ON THE DEVELOPMENT OF CLOSED CONVEX CURVES ON 3-POLYTOPES

By

Joseph O'Rourke

and

Catherine Schevon

IMA Preprint Series # 363

October 1987

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

ON THE DEVELOPMENT OF CLOSED CONVEX CURVES ON 3-POLYTOPES

JOSEPH O'ROURKE† CATHERINE SCHEVON‡

Abstract. It is shown that a closed convex polygonal curve on the surface of a 3-polytope develops in the plane to a simple path: it does not self-intersect.

Key words. 3-polytope, development, closed convex curve, unfolding

1. Introduction.

We prove in this paper that a closed convex polygonal curve on the surface of a 3-polytope develops without self-intersection.[1] This is a restriction to the polyhedral world of a suggestion of Thurston, who asked [T]: is it true that a simple closed curve on a convex body whose curvature is of one sign only develops to a simple path? By "develop" we mean, intuitively, rolling the convex body on the plane without slippage such that the curve is always the point of contact. The development of a curve is its trace on the plane. We believe our proof extends to smooth curves on smooth convex bodies via polytope approximations theorems, but we have made no attempt to detail this extension.

We now make the notion of "development" precise. Throughout we abbreviate 3-polytope to polytope. A *polygonal curve* C is a sequence of line segments on the surface of a polytope P. The point where two line segments of the curve meet will be called a *corner* c_i of the curve to distinguish it from a vertex of the polytope. We consider a closed curve to be directed counterclockwise so that the interior is to the left of the curve.

Let $C = (c_0, c_1, ..., c_{n-1})$, with the indices increasing counterclockwise. All index arithmetic is to be understood as $\bmod n$. The *surface angle* α_i at c_i is the sum of the angles from $c_{i-1}c_i$ to c_ic_{i+1} measured on the surface of P. The angle may be obtained by flattening all the faces of P that are to the left of C in a neighborhood of c_i, and measuring the angle between the two unfolded segments $c_{i-1}c_i$ and c_ic_{i+1}. A directed polygonal curve on the surface of a polytope is a *convex curve* iff $0 < \alpha_i \leq \pi$ for all i.

The *development* of a curve on a polytope is a drawing of the curve in the plane such that the interior angle between the two segments adjacent to the image of c_i on the plane is precisely α_i. Note that the development of a curve C is well-defined even when C passes through vertices of P, or travels along edges of P, although the intuitive "rolling

†Johns Hopkins University, Department of Computer Science, Baltimore, MD, 21218; Currently at The Institute for Mathematics and Its Applications, University of Minnesota, 514 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455

‡Johns Hopkins University

[1] Our first proof of this theorem [OS] was rather complex. The proof presented here is comparatively simple.

without slippage" image is no longer unambiguous in these cases. We let $\Delta(Z)$ represent the image of object Z under development. Two developments are considered equal if they are congruent. Development is defined with respect to a given starting point on the curve, called the *cutpoint* . The development of a curve is dependent on the cutpoint: curves developed from different cutpoints are not necessarily congruent.

If C encloses no curvature on P, for example, if C lies entirely on one face, then it develops to a simple polygon in the plane. We define self-intersection so that this developed curve does not self-intersect. $\Delta(C)$ *self-intersects* iff two distinct points $x \neq y$ on C develop to the same location on the plane: $\Delta(x) = \Delta(y)$.

2. Proof Sketch.

We first remove the "spherical cap" enclosed within C and replace it by the hull of C, as illustrated in Fig. 1. Let P' be the convex hull of the corners of C and the vertices of P that are not properly contained within C, and let C' be the curve C on P'. Note that although $C = C'$ as three-dimensional curves, in general $\Delta(C) \neq \Delta(C')$ because the corresponding surface angles of the curves are different. First we establish in Lemma 1 that the angles α'_i on C' are no larger than their counterparts α_i on C. Then Lemma 2 shows that $\Delta(C')$ is a convex polygon. Lemma 3 is taken from Cauchy's famous proof of his congruent polytopes theorem, and shows that increasing the angles of a convex polygon while keeping all but one edge length fixed, necessarily increases the length of the remaining edge.[2]

The proof of our theorem follows easily from these lemmas by contradiction: self-intersection requires two points of $\Delta(C)$ to be coincident, but opening up $\Delta(C')$ to match $\Delta(C)$ via Lemma 3 shows that their original non-zero separation can only increase.

