O(2)-EQUIVARIANT BIFURCATION EQUATIONS
WITH TWO MODES INTERACTION

By

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O(2)-EQUIVARIANT BIFURCATION EQUATIONS
WITH TWO MODES INTERACTION

HISASHI OKAMOTO†

§1. Introduction. Bifurcation equations of two distinct modes interaction in the
presence of symmetry group O(2) are considered in the present paper. The bifurcation
equation which will be considered is defined in 4-dimensional space, hence it is not im-
mediately clear what order of Taylor expansion suffices to the analysis of the zero points
set of the bifurcation equation. In other words, it is not an easy task to decide where we
should truncate the Taylor expansion of the bifurcation equation without changing topo-
logical nature of the zero points set. We think singularity theoretic approach by Golubitsky
and Schaeffer [3,4] is very useful in order to understand the structure of the bifurcation
equations.

What we do in this paper is as follows. In §2, we introduce a concept of O(2)-
equivariance which is supposed to be satisfied by a mapping of the form

\[ G: \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2. \]

In §3–§5 we consider the following problem:

i) Under what conditions \( G \) is O(2)-equivariant to a certain normal form

ii) What is universal unfolding of the normal form.

In these sections we consider both the simplest case and some degenerate cases, since
these are required in practical problems like Taylor problem or water wave problem ( see

§2. O(2)-equivariant bifurcation equations. We begin this section by

DEFINITION 2.1. Given two distinct integers \( m \) and \( n \) satisfying \( 0 < m < n \) and a
mapping

\[ G = (G_1, G_2): \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2, \]

we say that \( G \) is O(2)-equivariant with mode \((m, n)\) if the following conditions (2.1,2)
are satisfied.

(2.1) \[ G(\lambda; e^{im\alpha}\xi, e^{in\alpha}\zeta) = (e^{i\alpha}G_1(\lambda; \xi, \zeta), e^{i\alpha}G_2(\lambda; \xi, \zeta)) \quad (\alpha \in [0, 2\pi]) \]

for any \( \lambda \in \mathbb{R} \) and \( (\xi, \zeta) \in \mathbb{C}^2 \),

(2.2) \[ G(\lambda; \xi, \zeta) = (G_1(\lambda; \xi, \zeta), G_2(\lambda; \xi, \zeta)) \]

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for any $\lambda \in \mathbb{R}$ and $(\xi, \zeta) \in \mathbb{C}^2$.

Let us clarify this definition by a motivation. In a variety of bifurcation problems, we are given an equation like

$$(2.3) \quad F(a, u) = 0 \quad (a \in \mathbb{R}^k, u \in X)$$

where $k$-dimensional parameter $a$ represents physical parameters ($k \geq 2$) and $X$ is a space of functions of period $2\pi$. Physical circumstance put a natural action of the orthogonal group $O(2)$ on the function space $X$. For example, we frequently encounter the following action of $O(2)$: if we parametrize $O(2)$ by $\alpha \in [0, 2\pi)$ and the reflection with respect to the horizontal axis,

$$(2.4) \quad u(x) \to u(x + \alpha) \quad \text{for } \alpha \in [0, 2\pi)$$

$$(2.5) \quad u(x) \to u(-x) \quad \text{for the reflection}$$

defines an action of $O(2)$ on $X$. We are concerned with the equations which commute with the action of $O(2)$, i.e.,

$$(2.6) \quad \gamma F(a, u) = F(a, \gamma u) \quad (\gamma \in O(2))$$

We assume that $F(a, 0) \equiv 0$. In general, we have a variety $V_n$ in $\mathbb{R}^k$ of codimension one on which the linearized operator $D_u F(a, 0)$ has a nontrivial kernel and generically this kernel is spanned exactly by $\cos(n\theta)$ and $\sin(n\theta)$, where $n$ is an integer. Since $k \geq 2$, we, generically, have a variety $V_{mn}$ of codimension two in $\mathbb{R}^k$ such that $V_{mn}$ is a subset of $V_m \cap V_n$ and the kernel of $D_u F(a, 0)$ is spanned exactly by $\cos(m\theta), \sin(m\theta), \cos(n\theta), \sin(n\theta)$ where $m$ and $n$ are integers satisfying $0 < m < n$. Since the global bifurcation diagram is strongly dominated by these singularities with 4-dimensional kernel, it is important to analyze bifurcations at these points. Let $N$ denote 4-dimensional space spanned by $\cos(m\theta), \sin(m\theta), \cos(n\theta), \sin(n\theta)$. In order to get nontrivial solutions of (2.6), we use the Lyapunov-Schmidt procedure. Let $P$ be a projection from $X$ onto $N$. Then (2.3) is equivalent to (2.7, 8) below.

$$(2.7) \quad PF(a, x\cos(m\theta) + y\sin(m\theta) + z\cos(n\theta) + w\sin(n\theta) + \phi(a, x, y, z, w)) = 0,$$

$$(2.8) \quad (I - P)F(a, x\cos(m\theta) + y\sin(m\theta) + z\cos(n\theta) + w\sin(n\theta) + \phi(a, x, y, z, w)) = 0,$$

where $x, y, z$ and $w$ is real variables ans $\phi$ is a function of $(a, x, y, z, w)$ taking its value in a orthogonal complement of $N$ in $X$. As is known, (2.8) determines the function $\phi$ uniquely.
Therefore (2.7) can be regarded as a mapping defined in some neighborhood of $R^4 \times R^4$ with its value in $N$. Introducing $\xi = x + iy$ and $\zeta = z + iw$, we identify $R^4$ with $C^2$. $N$ too is identified with $C^2$. In this setting, the action of $O(2)$ ( (2.4, 5) ) is rewritten as

\begin{equation}
(\xi, \zeta) \to (e^{in_\alpha} \xi, e^{in_\alpha} \zeta)
\end{equation}

\begin{equation}
(\xi, \zeta) \to (\overline{\xi}, \overline{\zeta}),
\end{equation}

Let us fix $a_0 = (a_0^1, a_0^2, \ldots, a_0^k) \in V_2$. For the simplicity, let us assume that moving $a^1$ near $a_0^1$ corresponds to moving $a$ transversally to $V_m$ and $V_n$. Then we put $\lambda = a^1 - a_0^1$ and fix $a_0^2, \ldots, a_0^k$. Now the right hand side of (2.7) is regarded as a mapping of $\lambda, \xi$ and $\zeta$. We write this as $G = (G_1(\lambda, \xi, \zeta), G_2(\lambda, \xi, \zeta))$. Since $F$ commute with $O(2)$-action, so does the equation (2.7) ( see, e.g., [4] or Sattinger [7]. Thus we get to a mapping satisfying the properties in DEFINITION 2.1. In what follows we analyze $G$ along the line of Golubitsky-Schaeffer theory.

