right\} . \square$$

3.2 The ground state case

In the remainder of Section 3 $\omega$ is a spectrally passive quasi-free state of $\mathcal{X}(X)^{(0)}$.

Because of the lack of faithfulness we have to deal with the ground states (and with the Fock and anti-Fock states) separately, which is what we do in Proposition 3.9. It is preceded by the main lemma of the paper, where it is shown how the assumptions that $\omega$ is passive and quasi-free yield a computational device which will allow us to determine the functions $\phi$ and $\psi$.

Lemma 3.8. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ be two sets of $n$ real numbers such that $\lambda_j \neq \lambda_k$ and $\mu_j \neq \mu_k$ when $j \neq k$, and $\sum_{k=1}^{n} \mu_k \leq \sum_{j=1}^{n} \lambda_j$. Then there exists a positive number $\epsilon$ such that, whenever $f_j = P(\lambda_j, \lambda_j + \epsilon) f_j$ and $g_k = P(\mu_k - \epsilon, \mu_k) g_k$ ($1 \leq j, k \leq n$), one has the inequality
\[
\prod_{j=1}^{n} \omega(a^{\ast}(f_j) a(f_j)) \prod_{k=1}^{n} \omega(a(g_k)a^{\ast}(g_k)) \\
\leq \prod_{j=1}^{n} \omega(a(f_j)a^{\ast}(f_j)) \prod_{k=1}^{n} \omega(a^{\ast}(g_k)a(g_k)) .
\]

**Proof:** Let \( \lambda_j, \mu_k \) \((1 \leq j, k \leq n)\) satisfy the conditions in the statement. Choose \( \epsilon > 0 \) subject to the following requirements:

(i) \( 2\epsilon \leq \mu_k - \lambda_j \) if \( \mu_k > \lambda_j \) \((1 \leq j, k \leq n)\).

This implies that \( \omega(a^{\ast}(f_j) a(g_k)) = (g_k, Q f_j) = 0 \) whenever \( f_j = P(\lambda_j, \lambda_j + \epsilon)f_j \) and \( g_k = P(\mu_k - \epsilon, \mu_k)g_k \), by the \( \alpha \)-invariance of \( \omega \). Also \( \omega(a^{\ast}(g_k)a(f_j)) = 0 \).

(ii) \( \epsilon \leq |\lambda_j - \lambda_k| \)

(iii) \( \epsilon \leq |\mu_j - \mu_k| \)

for all \( j, k, j \neq k \). It follows that

\[
\omega(a^{\ast}(f_j) a(f_k)) = \omega(a(f_k) a^{\ast}(f_j)) = 0
\]

and

\[
\omega(a(g_j) a^{\ast}(g_k)) = \omega(a^{\ast}(g_k)a(g_j)) = 0
\]

when \( f_j = P(\lambda_j, \lambda_j + \epsilon)f_j, f_k = P(\lambda_k, \lambda_k + \epsilon)f_k, g_j = P(\mu_j - \epsilon, \mu_j)g_j, g_k = P(\mu_k - \epsilon, \mu_k)g_k \) and \( j \neq k \).

With \( \epsilon \) thus fixed, and the \( f \)'s and the \( g \)'s as above, the passivity condition implies that

\[
\omega(a^{\ast}(f_1) \ldots a^{\ast}(f_n) a(g_1) \ldots a(g_n) a^{\ast}(g_1) \ldots a^{\ast}(g_n) a(f_1) \ldots a(f_n))
\]

\[
\leq \omega(a^{\ast}(g_n) \ldots a^{\ast}(g_1) a(f_n) \ldots a(f_1) a^{\ast}(f_n) \ldots a^{\ast}(f_1) a(g_n) \ldots a(g_1))
\]

by (1.3) - (1.5). If we expand the two sides of this inequality in a sum of products of "two-point functions" according to (2.1), then because of (i), (ii) and (iii) all the terms but one vanish in each side, and the resulting inequality is exactly the one in the statement. \( \Box \)
Proposition 3.9. (i) If there exists \( \lambda_1 \) in \( \sigma(H) \setminus R \) with \( \psi(\lambda_1) = 1 \), then for all \( \mu \) in \( \sigma(H) \setminus L \), either \( \phi(\mu) = 1 \) or \( \phi(\mu) = 0 \).

(ii) If there exists \( \mu_1 \) in \( \sigma(H) \setminus L \) with \( \phi(\mu_1) = 0 \), then either \( \psi(\lambda) = 0 \) or \( \psi(\lambda) = 1 \) for all \( \lambda \) in \( \sigma(H) \setminus R \).

Proof: Both statements have similar proofs, so we only show (i). Hence we suppose \( \lambda_1 \in \sigma(H) \setminus R \) and \( \psi(\lambda_1) = 1 \). Define \( \mu_0 = \sup\{\mu : \phi(\mu) = 1\} \) (\( \mu_0 \geq \lambda_1 \) by Lemma 3.3). If \( \mu_0 = +\infty \) there is nothing to prove. If \( \mu_0 < +\infty \) then \( \phi(\mu_0) = 1 \) (by left continuity of \( \phi \)) and \( \mu_0 \in \sigma(H) \setminus R \) (otherwise \( P(-\infty, \mu_0) = P(-\infty, \mu_0 + \epsilon) \) for some \( \epsilon > 0 \) and \( \phi(\mu_0 + \epsilon) = \phi(\mu_0) = 1 \), contradicting the definition of \( \mu_0 \)). By Remark 2.4 we can suppose \( \mu_0 = 0 \).

Let \( \mu \) be a positive element of \( \sigma(H) \setminus L \). We have to show that \( \phi(\mu) = 0 \).