3. Proof.

We will not repeat the definitions of the symbols defined above. Let a prime on a symbol indicate corresponding quantities on the polytope P': thus C' on P' has corners c'_i and angles α'_i.

LEMMA 1. *For all i, $\alpha'_i \leq \alpha_i$.*

Proof. Let T be the triangle that is the convex hull of c_{i-1}, c_i, and c_{i+1}. By construction, we have $P' \subseteq P$, and by convexity, $T \subseteq P' \subseteq P$. Thus the interiors of C and C' in the vicinity of c_i are to the same side of the plane A containing T.

Let H be the halfspace bounded by A that includes the interior of C in the neighborhood of c_i, that is, includes the side to which $c_{i-1}c_i \times c_ic_{i+1}$ points. Let K be the cone with apex at c_i, formed by the extension of all faces of P incident to c_i, intersected with H. Note that the faces angles of K at c_i sum to α_i plus the space angle at c_i,

[2]Cauchy's proof was incorrect, and was later corrected by Steinitz [L].

$\theta = angle(c_{i-1}, c_i, c_{i+1})$. Define K' analogously on P'. Then $K' \subseteq K$ because $P' \subseteq P$. See Fig. 2.

Consider the "spherical image" S of K: the set of points on the unit sphere representing normals of planes supporting K at c_i. Define S' analogously for K'. Then because $K' \subseteq K$, $S' \supseteq S$, and thus $area(S') \geq area(S)$. But the area of S is just the curvature of K at c_i, which is the "angle deficit" [AZ, p.8] $2\pi - (\alpha_i + \theta) = area(S)$, and similarly $2\pi - (\alpha'_i + \theta) = area(S')$. Thus

$$2\pi - (\alpha'_i + \theta) \geq 2\pi - (\alpha_i + \theta)$$

or $\alpha'_i \leq \alpha_i$. □

LEMMA 2. *$\Delta(C')$ is a convex polygon.*

Proof. By definition of P', C' encloses no vertices, and therefore no curvature. Thus it develops to a simple polygon [A, p. 70]. We need only show that this polygon is convex.

By assumption, C is a convex curve, which by definition means that $0 < \alpha_i \leq \pi$. By Lemma 1, $\alpha'_i \leq \alpha_i$, and since clearly $0 < \alpha'_i$, we have $0 < \alpha'_i \leq \pi$. Thus $\Delta(C')$ is a convex polygon. □

The next Lemma is Cauchy's "angle opening" lemma.

LEMMA 3. *Let Q be a plane polygon with vertices v_i, $i = 0, ..., n-1$, with interior angles $\beta_i = angle(v_{i-1}, v_i, v_{i+1})$, $i = 0, ..., n-1$. Then if Q is transformed into another polygon Q' so that all but one side length is fixed,*

$$|v'_i v'_{i+1}| = |v_i v_{i+1}| \; , \; i = 0, ..., n-2$$

and the angles increase or stay the same,

$$\beta'_i \geq \beta_i \; , \; i = 1, ..., n-2$$

then the last side length must increase or stay the same:

$$|v'_0 v'_{n-1}| \geq |v_0 v_{n-1}|$$

Moreover, if at least one of the angle inequalities is strict, then the lengthening of $v_0 v_{n-1}$ is strict.