**Remark 2.1.** In DEFINITION 2.1 we write as if $G$ were defined in a whole space $R \times C^2$. In practice, the bifurcation equation $G$ is defined only in some neighborhood of the origin. In this sense, it is very convenient to use the notion of germs as in [4]. Hereafter we consider the mapping germs. However, we believe that writing as if $G$ were defined in a whole space makes no confusion. Therefore we say mapping, although it is actually a mapping germ.

**Proposition 2.1.** Let $G$ be a $C^\infty$-mapping which is $O(2)$-equivariant with mode $(m,n)$. Then it must be of the following form

\begin{equation}
G_1 = f_1(\lambda, u, v, r)\xi + f_2(\lambda, u, v, r)\overline{\xi}^{-1} \zeta
\end{equation}

\begin{equation}
G_2 = f_3(\lambda, u, v, r)\zeta + f_4(\lambda, u, v, r)\overline{\zeta}^{-1}
\end{equation}

where $m'$ and $n'$ are positive integers with no common divisor such that $n'/m' = n/m$. $f_j(\lambda, u, v, r)$ are real-valued functions of $\lambda, u \equiv |\xi|^2, v \equiv |\zeta|^2$ and $r \equiv Re[\xi^{n'}\overline{\zeta}^{m'}]$.

This proposition is proved in Fujii, Mimura and Nishiura [3] ( see also Dangelmayr and Armbruster [2] ) or you can prove it in the way similar to the proof of Proposition 2.3. Let $E$ be the set of all real-valued $C^\infty$-functions $f : R \times C^2 \rightarrow R$ which is of the following form

\[ f = g(\lambda, u, v, r) \quad (g \in C^\infty(R^4)). \]

Then $E$ is a commutative ring with a unit. PROPOSITION 2.1 is equivalently stated as follows.
PROPOSITION 2.1'. The set of all the $O(2)$-equivariant $C^\infty$-mappings with mode $(m, n)$ are $E$-module and generated by $(\xi, 0), (0, \zeta), (\xi_n^{-1}\zeta_m', 0)$ and $(0, \xi_n'\zeta_m'^{-1})$.

Since we are considering bifurcation equations, the Jacobian of $G$ must vanish at the origin. This means that

$$f_1(0; 0, 0, 0) = f_3(0; 0, 0, 0) = 0$$

Under this condition, we may generically assume that

$$f_2(0; 0, 0, 0) \neq 0 \quad \text{and} \quad f_4(0; 0, 0, 0) \neq 0.$$

Of course, if we have more than two parameters, the condition (2.14) may be violated at a critical point in the parameter space. We encounter this situation in the Taylor problem ( see [5] ). Therefore we consider the following three cases separately:

(I) \hspace{1cm} f_2(0; 0, 0, 0) \neq 0, \quad f_4(0; 0, 0, 0) \neq 0

(II) \hspace{1cm} f_2(0; 0, 0, 0) = 0, \quad f_4(0; 0, 0, 0) \neq 0

(III) \hspace{1cm} f_2(0; 0, 0, 0) \neq 0, \quad f_4(0; 0, 0, 0) = 0

We now state a definition in [4] which defines coordinate changes preserving $O(2)$-equivariance

DEFINITION 2.2. If there are coordinates changes $\sim X, T$ and $\Lambda$ which satisfy

$$G(\lambda; z) = H(\Lambda(\lambda); \sim X(\lambda, z)) \quad (\lambda \epsilon R, z \equiv (\xi, \zeta) \epsilon C^2)$$

and the following (2.15-17), then we say $G$ and $H$ are $O(2)$-equivalent.

$$\sim X : R \times C^2 \rightarrow C^2 \quad (\lambda \epsilon R, z \equiv (\xi, \zeta) \epsilon C^2)$$

$$\sim X(\lambda, \gamma z) = \gamma \sim X(\lambda, z) \quad (\gamma \epsilon O(2))$$

The Jacobian matrix of $X$ with respect to $z$ has positive determinant at the origin : $(\lambda, z) = (0, 0)$.

$$T : R \times C^2 \rightarrow M(4, R) \quad (= \text{the set of all } 4 \times 4 \text{ real matrices} )$$

$$T(\lambda, \gamma z) = \gamma T(\lambda, z)\gamma^{-1} \quad (\gamma \epsilon O(2))$$

$T$ is nonsingular at the origin : $(\lambda, z) = (0, 0)$.

$$\Lambda : R \rightarrow R, \quad \Lambda'(0) > 0.$$

Remark 2.2. $X$ and $T$ may depend on both $\lambda$ and $z$. But $\Lambda$ is a function of $\lambda$ only. One can easily check that, if $G$ is $O(2)$-equivariant and $H$ is $O(2)$-equivariant to $G$, then $H$ is also $O(2)$-equivariant.

By Definition 2.1' and Definition 2.2 we have
**Proposition 2.2.** If $\alpha$ and $\beta$ are real-valued functions of $(\lambda, u, v, r)$, then

$$H_1(\lambda, \xi, \zeta) = G_1(\lambda, \xi + \alpha \zeta^{n-1} \zeta', \zeta + \beta \zeta^{n-1} \zeta'^{-1}),$$

$$H_2(\lambda, \xi, \zeta) = G_2(\lambda, \xi + \alpha \zeta^{n-1} \zeta', \zeta + \beta \zeta^{n-1} \zeta'^{-1}),$$

is $O(2)$-equivalent to $G$.