Fix a positive integer \( n \) such that \( \lambda_1 + (n-1)\mu/2 \geq 1 \). Choose \( \lambda_2, \lambda_3, \ldots, \lambda_n \) such that

\[
\lambda_2 \in (\mu/2, \mu) \cap \sigma(H) \setminus R
\]

\[
\lambda_j \in (\lambda_{j-1}, \mu) \cap \sigma(H) \setminus R
\]

(\( 3 \leq j \leq n \)). (This can be done because, for instance, \( (\mu/2, \mu) \cap \sigma(H) \) is uncountable, whereas \( R \) is at most countable.) Furthermore choose \( \mu_1, \mu_2, \ldots, \mu_n \) such that

\[
\mu_1 \in (0, 1/2) \cap \sigma(H) \setminus L
\]

\[
\mu_k \in (0, \min(\mu_{k-1}, 2^{-k})) \cap \sigma(H) \setminus L
\]

(\( 2 \leq k \leq n \)). Then \( \sum_{k=1}^{n} \mu_k < 1 \leq \lambda_1 + (n-1)\mu/2 \leq \sum_{j=1}^{n} \lambda_j \). Hence Lemma 3.6 guarantees the existence of some positive \( \epsilon \), which we can assume to be smaller than \( \mu - \lambda_n \), such that

\[
\prod_{j=1}^{n} \omega(a^*(f_j)a(f_j)) \prod_{k=1}^{n} \omega(a(g_k)a^*(g_k)) < \prod_{j=1}^{n} \omega(a(f_j)a^*(f_j)) \prod_{k=1}^{n} \omega(a^*(g_k)a(g_k))
\]
whenever $f_j = P(\lambda_j, \lambda_j + \epsilon) f_j$ and $g_k = P(\mu_k - \epsilon, \mu_k) g_k$ (1 ≤ j, k ≤ n). But by Lemma 3.6(ii) there exists a sequence $\{f^{(m)}\}$ with $\|f^{(m)}\| = 1$, $f^{(m)} = P(\lambda_1, \lambda_1 + \epsilon) f^{(m)}$ and $\omega a(f^{(m)} a(f^{(m)})) \to 1$ (or, equivalently, $\omega a(f^{(m)} a(f^{(m)})) \to 0$) as $m \to \infty$. Substituting $f^{(m)}$ for $f_1$ in the previous inequality and passing to the limit we obtain

$$\prod_{j=2}^n \omega a(g_j a(f_j)) \prod_{k=1}^n \omega a(g_k a(f_k)) = 0.$$ 

Since $\phi(\mu_k) < 1$ for $k = 1, 2, \ldots, n$ we can choose $g_k$ such that $\omega a(g_k a^*(g_k)) \neq 0$ (using Lemma 3.6(i)). Hence for a fixed choice of nonzero $f_2, f_3, \ldots, f_n$ (which is possible because $\lambda_j \in \sigma(H) \setminus \mathbb{R}$) one has $\omega a(f_j a(f_j)) = 0$ for some $j$. It follows that $\phi(\lambda_j + \epsilon) = 0$. But then $\phi(\mu) = 0$, since $\mu > \lambda_j + \epsilon$ (Lemma 3.2(ii)). \square

Proposition 3.9 says the following: If $\phi$ (or $\psi$) reaches the value 0 or 1 anywhere on $S$, then the state $\omega$ is either one of the ground states $\omega_{\infty, \mu'}$ or the Fock or anti-Fock state.

### 3.3 The main result

This result reads as follows.

**Theorem 3.10.** Let $(\mathcal{U}(\mathcal{H})^{(0)}, \alpha)$ be the $C^*$-dynamical system corresponding to the noninteracting Fermi gas with one-particle space $\mathcal{H}$ and one-particle Hamiltonian $H$. Suppose $H$ has no point spectrum. Then $\omega$ is a spectrally passive quasi-free state of $(\mathcal{U}(\mathcal{H})^{(0)}, \alpha)$ if and only if one of the following holds:

1. $\omega = \omega_{\beta, \mu}$ for some $\beta$ and $\mu$ ($0 < \beta < +\infty$, $-\infty < \mu < +\infty$)

2. $\omega = \omega_{\infty, \mu}$ for some $\mu$ ($-\infty < \mu < +\infty$)

3. $\omega = \omega_0$ or $\omega = \omega_1$

4. $\omega = \omega_c$ for some $c$ ($0 < c < 1$). \square

The "if" part of the statement is clear, because all the states in the above list are KMS states with respect to $\alpha$ (as states of $\mathcal{U}(\mathcal{H})^{(0)}$) or limits of such states ($\beta = 0, +\infty$; cf. Section 2).
To prove the "only if" part we show that, if we assume that neither (ii) nor (iii) holds, then (i) or (iv) has to be the case. From this assumption it follows by Proposition 3.9 that \( \phi(\lambda) \neq 0 \) and \( \phi(\lambda) \neq 1 \) for all \( \lambda \) in \( S \). So we can define a nowhere vanishing real valued function \( \chi \) on \( S \) by

\[
\chi(\lambda) = \frac{(1 - \phi(\lambda))}{\phi(\lambda)}.
\] (3.4)

We also put \( \tilde{\chi} = \chi(0)^{-1} \chi \). Choosing the origin of the energy scale in such a way that \( 0 \in \mathcal{S} \) (Remark 2.4) we will be able to extend \( \tilde{\chi} \) to an increasing (hence continuous) homomorphism of the additive group \( \mathbb{R} \) into the multiplicative group \( \mathbb{R}^+ \) of nonzero positive real numbers. Then \( \tilde{\chi}(\lambda) \) has to be equal to \( e^{\beta \lambda} \) for some \( \beta \), \( 0 \leq \beta < +\infty \). If \( \beta \neq 0 \) we have \( \phi(\lambda) = (1 + e^{\beta(\lambda - \mu)})^{-1} \) (\( \lambda \in \mathcal{S} \)) with \( \mu = -\beta^{-1} \log \chi(0) \).

If \( \beta = 0 \) then \( \phi \) has the constant value \( (1 + \chi(0))^{-1} \) on \( S \), so that \( \omega = \omega_c \) with \( c = (1 + \chi(0))^{-1} \).

To implement this program we need the following lemma.

**Lemma 3.11.** If \( \lambda_j, \mu_k \in \mathcal{S} \) (1 \( \leq \) \( j, k \leq n \)) and

\[
\sum_{k=1}^{n} \mu_k \leq \sum_{j=1}^{n} \lambda_j \quad \text{then} \quad \prod_{k=1}^{n} \chi(\mu_k) \leq \prod_{j=1}^{n} \chi(\lambda_j).
\]

**Proof:** Since \( \chi \) is continuous (where it is defined, i.e., on \( \mathcal{S} \)) and no element of \( \mathcal{S} \) is right (resp. left) isolated in \( \mathcal{S} \), we can assume that all the \( \lambda_j \) (resp. the \( \mu_k \)) are pairwise distinct. By Lemma 3.8 there exists a positive number \( \epsilon \) such that, for all \( f_j \) in \( P(\lambda_j, \lambda_j + \epsilon) \mathcal{H} \) and \( g_k \) in \( P(\mu_k - \epsilon, \mu_k) \mathcal{H} \), one has

\[
\prod_{j=1}^{n} \omega(a^{\ast}(f_j)a(f_j)) \prod_{k=1}^{n} \omega(a^{\ast}(g_k)a(g_k)) \leq \prod_{j=1}^{n} \omega(a^{\ast}(f_j)a(f_j)) \prod_{k=1}^{n} \omega(a^{\ast}(g_k)a(g_k)).
\]