Proof. See Lyusternick [L, Lemma 1, p. 67] or Stoker [S, Lemma A]. □

THEOREM 1. *A closed convex polygonal curve on the surface of a 3-polytope develops in the plane without self-intersection.*

Proof. Suppose, in contradiction to the theorem, that $\Delta(C)$ self-intersects. Then by definition $\Delta(x) = \Delta(y)$ for distinct points $x \neq y$ of C. Now form P' and consider the development of C' as an intermediate stage towards the development of C. Because $x \neq y$, $\Delta(x') \neq \Delta(y')$, so the chord $h = \Delta(x')\Delta(y')$ of $\Delta(C')$ has non-zero length, since by Lemma 2 this is chord of a convex polygon.

Let z be the cutpoint on C, and consider the chain of $\Delta(C')$ counterclockwise between $\Delta(x')$ to $\Delta(y')$ that does not include $\Delta(z')$. Say this chain is

$$(\Delta(x'), \Delta(c'_{i_1}), ..., \Delta(c'_{i_k}), \Delta(y')) .$$

See Fig. 3. Then by Lemma 1, the angles at $\Delta(c'_{i_j})$ must open to transform $\Delta(C')$ to $\Delta(C)$: $\alpha'_{i_j} \leq \alpha_{i_j}$. But applying Lemma 3 to this chain (with $h = v_0 v_{n-1}$) shows that h cannot shrink to zero length in $\Delta(C)$, a contradiction to the assumption of self-intersection. □

We observe finally that $\Delta(C)$ is a closed curve iff C encloses no vertices (and therefore curvature), because Lemma 3 shows that if any of the angles properly open, then the chord must properly lengthen.

Acknowledgements.

This research was partially supported by NSF grant DCR-85-51468, the Guggenheim Foundation, and the Institute for Mathematics and Its Applications.

REFERENCES

[A] A. D. ALEKSANDROV, *Konvexe Polyeder*, Akademie Verlag, Berlin, 1958.

[AZ] A. D. ALEKSANDROV AND U. A. ZALGALLER, *Intrinsic Geometry of Surfaces*, American Mathematical Society, Providence, RI, 1967.

[L] L. A. LYUSTERNIK, *Convex Figures and Polyhedra*, D. C. Heath, Boston, MA, 1966.

[OS] J. O'ROURKE AND C. SCHEVON, *A theorem on the development of closed convex curves*, Johns Hopkins University Technical Report JHU-87/16 (1987).

[S] J. J. STOKER, *Geometrical problems concerning polyhedra in the large*, Comm. on Pure and Applied Math., 21 (1968), pp. 119–168.

[T] W. THURSTON, *personal communication*, February 2, 1987.