**Proposition 2.3.** The set of all $\mathbb{R}$-linear mappings on $\mathbb{C}^2$ which satisfy (2.16) but not necessarily nonsingular, is a finitely generated $E$-module generated by the following 16 elements.

$$T_1 : (H_1, H_2) \to (H_1, 0), \quad T_2 : (H_1, H_2) \to (\xi^2 \overline{H_1}, 0)$$

$$T_3 : (H_1, H_2) \to (\xi \zeta H_2, 0), \quad T_4 : (H_1, H_2) \to (\xi \zeta \overline{H_2}, 0)$$

$$T_5 : (H_1, H_2) \to (0, \xi \zeta H_1), \quad T_6 : (H_1, H_2) \to (0, \xi \zeta \overline{H_1})$$

$$T_7 : (H_1, H_2) \to (0, \xi^2 \overline{H_2}), \quad T_8 : (H_1, H_2) \to (0, H_2)$$

$$T_{j+8} = \text{Im}[\xi^n \overline{\zeta^{m'}}]T_j \quad (1 \leq j \leq 8)$$

**Proof.** We represent $T$ as follows:

$$T(\lambda, \xi, \zeta)(H_1, H_2) = (t_{11} H_1 + t_{12} \overline{H_1} + t_{13} H_2 + t_{14} \overline{H_2}, t_{21} H_1 + t_{22} \overline{H_1} + t_{23} H_2 + t_{24} \overline{H_2})$$

By (2.16) we have

$$T(\lambda, e^{i m \alpha} \xi, e^{i n \alpha} \zeta)(e^{i m \alpha} H_1, e^{i n \alpha} H_2) = (e^{i m \alpha}(t_{11} H_1 + t_{12} \overline{H_1} + t_{13} H_2 + t_{14} \overline{H_2}),$$

$$e^{i n \alpha}(t_{21} H_1 + t_{22} \overline{H_1} + t_{23} H_2 + t_{24} \overline{H_2})) \quad (\alpha \in O(2))$$

$$T(\lambda, \xi, \zeta)(H_1, H_2) = T(\lambda, \xi, \zeta)(H_1, H_2)$$

If we put $\hat{t}_{ij} = t_{ij}(\lambda, e^{i m \alpha} \xi, e^{i n \alpha} \zeta)$, (2.18, 19) are rewritten as follows.

$$\hat{t}_{11} = t_{11}, \quad \hat{t}_{12} = e^{2i m \alpha} t_{12}, \quad \hat{t}_{13} = e^{i (m-n) \alpha} t_{13}, \quad \hat{t}_{14} = e^{i (m+n) \alpha} t_{14},$$

$$\hat{t}_{21} = e^{i (m-n) \alpha} t_{21}, \quad \hat{t}_{22} = e^{i (m+n) \alpha} t_{22}, \quad \hat{t}_{23} = t_{23}, \quad \hat{t}_{24} = e^{2i m \alpha} t_{24}$$

$$t_{ij}(\lambda, \xi, \zeta) = t_{ij}(\lambda, \xi, \zeta)(i = 1, 2; \quad j = 1, 2, 3, 4),$$

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respectively. If we put \( r + is = \xi^{n'}\bar{\zeta}^{m'} \), we can show that

\[
\begin{align*}
t_{11} &= f_1 + is f_2, \quad t_{12} = \xi^2 f_3 + is \xi^2 f_4, \quad t_{13} = \xi \bar{\zeta} f_5 + is \xi \bar{\zeta} f_6, \quad t_{14} = \xi \zeta f_7 + is \xi \zeta f_8, \\
t_{21} &= \bar{\zeta} f_9 + is \bar{\zeta} f_{10}, \quad t_{22} = \xi \zeta f_{11} + is \xi \zeta f_{12}, \quad t_{23} = f_{13} + is f_{14}, \quad t_{13} = \xi^2 f_{15} + is \xi^2 f_{16},
\end{align*}
\]

where \( f_j \in E \). In order to prove these, let us consider \( t_{12} \) and expand it as

\[
(2.22) \quad t_{12} = \sum a_{pqkt}t(\lambda)\xi^p\bar{\zeta}^q\zeta^k\bar{\zeta}^\ell
\]

As in [4], we may, without loss of generality, assume that (2.22) is a finite sum. By (2.21) \( a_{pqkt} \) is real. By \( t_{12} = e^{2i\alpha t_{12}} \), we have

\[
\sum a_{pqkt}e^{i(p+q)\alpha + in(k-\ell)\alpha} = \sum a_{pqkt}t(\lambda)\xi^p\bar{\zeta}^q\zeta^k\bar{\zeta}^\ell,
\]

which means \( e^{i(p+q-2)\alpha + in(k-\ell)\alpha} = 1 \) for all \( \alpha \in [0, 2\pi) \), if \( a_{pqkt} \neq 0 \). Therefore \( m'(p - q - 2) = -n'(k - \ell) \) if \( a_{pqkt} \neq 0 \). This means \( k - \ell \) must be a multiple of \( m' \). We put \( \ell - k = jm' \) (\( j \in \mathbb{Z} \)). Then we have \( p = q + 2 + jn' \). Therefore \( t_{12} \) is represented as a sum of the following functions with \( C^\infty \)-functions of \( \lambda \) only as coefficients:

\[
\xi^{q+2+jn'}\bar{\zeta}^\ell \zeta^k\bar{\zeta}^j,
\]

This function is equal to \( u^qv^k(r + is)^j\xi^2 \), if \( j \geq 0 \), and \( u^{q+jn'}v^{k+jm'}(r - is)^{-j}\xi^k \), if \( j < 0 \). By the binomial theorem and \( r^2 + s^2 = u^nv^m \), \( t_{12} \) can be represented by an \( E \)-combination of \( \xi^2 \) and \( is\xi^2 \). Other equalities are proved similarly. \( \Box \)

**Corollary 2.4.** If \( h_1, h_2 : \mathbb{R}^4 \to \mathbb{R} \) satisfy \( h_1(0; 0, 0, 0) \neq 0, h_2(0; 0, 0, 0) \neq 0 \) and \( \alpha_j, \beta_j, \gamma_j \) (\( j = 1, 2 \)) are real constants, then

\[
H_1(\lambda, \xi, \zeta) = h_1(\lambda, u, v, r)G_1(\lambda, \xi, \zeta) + \alpha_1\xi^2\overline{G_1} + \beta_1\xi(\overline{\zeta}G_2 + \zeta\overline{G_2}) + \gamma_1\xi(\overline{G_2} - \zeta\overline{G_2}),
\]

\[
H_2(\lambda, \xi, \zeta) = h_2(\lambda, u, v, r)G_2(\lambda, \xi, \zeta) + \alpha_2\xi^2\overline{G_2} + \beta_2\zeta(\xi\overline{G_1} + \xi\overline{G_1}) + \gamma_2(\xi\overline{G_1} - \xi\overline{G_1}),
\]

is \( O(2) \)-equivalent to \( G \).