Replacing \( \epsilon \) by a smaller number if necessary we can assume that \( \psi(\lambda_j + \epsilon) > 0 \) (1 \( \leq \ j \leq n \)). This is possible because \( \psi(\lambda_j) > 0 \) (Proposition 3.9) and \( \psi \) is right continuous. It follows that \( \omega(a^{\ast}(f_j)a(f_j)) > 0 \) when \( f_j = 0 \) (otherwise \( \psi(\lambda_j + \epsilon) = \phi(\lambda_j + \epsilon) = 0 \) by Lemma 3.3). Hence for nonzero \( f_j \) and \( g_k \) as above one has
\[ \prod_{k=1}^{n} \frac{\|g_k\|^2 - \omega(a^{*}(g_k)a(g_k))}{\omega(a^{*}(g_k)a(g_k))} \leq \prod_{j=1}^{n} \frac{\|f_j\|^2 - \omega(a^{*}(f_j)a(f_j))}{\omega(a^{*}(f_j)a(f_j))} \]

\((g_k \neq 0 \text{ and } \omega(a^{*}(g_k)a(g_k)) = 0 \text{ would imply } \phi(\mu_k) = 0, \text{ which is not the case}).\)

Taking the infimum (resp. the supremum) over all \(f_j\) (resp. all \(g_k\)) subject to the conditions \(f_j = P(\lambda_j, \lambda_j + \epsilon)f_j \neq 0\) (resp. \(g_k = P(\mu_k - \epsilon, \mu_k)g_k \neq 0\)) yields

\[ \prod_{k=1}^{n} \left(1 - \psi(\mu_k)\right)/\psi(\mu_k) \leq \prod_{j=1}^{n} \left(1 - \phi(\lambda_j)\right)/\phi(\lambda_j) \]

(one uses Lemma 3.6). In view of the equation (3.4) defining \(\chi\) this concludes the proof, since \(\phi\) and \(\psi\) coincide on \(S\). □

We resume the proof of Theorem 3.10. Let \(G\) be the subgroup of \(\mathbb{R}\) generated by \(S\), i.e., the set of all real numbers of the form \(\sum_{j=1}^{m} \lambda_j - \sum_{k=1}^{n} \mu_k\) with \(\lambda_j, \mu_k \in S\). \(G\) is dense in \(\mathbb{R}\) because \(S\) is uncountable. Now we extend \(\tilde{\chi}\) to a function on \(G\) with values in \(\mathbb{R}_0^+\) by defining

\[ \tilde{\chi}\left(\sum_{j=1}^{m} \lambda_j - \sum_{k=1}^{n} \mu_k\right) = \chi(0)^{n-m} \prod_{j=1}^{m} \chi(\lambda_j) \prod_{k=1}^{n} \chi(\mu_k)^{-1}. \]

It follows from Lemma 3.11 (and from the assumption \(0 \in S\)) that \(\tilde{\chi}\) is well defined and increasing. On the other hand, by its very definition, \(\tilde{\chi}\) is a homomorphism of \(G\) into \(\mathbb{R}_0^+\). As \(G\) is dense, \(\tilde{\chi}\) finally extends to a continuous increasing homomorphism of \(\mathbb{R}\) into \(\mathbb{R}_0^+\). By the discussion preceding Lemma 3.11, this concludes the proof of Theorem 3.10.

Remark 3.12. The technique of the above proof is very close in spirit to the arguments used in [6, Section 4]. Now Batty has given a sweeping simplification of these latter arguments [4, Section 2]. It is not clear, however, how his ideas might be applicable in the present context. □
Remark 3.13. In the same paper, improving on a previous result of Pusz and Woronowicz [18, Theorem 1.3] and the author [8, Theorem 4.11], Batty proves the following: if a state $\omega$ of a $C^*$-dynamical system $(\mathcal{A}, \alpha)$ is spectrally passive, and at the same time extremal invariant ("ergodic") for an action $\gamma$ of some group $G$ on $\mathcal{A}$ which commutes with $\alpha$, then $\omega$ is $\beta$-KMS with respect to $\alpha$ ($0 \leq \beta \leq +\infty$). This result can be used to give a different (although conceptually no simpler) proof of Theorem 3.10 in special cases. Suppose, e.g., that $\mathcal{A} = L^2(\mathbb{R}^N)$ and that $H$ is translation invariant. If the one-particle density $Q$ corresponding to a quasi-free state $\omega$ of $\mathcal{A}(L^2(\mathbb{R}^N))^{(0)}$ is a function of $H$ (either as a consequence of Proposition 3.5 or for some other reason), then $\omega$ is also translation invariant.

It is well known and easy to show that $\omega$ has to be strongly clustering (or strongly mixing [7, Example 4.3.24]) for the translation group and hence extremal translation invariant [7, Section 4.3.2]. If that state $\omega$ is moreover spectrally passive with respect to $\alpha$ it has to be KMS, by Batty's theorem. One then only has to show that Theorem 3.10 gives a complete list of the quasi-free KMS states ($\beta = 0$, $+\infty$ included). The same reasoning applies if $H$ has only absolutely continuous spectrum: in that case any $\alpha$-invariant quasi-free state of $\mathcal{A}(\mathcal{H})^{(0)}$ is extremal $\alpha$-invariant (in fact strongly clustering, by the Riemann–Lebesgue lemma: $\lim_{|t| \to \infty} (f, U_t f) = 0$ for all $f$ in $\mathcal{H}$).

This leads to an interesting problem: give necessary and sufficient conditions on $H$ that guarantee that every $\alpha$-invariant quasi-free state of $\mathcal{A}(\mathcal{H})^{(0)}$ (or gauge-invariant such state of $\mathcal{A}(\mathcal{H})$) is extremal $\alpha$-invariant. (H cannot have eigenvalues, as one can prove using the method of [13, p. 431].)

Remark 3.14. If in the hypothesis of Theorem 3.10 $\omega$ is assumed to be spectrally passive with respect to $\alpha$ as a state of $\mathcal{A}(\mathcal{H})$, then the conclusion is more precise. One of the following possibilities has to hold:

(i) $\omega = \omega_{\beta, 0}$ for some $\beta$ ($0 < \beta < +\infty$).

(ii) $\omega = \omega_{\infty, 0}$, i.e., $\omega$ is a ground state, which is unique because $H$ is injective [7, Example 5.3.20].
(iii) \( \omega = \omega_{1/2} \) i.e., \( \omega \) is the unique tracial state of \( \mathcal{A}(\mathcal{H}) \).