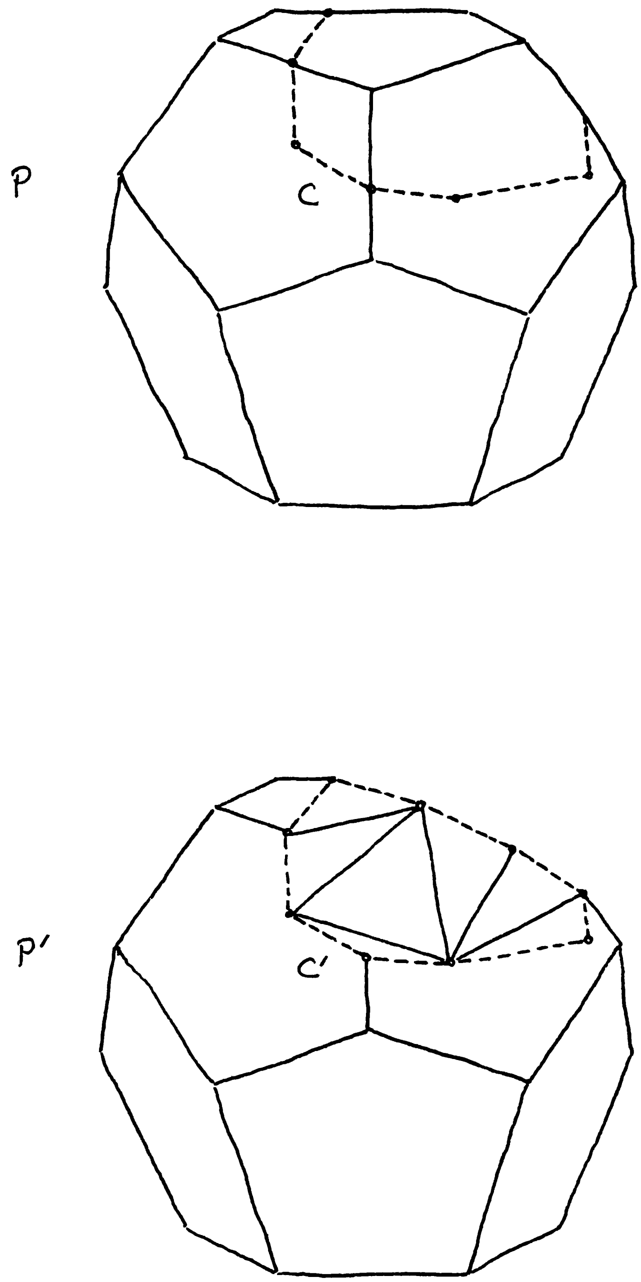


Fig. 1

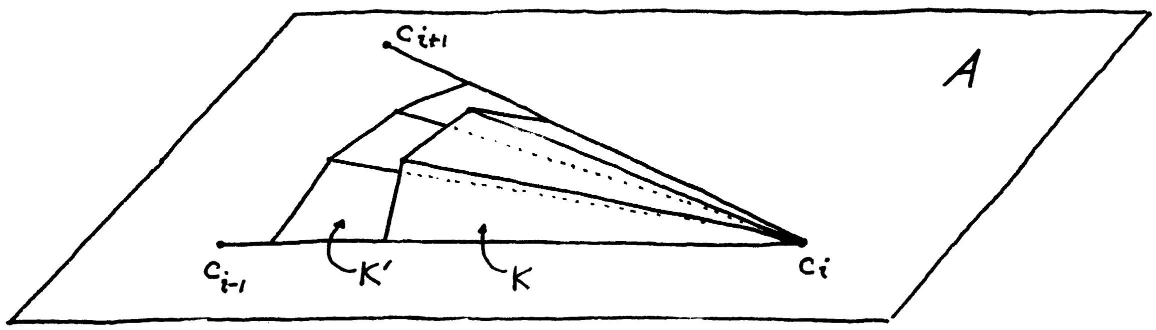
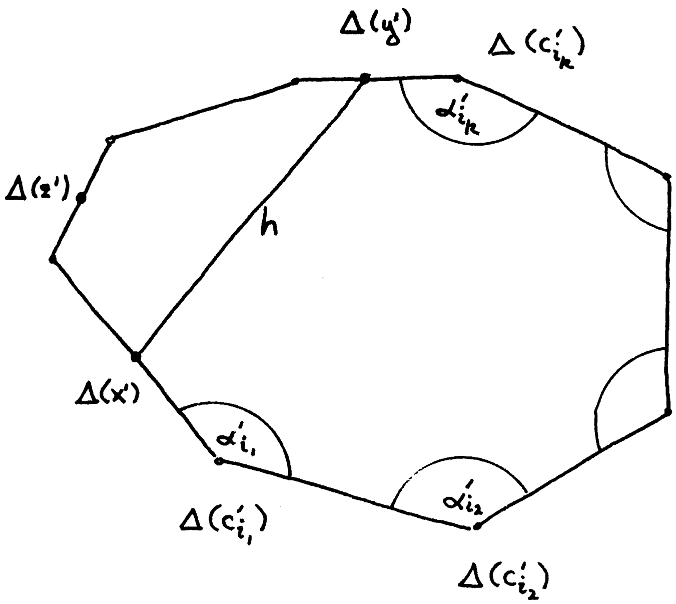


Fig. 2



$\Delta(y')$
$\Delta(c'_{i_k})$
α'_{i_k}
$\Delta(z')$
h
$\Delta(x')$
α'_{i_1}
α'_{i_2}
$\Delta(c'_{i_1})$
$\Delta(c'_{i_2})$

Fig. 3