In what follows we will transform a given \( G \) into a simpler form by coordinate changes which are allowed by **Definition 2.2.** From now on we assume that

\[
(2.22) \quad n/m = 2.
\]

Therefore we have \( n' = 2, m' = 1 \). What we will consider is, therefore,

\[
(2.23) \quad G_1 = f_1\xi + f_2\bar{\xi}\zeta
\]

\[
(2.24) \quad G_2 = f_3\zeta + f_4\xi^2,
\]

where \( f_j \) (\( j = 1, 2, 3, 4 \)) are in \( E \). We fix this \( G \). We use the following theorem as a basic tool.
THEOREM 2.1 ([4]). Let \( \hat{E} \) be a set of all \( C^\infty \)-mappings \( X : \mathbb{R} \times C^2 \rightarrow C^2 \) such that 
\[ X(\lambda, \gamma(\xi, \zeta)) = \gamma X(\lambda, \xi, \zeta) \text{ for all } \gamma \in O(2). \]
Define \( \tilde{\Gamma}G \) and \( \Gamma G \) by the followings:
\[
\tilde{\Gamma}G = \{ dG(X); X \in \hat{E} \} + \{ TG; T \text{ satisfies } (2.16) \text{ but may be singular matrix} \},
\]
where \( dG \) is the Jacobian matrix of \( G \).
\[
\varepsilon_\lambda = \{ \Lambda; \Lambda(\lambda) \text{is a } C^\infty \text{ - function} \}
\]
\[
\Gamma G = \tilde{\Gamma}G + \varepsilon_\lambda \frac{\partial G}{\partial \lambda}.
\]
If \( \Gamma(G + tP) = \Gamma G \) for all \( t \in [0, 1] \), then \( G + P \) is \( O(2) \)-equivalent to \( G \). If \( \Gamma G \) is an \( \mathbb{R} \)-linear subspace of \( \hat{E} \) of finite codimension and \( q_1, \ldots, q_k \in \hat{E} \) are such that
\[
\Gamma G \oplus \mathbb{R}q_1 \oplus \mathbb{R}q_2 \oplus \cdots \oplus \mathbb{R}q_k = \hat{E},
\]
then
\[
F(\alpha_1, \alpha_2, \ldots, \alpha_k, \lambda, \xi, \zeta) = G(\lambda, \xi, \zeta) + \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_k q_k
\]
is a universal unfolding of \( G \).

REMARK 2.3. By a universal unfolding, we mean that all small perturbation of \( G \) is \( O(2) \)-equivalent to \( F \) with an appropriate choice of \( \alpha_j \)'s, see [6] for the precise definition.

Our main results are roughly stated in the following way

1. Normal forms for (2.23,24) in the case of (I), (II),(III) are respectively

\[
(\varepsilon \lambda + bv)\xi + \bar{\zeta} \zeta
\]
\[
(\delta \lambda + bv)\zeta + \xi^2,
\]
\[
(\varepsilon \lambda + au + bv + cr)\xi + ev\bar{\zeta} \zeta
\]
\[
(\delta \lambda + au + bv)\zeta + \xi^2,
\]
\[
(\varepsilon \lambda + au + bv)\xi + \bar{\zeta} \zeta
\]
\[
(\delta \lambda + au + bv + cr)\zeta + du\xi^2,
\]

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where $\epsilon, \delta a, b, c, \hat{a}, \hat{b}, \hat{c}, d$ and $e$ are real constants.

2) According to cases (I), (II), (III), the followings are universal unfolding

\[(\epsilon \lambda + \alpha + (b + s_0)\nu)\epsilon + \xi \zeta\]

\[(\delta \lambda + \nu)\epsilon + \bar{\zeta} \xi^2,\]

\[(\epsilon \lambda + \alpha + (a + s_1)u + (b + s_2)v + (c + s_3)r)\epsilon + (\nu + \beta)\bar{\zeta} \xi \zeta\]

\[(\delta \lambda + \bar{\epsilon} u + \nu)\epsilon + \xi^2,\]

\[(\epsilon \lambda + \alpha + au + \nu)\epsilon + \bar{\zeta} \xi \zeta\]

\[(\delta \lambda + (\hat{a} + s_1)u + (\hat{b} + s_2)v + (\hat{c} + s_3)r)\epsilon + (\nu + \beta)\epsilon^2 \xi^2,\]

where $\alpha, \beta, s_j$ are unfolding parameters. One can show that only $\alpha$ and $\beta$ are essential parameters and $s_j$ are modal parameters.
§3. Computation of $\Gamma G$: Case (I). In this section we consider the case (I) and compute $\Gamma G$. We first transform $G$ into $(G_1/f_2, G_2/f_4)$. This is permitted by COROLLARY 2.4. We use the following notation.

$$\mathcal{M} = \{ f \in E; f(0; 0, 0, 0) = 0 \}$$

For $g \in E$, we put $< g > = \{ fg; f \in E \}$, which is an ideal in $E$ generated by $g$. Now we may assume without loss of generality that

(3.1) \hspace{1cm} G_1 = (\epsilon \lambda + au + bv + g_1)\zeta + \overline{\zeta}\zeta$

(3.2) \hspace{1cm} G_2 = (\delta \lambda + \hat{a}u + \hat{b}v + g_2)\zeta + \xi^2,$

where $\epsilon, \delta, a, \hat{a}, b, \hat{b}$ are real constants and $g_1, g_2 \in \mathcal{M}^2 < r >$.