[If \( 0 \in S \) this follows from Theorem 3.10 and Remark 3.4 \( \phi(0) = \psi(0) = 1/2 \). In general it seems that one has to give an independent proof, which however proceeds along the same lines as that of Theorem 3.10.]

Fermi-Dirac states for a chemical potential \( \mu \neq 0 \) can be obtained by replacing \( H \) by \( H - \mu \) (Remark 3.4). Replacing the observable algebra \( \mathcal{A}(\mathcal{H})^{(0)} \) with the field algebra \( \mathcal{A}(\mathcal{H}) \) is somewhat analogous to passing from the Gibbs canonical ensemble to the grand canonical ensemble in statistical mechanics. In the first case the number of particles is fixed, in the second case it is not, but the chemical potential \( \mu \) is given as an \textit{a priori} thermodynamic parameter. For the so-called "algebraic theory of the chemical potential," which gives a general explanation of its occurrence as a thermodynamic variable, we refer to [1, 2, 15].

\[ \square \]

Remark 3.15. There are of course other "physical" principles, besides passivity, that can be used to derive the Fermi-Dirac distribution law. One of these is the concept of \textit{stability}, introduced by Haag, Kastler and Trych-Pohlmeyer [11, Appendix]. However stability, like passivity, is a \textit{linear} condition on the state, hence it has to be supplemented with a nonlinear condition (clustering in [11]; in the present paper the linearity is broken by the requirement that the state be quasi-free). \[ \square \]

This ends the discussion of Theorem 3.10. In the next lemma we prepare the extension of our results to more general \( H \).

\textbf{Lemma 3.16.} Let \( H \) have an uncountable spectrum. For every \( \lambda \in \mathbb{R}, \; \epsilon > 2 \), there exists a nonzero \( x \) in \( \text{R}(\lambda - \epsilon, \lambda + \epsilon) \cap \mathcal{A}(\mathcal{H})^{(d)} \). (In fact, \( x \) can be chosen to be a monomial in the creation and annihilation operators.)

\textbf{Proof:} If, for a well chosen \( \mu \), we replace \( H \) with \( H - \mu \) and \( \lambda \) with \( \lambda - \mu \), we can assume that 0 is an accumulation point of the spectrum. As in the proof of Theorem 3.10 we can find pairwise distinct numbers \( \lambda_1, \lambda_2, \ldots, \lambda_{m+n} \) in \( \sigma(H) \) such that

\[
\sum_{j=1}^{m} \lambda_j - \sum_{j=m+1}^{m+n} \lambda_j \in (\lambda - \epsilon/2, \lambda + \epsilon/2). \]

Using the assumption that 0 is an
accumulation point of $\sigma(H)$ we can moreover arrange it so that $m-n = d$. Now we choose $f_j$ in $H$ such that $\|f_j\| = 1$ and $f_j = P(\lambda_j - \delta, \lambda_j + \delta)f_j$, where $\delta = \min\left\{\epsilon/2(m+n), \frac{|\lambda_j - \lambda_k|}{2} \middle| 1 \leq j < k \leq m+n \right\}$. The element $x = a^*(f_1)\ldots a^*(f_m)a(f_{m+1})\ldots a(f_{m+n})$ is nonzero (because all the $f_j$ are pairwise orthogonal) and clearly belongs to $\mathbb{R}(\lambda - \epsilon, \lambda + \epsilon) \cap \mathfrak{z}(H)^{(d)}$. □
4. The general case, and counterexamples

In this section we investigate the question whether Theorem 3.10 still holds if we remove the assumption that \( H \) has no point spectrum. Following [19] we introduce the notation

\[
\sigma_{\text{cont}}(H) \quad \text{for the continuous spectrum of } H
\]

\[
\sigma_{\text{pp}}(H) \quad \text{for the set of eigenvalues of } H.
\]

Hence \( \sigma(H) = \sigma_{\text{cont}}(H) \cup \sigma_{\text{pp}}(H) \). We also write \( \mathfrak{H} = \mathfrak{H}_{\text{cont}} \oplus \mathfrak{H}_{\text{pp}} \), where \( \mathfrak{H}_{\text{pp}} \) is the closed subspace of \( \mathfrak{H} \) generated by all the eigenvectors of \( H \). Notice that \( \sigma_{\text{cont}}(H) \) is exactly the spectrum of the restriction of \( H \) to \( \mathfrak{H}_{\text{cont}} \).

We now discuss a simple construction which is the basis for all the subsequent arguments. Let \( \tilde{\mathfrak{H}} \) be the orthogonal complement of some unit vector \( e \) in \( \mathfrak{H} \). We write \( \tilde{\mathfrak{A}}(f), \tilde{\mathfrak{A}}(f) \) (\( f \in \tilde{\mathfrak{H}} \)) for the generators of the algebra \( \mathfrak{M}(\tilde{\mathfrak{H}}) \). It is well known that \( \mathfrak{M}(\tilde{\mathfrak{H}}) \) can be identified with the algebra of 2x2 matrices with entries in \( \mathfrak{M}(\mathfrak{H}) \). Specifically one has

\[
a(f + ce) = \begin{pmatrix}
\tilde{\mathfrak{A}}(f) & \mathcal{C} \\
0 & -\tilde{\mathfrak{A}}(f)
\end{pmatrix} \quad (f \in \tilde{\mathfrak{H}}, \ c \in \mathcal{C}). \quad (4.1)
\]

Moreover

\[
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{pmatrix} \in \mathfrak{M}(\mathfrak{H})^{(d)}
\]

if and only if \( x_{ij} \in \mathfrak{M}(\tilde{\mathfrak{H}})^{(d-i+j)} \) (\( i, j = 1, 2 \)). Abusing notation somewhat, we shall also identify \( \mathfrak{M}(\tilde{\mathfrak{H}}) \) with the \( C^* \)-subalgebra of \( \mathfrak{M}(\mathfrak{H}) \) generated by the \( a(f), f \in \tilde{\mathfrak{H}} \).

The next lemma follows immediately from the definitions.

**Lemma 4.1.** (i) With the above notation, let \( \tilde{\omega} \) be a gauge-invariant quasi-free state of \( \mathfrak{M}(\tilde{\mathfrak{H}}) \) with one-particle density \( \tilde{Q} \), and \( q \) a real number between 0 and 1. Then the linear functional \( \omega \) on \( \mathfrak{M}(\mathfrak{H}) \) defined by
\[
\omega \begin{pmatrix}
    x_{11} & x_{12} \\
    x_{21} & x_{22}
\end{pmatrix} = (1-q)\tilde{\omega}(x_{11}) + q\tilde{\omega}(x_{22})
\] (4.2)

\((x_{ij} \in \mathcal{U}(\tilde{\mathcal{H}}))\) is a gauge-invariant quasi-free state of \(\mathcal{U}(\mathcal{H})\) with one-particle operator \(Q = \tilde{Q} \oplus q\) (with respect to the decomposition \(\mathcal{H} = \tilde{\mathcal{H}} \oplus \mathcal{H}_e\)).

(ii) Conversely, suppose that \(\omega\) is a gauge-invariant quasi-free state of \(\mathcal{U}(\mathcal{H})\), and suppose that \(e\) is an eigenvector of its one-particle density \(Q\) with eigenvalue \(q\). Then \(\omega\) has the form (4.2), where \(\tilde{\omega}\) is the restriction of \(\omega\) to \(\mathcal{U}(\tilde{\mathcal{H}})\). □

Now we assume that \(He = \lambda e\) for some real \(\lambda\). Then \(\alpha\) leaves \(\mathcal{U}(\tilde{\mathcal{H}})\) invariant. We write \(\tilde{\alpha}\) to denote the restriction of \(\alpha\) to \(\mathcal{U}(\tilde{\mathcal{H}})\). It follows from (4.1) that

\[
\tilde{\alpha}_t \begin{pmatrix}
    x_{11} & x_{12} \\
    x_{21} & x_{22}
\end{pmatrix} = \begin{pmatrix}
    \tilde{\alpha}_t(x_{11}) & e^{-i\lambda t} \tilde{\alpha}_t(x_{12}) \\
    e^{i\lambda t} \tilde{\alpha}_t(x_{21}) & \tilde{\alpha}_t(x_{22})
\end{pmatrix}
\] (4.3)

Let us observe that the spectral subspaces of \(\mathcal{U}(\tilde{\mathcal{H}})\) associated with \(\tilde{\alpha}\) are simply of the form \(R(\cdot) \cap \mathcal{U}(\tilde{\mathcal{H}})\). Thus it is clear from (4.3) that

\[
\begin{pmatrix}
    x_{11} & x_{12} \\
    x_{21} & x_{22}
\end{pmatrix} \in R(-\infty, 0) \iff x_{ij} \in R(-\infty, (j-i)\lambda) \cap \mathcal{U}(\mathcal{H})
\]

\((i, j = 1, 2)\). Hence for a state \(\omega\) of \(\mathcal{U}(\mathcal{H})\) of the form (4.2) to be spectrally passive with respect to \(\alpha\) it is necessary and sufficient that

\[
(1-q)\omega(x_{11}^{*} x_{11} + x_{21}^{*} x_{21}) + q\tilde{\omega}(x_{12}^{*} x_{12} + x_{22}^{*} x_{22})
\leq (1-q)\tilde{\omega}(x_{11}^{*} x_{11} + x_{21}^{*} x_{21}) + q\tilde{\omega}(x_{12}^{*} x_{12} + x_{22}^{*} x_{22})
\]

25
for all $x_{ij} \in \mathbb{R}(\infty, (j-i)\lambda) \cap \mathcal{M}(\mathfrak{H})$ $(i, j = 1, 2)$. This is equivalent with the three conditions

\[
(1-q)\tilde{\omega}(xx^*) \leq q \omega(xx^*) , \quad x \in \mathbb{R}(\infty, -\lambda) \cap \mathcal{M}(\mathfrak{H}) \tag{4.4}
\]

\[
\tilde{\omega}(yy^*) \leq (1-q)\tilde{\omega}(yy^*) , \quad y \in \mathbb{R}(\infty, \lambda) \cap \mathcal{M}(\mathfrak{H}) \tag{4.5}
\]

\[
\tilde{\omega}(zz^*) \leq \tilde{\omega}(zz^*) , \quad z \in \mathbb{R}(\infty, 0) \cap \mathcal{M}(\mathfrak{H}).
\]

Finally we observe that the third condition is verified if $\tilde{\omega}$ is known to be spectrally passive with respect to $\mathfrak{H}$. Summing up:

**Lemma 4.2.** Let $\omega$ be a quasi-free state of $\mathcal{M}(\mathfrak{H})^{(0)}$ of the form (4.2), and suppose that its restriction $\tilde{\omega}$ to $\mathcal{M}(\mathfrak{H})^{(0)}$ is spectrally passive with respect to $\mathfrak{H}$. Then $\omega$ is spectrally passive with respect to $\mathfrak{H}$ if and only if

(i) $\quad (1-q)\tilde{\omega}(xx^*) \leq q \omega(xx^*) , \quad x \in \mathbb{R}(\infty, -\lambda) \cap \mathcal{M}(\mathfrak{H})^{(-1)}$

(ii) $\quad q \tilde{\omega}(yy^*) \leq (1-q)\tilde{\omega}(yy^*) , \quad y \in \mathbb{R}(\infty, \lambda) \cap \mathcal{M}(\mathfrak{H})^{(+1)}$.

**Theorem 4.3.** Let $(\mathcal{M}(\mathfrak{H})^{(0)}, \alpha)$ be the $C^*$-dynamical system representing the noninteracting Fermi gas, and suppose that the one-particle hamiltonian $H$ has nonempty continuous spectrum and that its eigenvalues have finite multiplicity. Then every faithful, spectrally passive, quasi-free state $\omega$ of $(\mathcal{M}(\mathfrak{H})^{(0)}, \alpha)$ is either a Fermi-Dirac state $\omega_{\beta, \mu}$ $(0 < \beta <+\infty, -\infty < \mu <+\infty)$ or a Powers state $\omega_c$ $(0 < c < 1)$.

**Proof:** Let $\tilde{\omega}$ be the restriction of $\omega$ to $\mathcal{M}(\mathfrak{H}_{cont})$ (considered as a subalgebra of $\mathcal{M}(\mathfrak{H})$ in the obvious way). By Theorem 3.10, we know that $\tilde{\omega}$ (which is faithful) is either a Fermi-Dirac state $\tilde{\omega}_{\beta, \mu}$ or a Powers state $\tilde{\omega}_c$. Let $\lambda$ be an eigenvalue of $H$. The one-particle density $Q$ leaves the corresponding (finite dimensional) eigenspace invariant, hence it can be diagonalized in that space. Let $e$ be one of the resulting (normalized) eigenvectors of $Q$, i.e., $Qe = qe$, $He = \lambda e$. It is sufficient to show that $q = (1+e^{\beta(\lambda-\mu)})^{-1}$ (if $\tilde{\omega} = \tilde{\omega}_{\beta, \mu}$) or $q = c$ (if $\tilde{\omega} = \tilde{\omega}_c$).
Replacing $\mathcal{K}$ by $\mathcal{K}_{\text{cont}} \oplus \mathfrak{c}$ we find ourselves in the situation described in the beginning of this section (with $\tilde{\mathcal{K}} = \mathcal{K}_{\text{cont}}$).