**Proposition 3.1.** $G = (G_1, G_2)$ is $O(2)$-equivalent to the equation (3.1, 2) with $a = \hat{a} = 0$, although $b, \hat{b}, g_1$ and $g_2$ may be changed.

**Proof.** We change the variables: $(\xi, \zeta) \rightarrow (\xi', \zeta')$, where $\xi = \xi' + \alpha \xi' \zeta', \zeta = \zeta' + \beta \xi' \zeta', \alpha$ and $\beta$ are real constants ( see Proposition 2.2 ). We put

$$H_1(\lambda, \xi', \zeta') = G_1(\lambda, \xi' + \alpha \xi' \zeta', \zeta' + \beta \xi' \zeta'),$$

$$H_2(\lambda, \xi', \zeta') = G_2(\lambda, \xi' + \alpha \xi' \zeta', \zeta' + \beta \xi' \zeta'),$$

Writing $(\xi, \zeta)$ instead of $(\xi', \zeta')$, we have

$$H_1 = (\epsilon \lambda + (a + \beta)u + (b + \alpha)v + g_3)\xi + (1 + g_4)\overline{\xi}\zeta,$$

where $g_3 \in \mathcal{M}^2 < r >$ and $g_4 \in \mathcal{M}$. Therefore

$$H_1/(1 + g_4) = (\epsilon \lambda + (a + \beta)u + (b + \alpha)v + g_3)\xi + \overline{\xi}\zeta,$$

where $g_5 \in \mathcal{M}^2 < r >$. Similarly we have

$$H_2 = (\delta \lambda + (\hat{a} + 2\alpha)u + \hat{b}v + g_6)\xi + (1 + g_7)\xi^2,$$

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where \( g_5 \in \mathcal{M}^2 + < r > \), \( g_7 \in \mathcal{M} \). Therefore, if we choose \( \alpha \) and \( \beta \) so that \( 2\alpha + \hat{\alpha} = 0 \), and \( \alpha + \beta = 0 \), then \( (G_1, G_2) \) are equivalent to

\[
H_1/(1 + g_4) = (\epsilon \lambda + (b - \hat{\alpha}/2)v + g_5)\xi + \bar{\xi}\zeta, \\
H_2/(1 + g_7) = (\delta \lambda + \hat{\nu} + g_8)\xi + \xi^2, \\
g_5, \quad g_9 \in \mathcal{M}^2 + < r > .
\]

We now put

\[
(3.3) \quad G_1 = (\epsilon \lambda + bv + g_1)\xi + \bar{\xi}\zeta
\]

\[
(3.4) \quad G_2 = (\delta \lambda + \hat{\nu} + g_2)\xi + \xi^2, \\
g_1, \quad g_2 \in \mathcal{M}^2 + < r > .
\]

**Proposition 3.2.** If \( \epsilon \neq 0, \delta \neq 0 \), then we can assume that in (3.3, 4)

\[
(3.5) \quad \epsilon = \pm 1, \quad \delta = \pm 1
\]

\[
(3.6) \quad g_1(\lambda, 0, 0, 0) \equiv 0, \quad g_2(\lambda, 0, 0, 0) \equiv 0
\]

**Proof.** Consider the following transformation:

\[
(G_1, G_2) \to (\gamma^{-1}G_1(\frac{\lambda}{|\epsilon|}, \gamma\xi, \zeta), \gamma^{-2}G_2(\frac{\lambda}{|\epsilon|}, \gamma\xi, \zeta)).
\]

with \( \gamma = (|\delta|/|\epsilon|)^{1/2} \). Then we have (3.5). In order to prove that we can assume (3.6), we prepare a symbol:

\[
M_\lambda = \{ f \in \mathcal{E}_\lambda; f(0) = 0 \}
\]

We change variables as follows:

\[
(3.7) \quad (\xi, \zeta) \to ((1 + \varphi(\lambda))\xi, (1 + \psi(\lambda))\zeta),
\]

where \( \varphi, \psi \in M_\lambda \). We put \( g_j(\lambda; 0, 0, 0) = \eta_j(\lambda) \quad (j = 1, 2) \). Note that \( \eta \in M_\lambda^2 \). Then we have

\[
G_1(\lambda, (1 + \varphi)\xi, (1 + \psi)\zeta)/(1 + \varphi(\lambda))(1 + \psi(\lambda)) = (1 + \psi(\lambda))^{-1}(\epsilon \lambda + bv + \eta_1(\lambda) + g_3)\xi + \bar{\xi}\zeta
\]
\[ G_2(\lambda, (1 + \varphi) \xi, (1 + \psi) \zeta)/(1 + \varphi(\lambda))^{-2} = (1 + \psi(\lambda))/(1 + \varphi(\lambda))^{-2}(\delta\lambda + \hat{b}v + \eta_2(\lambda) + g_4)\zeta + \xi^2. \]

where \( g_j \in \mathcal{M}^2 \times \mathbb{R}^3 \). Clearly we can choose \( \varphi \) and \( \psi \) so that
\[
\frac{\lambda + \epsilon^{-1}\eta_1(\lambda)}{1 + \psi(\lambda)} = (\lambda + \delta^{-1}\eta_2(\lambda))(1 + \varphi(\lambda))^{-2}(1 + \psi(\lambda))
\]
Let \( \Lambda(\lambda) \) denote this quantity. Then
\[
G_1/(1 + \varphi(\lambda))(1 + \psi(\lambda)) = (\epsilon\Lambda + bv + g_3)\xi + \bar{\xi}\zeta
\]
\[
G_2/(1 + \varphi(\lambda))^{-2} = (\delta\Lambda + \hat{b}v + g_4)\zeta + \xi^2.
\]

Let after we assume (3.5,6)

**Proposition 3.3.** In (3.3,4) we can assume that, in addition to (3.6), \( g_1 \) and \( g_2 \) satisfy

\[
g_1, \quad g_2 \in < \lambda^2 u, \lambda^2 v, u^2, uv, v^2, r >.
\]

In other words, the coefficients of \( \lambda^2, \lambda u, \lambda v \) in \( g_1, g_2 \) can be assumed to vanish.

**Proof.** We already know that we may assume the coefficients of \( \lambda^2 \) in \( g_1 \) and \( g_2 \) are zero. We perform the coordinates change in COROLLARY 2.4 with \( h_1 \equiv 1, h_2 \equiv 1, \alpha_1 = \alpha_2 = 0 \). We express \( G \) as follows:

\[
G_1 = (\epsilon\lambda + bv + c_1 \lambda u + c_2 \lambda v + d_1 u^2 + d_2 uv + d_3 v^2 + g_3)\xi + \bar{\xi}\zeta
\]
\[
G_2 = (\delta\lambda + \hat{b}v + c_3 \lambda u + c_4 \lambda v + d_4 u^2 + d_5 uv + d_6 v^2 + g_4)\zeta + \xi^2,
\]

where \( g_3, g_4 \in \mathcal{M}^3 + < r >, g_3(\lambda, 0, 0, 0) \equiv 0, g_4(\lambda, 0, 0, 0) \equiv 0 \). We have only to show that we can make \( c_j \) 's vanish. By the coordinates change stated above, we have

\[
H_1 = (\epsilon\lambda + bv + c_1 \lambda u + (c_2 + 2\delta \beta_1)\lambda v + d_1 u^2 + d_2 uv + (d_3 + 2\hat{b} \beta_1)v^2 + g_5)\xi
\]
\[
+ (1 - 2\gamma_1 u)\bar{\xi}\zeta
\]

where \( g_5 \in \mathcal{M}^3 + < r >. \) Similarly we have

\[
H_2 = (\delta\lambda + \hat{b}v + (c_3 + 2\beta_2 \epsilon)\lambda u + c_4 \lambda v + d_4 u^2 + (d_5 + 2\beta_2 b)uv + d_6 v^2 + g_6)\zeta
\]
\[
+ (1 - 2\gamma_2 v)\xi^2.
\]
where $g_6 \in M^3 + < r >$. It is important to note that by the transformation in question, we have $g_5(\lambda, 0, 0, 0) \equiv 0, g_6(\lambda, 0, 0, 0) \equiv 0$. Therefore we have
\[
H_1/(1 - 2\gamma_1 u) = (\epsilon\lambda + bv + (c_1 + 2\gamma_1\epsilon)\lambda u + (c_2 + 2\delta\beta_1)\lambda v + d_1u^2 + (d_2 + 2\gamma_1 b)uv + (d_3 + 2b\beta_1)u^2 + g_7)\xi + \bar{\xi}\zeta,
\]
\[
H_2/(1 - 2\gamma_2 v) = (\delta\lambda + \bar{\lambda}v + (c_3 + 2\beta_2\epsilon)\lambda u + (c_4 + 2\gamma_2\delta)\lambda v + d_4u^2 + (d_5 + 2\beta_2 b)uv + (d_6 + 2\gamma_2 b)u^2 + g_8)\zeta + \xi^2,
\]
where $g_7, g_8 \in M^3 + < r >, g_7(\lambda, 0, 0, 0) \equiv 0, g_8(\lambda, 0, 0, 0) \equiv 0$. Choosing constants so that
\[
c_1 + 2\gamma_1\epsilon = 0, \quad c_2 + 2\delta\beta_1 = 0, \quad c_3 + 2\beta_2\epsilon = 0, \quad c_4 + 2\gamma_2\delta = 0,
\]
we can complete the proof. \(\square\)

From now on we consider

\[G_1 = (\epsilon\lambda + bv + p)\xi + \bar{\xi}\zeta\]

(3.9)

\[G_2 = (\delta\lambda + \bar{\lambda}v + q)\zeta + \xi^2,
\]

(3.10)

under the conditions (3.5), and

\[p, \quad q \in < \lambda^2u, \lambda^2v, u^2, uv, v^2, r >.
\]

(3.12)

We compute $\Gamma G$ concretely in the remaining part of this section. We first consider $dG(X)$. We put $X_1 = (\xi, 0), X_2 = (\bar{\xi}\zeta, 0), X_3 = (0, \zeta), X_4 = (0, \xi^2)$. Then
\[
\{dG(X); X \in \hat{E}\}
\]
is a $E$-module generated by $dG(X_1), dG(X_2), dG(X_3)$ and $dG(X_4)$. In the complex notation we have
\[
dG(f, g) = (\frac{\partial G_1}{\partial \xi}f + \frac{\partial G_1}{\partial \zeta}\bar{f} + \frac{\partial G_1}{\partial \bar{\xi}}g + \frac{\partial G_1}{\partial \bar{\zeta}}\bar{g},
\]
\[
\frac{\partial G_2}{\partial \xi}f + \frac{\partial G_2}{\partial \zeta}\bar{f} + \frac{\partial G_2}{\partial \bar{\xi}}g + \frac{\partial G_2}{\partial \bar{\zeta}}\bar{g}).
\]

We compute the coefficients of $dG$ as follows:
\[
\frac{\partial G_1}{\partial \xi} = \epsilon\lambda + bv + p + up_u + pr_u\frac{\partial r}{\partial \xi}, \quad \frac{\partial G_1}{\partial \bar{\xi}} = \zeta + pu\xi^2 + pr_u\frac{\partial r}{\partial \bar{\xi}},
\]

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\[
\frac{\partial G_1}{\partial \zeta} = \bar{\zeta} + (b\zeta + p_v\zeta + p_r \frac{\partial r}{\partial \zeta})\zeta, \quad \frac{\partial G_1}{\partial \zeta} = (b\zeta + p_v\zeta + p_r \frac{\partial r}{\partial \zeta})\zeta,
\]
\[
\frac{\partial G_2}{\partial \zeta} = 2\xi + (q_u\bar{\zeta} + q_r \frac{\partial r}{\partial \zeta})\zeta, \quad \frac{\partial G_2}{\partial \zeta} = (q_u\xi + q_r \frac{\partial r}{\partial \zeta})\zeta,
\]
\[
\frac{\partial G_2}{\partial \zeta} = \delta \lambda + \hat{b}v + q + u v + q_r \frac{\partial r}{\partial \zeta}, \quad \frac{\partial G_2}{\partial \zeta} = (\hat{b} \xi + q_v \xi + q_r \frac{\partial r}{\partial \zeta})\zeta,
\]
where the subscript means differentiation. We also have the following formulas:

\[
(3.12) \quad \xi \frac{\partial r}{\partial \zeta} + \bar{\zeta} \frac{\partial r}{\partial \zeta} = 2r, \quad \bar{\zeta} \xi \frac{\partial r}{\partial \zeta} + \xi \bar{\zeta} \frac{\partial r}{\partial \zeta} = 2uv,
\]
\[
\xi \frac{\partial r}{\partial \zeta} + \bar{\zeta} \frac{\partial r}{\partial \zeta} = r, \quad \xi^2 \frac{\partial r}{\partial \zeta} + \bar{\zeta}^2 \frac{\partial r}{\partial \zeta} = u^2.
\]