We first suppose that $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{\beta, \mu}$. In particular $\tilde{\mathcal{O}}$ is $\beta$-KMS with respect to the restriction of the one-parameter group $t \mapsto \alpha_t \gamma - \mu t$ to $\mathcal{U}(\tilde{\mathcal{K}})$. By Lemma 3.16 and the fact that $\sigma_{\text{cont}}(\mathcal{H})$ is uncountable, for every positive $\epsilon$ there exists a non-zero element $x$ of $R(-\lambda - \epsilon, -\lambda) \cap \mathcal{U}(\tilde{\mathcal{K}})^{-1}$. For such $x$ one has

$$(1 - q)\tilde{\omega}(xx^*) \leq q\tilde{\omega}(xx^*) \leq q e^{\beta(\lambda + \mu + \epsilon)} \tilde{\omega}(xx^*) .$$

The first inequality is (i) of Lemma 4.2, the second one follows from $[8, \text{Theorem 1.1}]$ and the fact that, with respect to $t \mapsto \alpha_t \gamma - \mu t$, $x$ belongs to the spectral subspace corresponding to the interval $(-\lambda + \mu - \epsilon, -\lambda + \mu)$. Now $\tilde{\omega}(xx^*) \neq 0$ because $\omega$ is faithful and hence

$$q^{-1} - 1 \leq e^{\beta(\lambda - \mu - \epsilon)} .$$

By continuity this inequality is also valid for $\epsilon = 0$. Similarly choosing $y$ in $R(\lambda - \epsilon, \lambda) \cap \mathcal{U}(\tilde{\mathcal{K}})^{+1}$ and using Lemma 4.2(ii) and the KMS condition in the form of $[8, \text{Theorem 1.1}]$ we obtain $q^{-1} - 1 \geq e^{\beta(\lambda - \mu)}$. The conclusion is that $q = (1 + e^{\beta(\lambda - \mu)})^{-1}$, as desired.

The proof under the assumption that $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{\beta}$ is very similar. Now $\tilde{\mathcal{O}}$ is $\beta$-KMS with respect to $\gamma$, with $\beta = \log(c^{-1} - 1)$. If $x \in \mathcal{U}(\tilde{\mathcal{K}})^{-1}$ then $x$ is analytic with respect to $\gamma$ and the KMS condition implies

$$\tilde{\omega}(xx^*) = \tilde{\omega}(x^* \gamma_{\beta}(x)) = e^{\beta} \tilde{\omega}(xx^*) .$$

Considering $x \in R(-\omega, -\lambda) \cap \mathcal{U}(\tilde{\mathcal{K}})^{-1}$, $x \neq 0$, and using Lemma 4.2(i) one finds

$$q^{-1} - 1 \leq e^{\beta} = c^{-1} - 1 .$$

The opposite inequality is derived the same way, hence $q = c$. □
Remark 4.4. The assumption about the finite dimensionality of the eigenspaces of $\mathcal{H}$ can be dropped if $\omega$ is strongly spectrally passive (Remark 2.3), because in that case the restriction of $Q$ to each of these eigenspaces is a scalar (Remark 3.7). □

Now we want to discuss the necessity of the faithfulness assumption in Theorem 4.3. Suppose that $\omega$ is a spectrally passive quasi-free state of $\mathfrak{M}(\mathcal{H})(0)$ and that $\tilde{\omega}$ (i.e., the restriction of $\omega$ to $\mathfrak{M}(\mathcal{H}_{\text{cont}})(0)$) is a ground state $\tilde{\omega}_{\infty, \mu'}$. Is then $\omega$ itself necessarily a ground state? Define $\mu_1 = \sup(-\infty, \mu) \cap \sigma_{\text{cont}}(H) \in [-\infty, \mu]$ and $\mu_2 = \inf(\mu, +\infty) \cap \sigma_{\text{cont}}(H) \in [\mu, +\infty]$. Notice that, unless $\mu = \mu_1 = \mu_2$ (which means that $\mu$ is neither left nor right isolated in $\sigma_{\text{cont}}(H)$), the Fermi energy $\mu$ is not unambiguously determined by the condition $\tilde{\omega} = \tilde{\omega}_{\infty, \mu'}$; it can take any value in $[\mu_1, \mu_2]$. Only this interval as such has a precise meaning. With the above notation we have:

**Proposition 4.5.** Let $\omega$ be a spectrally passive state of $(\mathfrak{M}(\mathcal{H})(0), \sigma)$. If $\tilde{\omega} = \tilde{\omega}_{\infty, \mu}$ for some $\mu$, and if $H$ has at most one eigenvalue in $[\mu_1, \mu_2]$, then $\omega$ is a ground state of $(\mathfrak{M}(\mathcal{H})(0), \sigma)$ whose Fermi energy lies in $[\mu_1, \mu_2]$. (It is still assumed that $\sigma_{\text{cont}}(H) \neq \emptyset$, and that the eigenvalues of $H$ have finite multiplicity or that $\omega$ is strongly spectrally passive.)

**Proof:** As in the proof of Theorem 4.3 we consider a unit vector $e$ such that $He = \lambda e$ and $Qe = qe$. Suppose that $\lambda < \mu_1$; we then have to show that $q = 1$. By definition of $\mu_1$ we can find a nonzero $f$ in $P(\lambda, \mu_1)^{\mathfrak{M}_{\text{cont}}}$. Using Lemma 4.2(i) we obtain

$$
(1 - q\|f\|_2^2)^2 - (1 - q)\tilde{\omega}(a\ast(f)a(f)) \leq q\tilde{\omega}(a(f)a\ast(f)) = 0
$$

where the equalities follow from the fact that $\tilde{\omega} = \tilde{\omega}_{\infty, \mu} = \tilde{\omega}_{\infty, \mu_1}$. Hence $q = 1$.

If $\lambda > \mu_2$ a similar reasoning yields $q = 0$. If $\sigma_{\text{pp}}(H) \cap [\mu_1, \mu_2] = \emptyset$ we conclude that $\omega = \omega_{\infty, \mu}$. If $\sigma_{\text{pp}}(H) \cap [\mu_1, \mu_2] = \{\mu'\}$ then $\omega$ is a ground state with Fermi energy $\mu'$. □

Remark 4.6. There is an obvious analogue to Proposition 4.5 in case $\omega$ is a Fock or anti-Fock state ($\mu = -\infty$ or $\mu = +\infty$, respectively). □
Proposition 4.5 has the following interpretation: If \( \sigma_{\text{cont}}(H) \neq \phi \), and if \( \omega \) is a spectrally passive quasi-free state of \( (\mathfrak{A}(\mathcal{F})^{(0)}, \alpha) \), but is not faithful, then it behaves like a ground state (or a Fock or anti-Fock state) on the subspace \( P(\mathbb{R} \setminus [\mu_1, \mu_2]) \mathcal{H} \). Precisely we mean that, if \( f \) is a unit vector in \( P(-\infty, \mu_1) \mathcal{H} \), then the state \( f \) is occupied by some particle of the system with probability 1; if \( f \in P(\mu_2, +\infty) \mathcal{H} \) the same probability is zero. On the other hand we have no such control over the eigenstates of \( H \) with eigenvalue in the interval \( [\mu_1, \mu_2] \): it is a priori possible that they are excited in a random way, i.e., that they are occupied with a probability strictly between 0 and 1. In other words, if there is more than one of these eigenvalues the existence of a spectrally passive quasi-free state of the noninteracting Fermi gas which is not of one of the types listed in Theorem 3.10 cannot be ruled out. That such states do in fact occur is shown in the following example.

Example 4.7. Let the one-particle Hilbert space \( \mathcal{H} \) be of the form \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{C}f \oplus \mathcal{C}e \), where \( e \) and \( f \) are unit vectors. Suppose that the one-particle hamiltonian \( H \) has the following properties: \( He = \lambda e \) (\( \lambda > 0 \)), \( Hf = 0 \), \( \sigma(H_0) \cap (-\lambda, \lambda) = \emptyset \) (\( H_0 = H \bigg|_{\mathcal{H}_0} \)).

Let \( \omega \) be an \( \alpha \)-invariant and gauge-invariant quasi-free state of the corresponding \( \mathcal{C}^* \)-dynamical system \( (\mathfrak{A}(\mathcal{H}), \alpha) \) with one-particle density \( Q \) defined as follows: \( Q = Q_{\infty,0} \) on \( \mathcal{H}_0 \oplus \mathcal{C}f \) (in particular \( Qf = \frac{1}{2} f \)) and \( Qe = qe \). We claim that, for every \( q \) in \( [0, 1/2] \), \( \omega \) is a spectrally passive state of \( (\mathfrak{A}(\mathcal{H}), \alpha) \) (a fortiori also of \( (\mathfrak{A}(\mathcal{H})^{(0)}, \alpha) \)). However, if \( q > 0 \), the state \( \omega \) does not belong to any of the families described in Theorem 3.10.

To prove our claim we put \( \tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{C}f \) and denote the restriction of \( \omega \) to \( \mathfrak{A}(\tilde{\mathcal{H}}) \) by \( \tilde{\omega} \) as before. The conditions of spectral passivity are given by (4.4) and (4.5). But the first of these inequalities holds for every value of \( q \), since \( \mathcal{C}(x^{\infty}) \alpha = 0 \) whenever \( x \in \mathbb{R}(-\infty, -\lambda) \cap \mathfrak{A}(\mathcal{H}) \) because \( \tilde{\omega} \) is a ground state and \( \lambda > 0 \).

Let \( \omega_0 \) be the restriction of \( \tilde{\omega} \) (or \( \omega \)) to \( \mathfrak{A}(\mathcal{H}_0) \). Considering \( \mathfrak{A}(\tilde{\mathcal{H}}) \) as the algebra of 2x2 matrices over \( \mathfrak{A}(\mathcal{H}_0) \) and using the assumptions \( Hf = 0 \) and \( Qf = (1/2)f \) one finds that (4.5) is equivalent with
$$q\omega_0(z^\ast z) \leq (1-q)\omega_0(zz^\ast), \quad z \in \mathbb{R}(\lambda, \Omega) \cap \mathcal{A}(\mathcal{H}_0). \quad (4.6)$$

Substituting $z = 1$ yields the necessary condition $q \leq 1/2$. We want to show that this condition is sufficient as well. Let $(\mathcal{K}, \pi, \Omega)$ be the GNS representation of $\mathcal{A}(\mathcal{H}_0)$ induced by $\omega_0$. Then (4.6) can be written as

$$q\|\pi(z)\Omega\|^2 \leq (1-q)\|\pi(z^\ast)\Omega\|^2, \quad z \in \mathbb{R}(\lambda, \Omega) \cap \mathcal{A}(\mathcal{H}_0). \quad (4.7)$$

Let $K$ be the unique self-adjoint operator on $\mathcal{K}$ satisfying $K\Omega = 0$ and $e^{itK}\pi(x)e^{-itK} = \pi(\alpha_t(x))$. Using the facts that $\omega_0$ is a ground state and that $\sigma(H_0) \cap (-\lambda, \lambda) = \emptyset$ one can show that $\sigma(K) \cap (-\lambda, \lambda) = \{0\}$, where $0$ is a simple eigenvalue (the corresponding eigenvector is $\Omega$; cf. Lemma 4.8 below). Writing $E$ for the spectral measure associated with $K$, we have, in other words,

$$E(-\lambda, \lambda) = \text{orthogonal projection onto } \mathcal{C}\Omega.$$ 

Now suppose $z \in \mathbb{R}(\lambda, \Omega) \cap \mathcal{A}(\mathcal{H}_0)$. Then $\pi(z)\Omega \in E(-\lambda, \lambda)\mathcal{K}$, $[8, \text{Lemma 1.4(ii)}]$, hence

$$\pi(z)\Omega = (\Omega, \pi(z)\Omega)\Omega.$$ 

On the other hand $\pi(z^\ast)\Omega \in E(-\lambda, +\infty)\mathcal{K}$, or

$$\pi(z^\ast)\Omega = (\Omega, \pi(z^\ast)\Omega)\Omega + \tilde{\nu},$$

where $\tilde{\nu} = E[\lambda, +\infty]\tilde{\nu}$. In this notation (4.7) becomes

$$q|\langle \Omega, \pi(z)\Omega \rangle|^2 \leq (1-q)|\langle \Omega, \pi(z)\Omega \rangle|^2 + (1-q)\|\tilde{\nu}\|^2$$

which is clearly satisfied if $q \leq 1/2$. $\square$

For the sake of completeness we give a proof of the spectral property of $K$ used above (dropping the subscripts from $\mathcal{H}_0$, $H_0$ and $\omega_0$).
Lemma 4.8. Let $\omega$ be a ground state of the $C^*$-dynamical system $(\mathfrak{A}, \lambda)$ given by a one-particle hamiltonian $H$ such that $\sigma(H) \cap (-\lambda, \lambda) = \emptyset$ ($\lambda > 0$). If $iK$ is the infinitesimal generator of the time translation group in the GNS representation, then $\sigma(K) \subset \{0\} \cup [\lambda, +\infty)$ and $0$ is an eigenvalue of $K$ of multiplicity one.