Using these formulas, we compute \(dG(X_j)\) \((j = 1, 2, 3, 4)\) as follows:

\[
dG(X_1) = (\epsilon \lambda + bv + p + 2up_u + 2rp_r)X_1 + X_2 + (2uq_u + 2rq_r)X_3 + 2X_4,
\]
\[
dG(X_2) = (v + 2rp_u + 2uvp_r)X_1 + (\epsilon \lambda + bv + p)X_2 + (2u + 2rq_u + 2uvq_r)X_3,
\]
\[
dG(X_3) = (2bv + 2vp_v + rp_r)X_1 + X_2 + (\delta \lambda + 3bv + q + 2vg_v + rq_r)X_3,
\]
\[
dG(X_4) = (u + 2br + 2rp_v + u^2p_r)X_1 + (2br + 2rq_v + u^2q_r)X_3 + (\delta \lambda + \hat{b}v + q)X_4.
\]

We next consider \(\{T G; T \text{ satisfies } (2.16), \text{ but may be singular} \}.\) This space is an \(E\)-module generated by the following 8 elements:

\[
T_1G = (\epsilon \lambda + bv + p)X_1 + X_2
\]
\[
T_2G = [u(\epsilon \lambda + bv + p) + 2r]X_1 - uX_2
\]
\[
T_3G + T_4G = [2v(\delta \lambda + \hat{b}v + q) + 2r]X_1
\]
\[
T_3G - T_4G = 2rX_1 - 2uX_2
\]
\[
T_5G + T_6G = [2u(\epsilon \lambda + bv + p) + 2r]X_3
\]
\[
T_5G - T_6G = 2rX_3 - 2vX_4
\]
\[
T_7G = [v(\delta \lambda + \hat{b}v + q) + 2r]X_3 - vX_4
\]
\[
T_8G = (\delta \lambda + \hat{b}v + q)X_3 + X_4
\]
and is multiples of these 8 elements. Note that $T_2G = uT_1G + (T_3G - T_4G)$. Therefore we can omit $T_2G$ from the generators. Similarly we can omit $T_7G$. Since we have

\begin{equation}
(3.13) \quad \text{is}X_1 = rX_1 - uX_2, \quad \text{is}X_2 = uvX_1 - rX_2,
\end{equation}

\begin{equation}
\text{is}X_3 = -rX_3 + vX_4, \quad \text{is}X_4 = -u^2X_3 + rX_4,
\end{equation}

the $E$-module in question is generated by the following 8 elements:

\begin{align*}
(\varepsilon\lambda + bv + p)X_1 + X_2, [v(\delta\lambda + bv + q) + r]X_1, \quad rX_1 - uX_2, \\
[u(\varepsilon\lambda + bv + p) + r]X_3, rX_3 - vX_4, \quad (\delta\lambda + bv + q)X_3 + X_4, \\
 uvX_1 - rX_2, \quad u^2X_3 - rX_4.
\end{align*}

Using $T_1G$ and $T_8G$ to eliminate the term involving $\lambda$ in $X_1$ and $X_3$, we have now proved

**Proposition 3.4.** $E$-module $\tilde{T}G$ is generated by the following 12 elements

\begin{equation}
W_1 = (2up_u + 2rp_r)X_1 + (2uq_u + 2rq_r)X_3 + 2X_4,
\end{equation}

\begin{equation}
W_2 = (v + 2rp_u + uvpr)X_1 + (\varepsilon\lambda + bv + p)X_2 + (2u + 2rq_u + 2uvq_r)X_3,
\end{equation}

\begin{equation}
W_3 = (2bv + 2vp_v + rp_r)X_1 + X_2 + (2bv + 2vp_v + rq_r)X_3 - X_4,
\end{equation}

\begin{equation}
W_4 = (u + 2br + 2rp_v + u^2p_r)X_1 + (2bv + 2rq_v + u^2q_r)X_3 + (\delta\lambda + bv + q)X_4.
\end{equation}

\begin{equation}
W_5 = (\varepsilon\lambda + bv + p)X_1 + X_2,
\end{equation}

\begin{equation}
W_6 = [v((b - \varepsilon b)v + q - \varepsilon bp) + r]X_1 - \varepsilon vX_2
\end{equation}

\begin{equation}
W_7 = rX_1 - uX_2,
\end{equation}

\begin{equation}
W_8 = [u((b - \varepsilon b)v + p - \varepsilon q) + r]X_3 - \varepsilon vX_4
\end{equation}

\begin{equation}
W_9 = rX_3 - vX_4
\end{equation}

\begin{equation}
W_{10} = (\delta\lambda + bv + q)X_3 + X_4,
\end{equation}

\begin{equation}
W_{11} = uvX_1 - rX_2,
\end{equation}

\begin{equation}
W_{12} = u^2X_3 - rX_4.
\end{equation}

We then make 8 elements which belong to $\tilde{T}G$ and contain only $X_1$ and $X_3$.