Proof: The Hilbert space $\mathfrak{K}$ of the GNS representation is generated by $\Omega$ and all vectors of the form

$$\bar{\psi} = \pi(a^*(g_n) \ldots a^*(g_2)a^*(g_1)a(f_m) \ldots a(f_2)a(f_1))\Omega$$

$(f_j, g_k \in \mathfrak{K}, 0 \leq j \leq m, 0 \leq k \leq n, m \text{ and } n \text{ not both } 0)$. We shall show that such a vector can be written as a linear combination of $\Omega$ and of vectors of the same form (possibly with smaller $m$ and $n$), but with $f_j \in \mathcal{P}(-\infty, -\lambda][\mathfrak{K}$ ($1 \leq j \leq m$) and $g_k \in \mathcal{P}[\lambda, +\infty)$ ($1 \leq k \leq n$). These latter vectors belong to $E[(m+n)\lambda, +\infty)\mathfrak{K}$ (where $K = \int \mathcal{E}d\mathcal{E})$, which clearly yields the result.

First we observe that

$$f \in \mathcal{P}(0, +\infty)\mathfrak{K} \implies \|\pi(a(f))\Omega\|^2 = \omega(a^*(f)a(f)) = 0$$

and

$$g \in \mathcal{P}(-\infty, 0)\mathfrak{K} \implies \|\pi(a^*(g))\Omega\|^2 = \omega(a(g)a^*(g)) = 0$$

because $\omega$ is a ground state. Using the CAR (1.2) it follows from the first observation that

$$\pi(a(f_m) \ldots a(f_2)a(f_1))\Omega = \pi(a(f'_m) \ldots a(f'_2)a(f'_1))\Omega,$$

where $f'_j = \mathcal{P}(-\infty, 0]f_j = \mathcal{P}(-\infty, -\lambda]f_j$. Again by the CAR

$$\pi(a^*(g)a(f'_m) \ldots a(f'_2)a(f'_1))\Omega$$

$$= \sum_{j=1}^{m}(-1)^{m-j}(f'_j, g)\pi(a(f'_m) \ldots a(f'_{j+1})a(f'_{j+1}) \ldots a(f'_1))\Omega$$

$$+ (-1)^{m}\pi(a(f'_m) \ldots a(f'_2)a(f'_1)a^*(g))\Omega.$$
By the second observation above, the last term in the right hand side vanishes when $g \in P(-\infty, 0) \mathcal{H} = P(-\infty, \lambda) \mathcal{H}$. From this, and the CAR, it follows easily that $\tilde{\varphi}$ can be brought into the desired form. □

It is unclear to the author whether states of the type exhibited in Example 4.7 have any physical relevance, or are to be regarded as a mathematical oddity.

In a final example we show that no analogue of Theorem 3.10, or Theorem 4.3, can possibly hold if $H$ has no continuous spectrum at all.

**Example 4.9.** Keeping the notation used throughout this section, we suppose that $\mathcal{H} = \tilde{\mathcal{H}} \oplus \mathbb{C}e$, $He = \lambda e$, $\lambda \notin \mathbb{Z}$, and that the spectrum of $H|_{\tilde{\mathcal{H}}}$ is contained in $\mathbb{Z}$ (i.e., $U_t|_{\tilde{\mathcal{H}}}$ is periodic with period $2\pi$). It follows that $\tilde{\alpha}$ has the same period, i.e., its Arveson spectrum $[3, 16, 7]$ is contained in $\mathbb{Z}$. Now let $\omega$ be an $\alpha$-invariant quasi-free state of $(\mathfrak{A}(\mathcal{H}), \alpha)$ such that $\tilde{\omega}$ is $\beta$-KMS with respect to $\tilde{\alpha}$. Let $Q$ be the one-particle density of $\omega$, and $Qe = qe$. For $\omega$ to be spectrally passive it is necessary and sufficient that the inequalities (4.4) and (4.5) hold. But because of the periodicity

$$R(-\infty, -\lambda) \cap \mathfrak{A}(\tilde{\mathcal{H}}) = M(-\infty, -[\lambda + 1]) \cap \mathfrak{A}(\tilde{\mathcal{H}})$$

and

$$R(-\infty, \lambda) \cap \mathfrak{A}(\tilde{\mathcal{H}}) = M(-\infty, [\lambda]) \cap \mathfrak{A}(\tilde{\mathcal{H}}),$$

where $[\lambda]$ is the largest integer not exceeding $\lambda$. Since $\tilde{\omega}$ is $\beta$-KMS, we have the inequalities

$$\tilde{\omega}(x^*x) \leq e^{-\beta[\lambda + 1]} \tilde{\omega}(xx^*), \quad x \in R(-\infty, -\lambda) \cap \mathfrak{A}(\tilde{\mathcal{H}})$$

and

$$\tilde{\omega}(y^*y) \leq e^{\beta[\lambda]} \tilde{\omega}(yy^*), \quad y \in R(-\infty, \lambda) \cap \mathfrak{A}(\tilde{\mathcal{H}}).$$

[8, Theorem 1.1 and Remark 2.1(iii)]. It follows that (4.4) and (4.5) are valid for all values of $q$ such that

$$(1 + e^{\beta[\lambda + 1]})^{-1} \leq q \leq (1 + e^{\beta[\lambda]})^{-1}.$$

However, the state $\omega$, although faithful, is not KMS if $q \neq (1 + e^{\beta\lambda})^{-1}$. □
It should be interesting to know all passive (gauge-invariant) quasi-free states of a noninteracting Fermi "gas" of harmonic oscillators, for instance.
References


<table>
<thead>
<tr>
<th>Page</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>M. Slemrod, J. E. Marsden</td>
<td>Temporal and Spatial Chaos in a Van der Waals Fluid due to Periodic Thermal Fluctuations</td>
</tr>
<tr>
<td>34</td>
<td>J. Kirkwood, C. E. Wayne</td>
<td>Percolation in Continuous Systems</td>
</tr>
<tr>
<td>35</td>
<td>Luis Magalhaes</td>
<td>Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations</td>
</tr>
<tr>
<td>36</td>
<td>C. Eugene Wayne</td>
<td>The KAM Theory of Systems with Short Range Interactions II</td>
</tr>
</tbody>
</table>