\begin{equation}
Y_1 = \frac{1}{2}\varepsilon\delta uW_1 + W_8 = \varepsilon\delta(u^2p_u + ru_p_r)X_1
\end{equation}
\[ + [\epsilon \delta (u^2 q_u + r u q_r) + r + u v (b - \epsilon \delta \hat{b}) + u(p - \epsilon \delta q)] X_3 \]

\[ Y_2 = \frac{1}{2} v W_1 + W_9 = (u v p_u + r v p_r) X_1 + [u v q_u + r v q_r + r] X_3 \]

\[ Y_3 = u W_3 + W_7 - \epsilon \delta W_8 = [2 b u v + 2 u v p_v + r u p_r + r] X_1 \]

\[ + [2 b u v + 2 u v q_v + r u q_r - \epsilon \delta r - (\epsilon \delta \hat{b} - b) u v - u(\epsilon \delta p - q)] X_3 \]

\[ Y_4 = v W_3 + \epsilon \delta W_6 - W_9 \]

\[ = [2 b v^2 + 2 v^2 p_v + r u p_r + \epsilon \delta r + (\epsilon \delta \hat{b} - b) v^2 + (\epsilon \delta q - p) v] X_1 \]

\[ [2 b v^2 + 2 v^2 q_v + r v q_r - r] X_3 \]

\[ Y_5 = \frac{1}{2} r W_1 + W_{12} \]

\[ = [r u p_u + r^2 p_r] X_1 + [r u q_u + r^2 q_r + u^2] X_3 \]

\[ Y_6 = r W_3 + W_{11} - W_{12} \]

\[ = [2 b r u + 2 r u p_v + r^2 p_r + u v] X_1 + [2 b r v + 2 r v q_v + r^2 q_r - u^2] X_3 \]

\[ Y_7 = v W_2 + \epsilon \delta (\epsilon \lambda + b v + p) W_6 - (\epsilon \delta r + (\epsilon \delta \hat{b} - b) v^2 + v(\epsilon \delta q - p)) W_6 \]

\[ - \epsilon \delta W_{11} - (v(b - \epsilon \delta b) + q - \epsilon \delta p) W_6 \]

\[ = \{v^2 + 2 r u p_u + 2 u v^2 p_r - \epsilon \delta u v - [(\hat{b} - \epsilon \delta b)v + q - \epsilon \delta][r + (\hat{b} - \epsilon \delta b)v^2 + (q - \epsilon \delta p)v]\} X_1 \]

\[ + [2 u v + 2 r u q_u + 2 u v^2 q_r] X_3 \]

\[ Y_8 = u W_4 + \epsilon \delta (\delta \lambda + \hat{b} u + q) W_8 - (\epsilon \delta r + (\epsilon \delta b - \hat{b}) u v + u(\epsilon \delta p - q)) W_{10} \]

\[ - \epsilon \delta W_{12} - (v(b - \epsilon \delta b) + p - \epsilon \delta q) W_8 \]

\[ = (u^2 + 2 b r u + 2 r u p_v + u^3 p_r) X_1 + [2 b r u + 2 r u q_v + u^3 q_r - \epsilon \delta u^2 \]

\[ - [v(b - \epsilon \delta b) + p - \epsilon \delta q] [r + (b - \epsilon \delta b) u v + u(p - \epsilon \delta q)] \} X_3 \]

We are now in a position to prove
Proposition 3.5. If $\hat{b} \neq 0$, $\hat{b} \neq \varepsilon \delta \hat{b}$, then
\[ u^2 X_j, u v X_j, v^2 X_j, r X_j \quad (j = 1, 3) \]
belong to $\widetilde{\Gamma} G$.

Proof. We put $V_j = u^{3-j} v^j X_1 \quad (j = 1, 2, 3)$, $V_{j+3} = u^{3-j} v^{j+1} X_3 \quad (j = 1, 2, 3)$, $V_7 = r X_1$, $V_8 = r X_3$. By the assumption on $p$ and $q$, and the fact that $Y_1 \in \Gamma G$, we have
\[ V_8 + (b - \varepsilon \delta \hat{b})V_5 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >. \]
Similarly, by $Y_2, \cdots, Y_8$, we have
\[ V_8 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ 2b V_2 + V_7 + (3 \hat{b} - \varepsilon \delta \hat{b})V_5 - \varepsilon \delta V_8 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ (b + \varepsilon \delta \hat{b})V_3 + \varepsilon \delta V_7 + 2b V_6 - V_8 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ V_4 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ V_2 - V_4 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ V_3 - \varepsilon \delta V_2 + 2V_5 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
\[ V_1 - \varepsilon \delta V_4 \in \widetilde{\Gamma} G + M < V_1 > + M < V_2 > + \cdots + M < V_8 >, \]
respectively. These relations are rewritten in the following form:
\[ AV \in (\widetilde{\Gamma} G)^8, \]
where $V = (V_1, V_2, \cdots, V_8)$, and $A$ is a $8 \times 8$ matrix whose entries are in $E$ and is reduced to a $R$-matrix at $(\lambda, u, v, r) = (0, 0, 0, 0)$ which we denote by $A_0$. $A_0$ is given by

\[
A_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & b - \varepsilon \delta \hat{b} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 2b & 0 & 0 & 3 \hat{b} - \varepsilon \delta \hat{b} & 0 & 1 & -\varepsilon \delta \\
0 & 0 & b + \varepsilon \delta \hat{b} & 0 & 0 & 2b & \varepsilon \delta & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -\varepsilon \delta & 1 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & 0 & -\varepsilon \delta & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In order to prove the present proposition, it is sufficient to show that $\det A_0 \neq 0$ under the conditions $\hat{b} \neq 0$, $\hat{b} \neq \varepsilon \delta \hat{b}$. This is indeed the case, since we have by the direct computation
\[ \det A_0 = 2b(b - \varepsilon \delta \hat{b}). \]
PROPOSITION 3.6. Under the same conditions as in the previous proposition,

\[ M^2 X_1, \quad M^2 X_3, \quad M X_2, \quad M X_4, \quad < r > X_1 \text{ and } < r > X_3 \]

are contained in \( \tilde{\Gamma} G \).

Proof. Since \( \tilde{\Gamma} G \) is an E-module, we have only to show that

\[ \lambda^2 X_j, \quad \lambda u X_j, \quad \lambda v X_j \quad (j = 1, 3); \quad \lambda X_k, \quad u X_k, \quad v X_k \quad r X_k \quad (k = 2, 4) \]

belong to \( \tilde{\Gamma} G \). By \( uW_1, vW_1 \) and \( rW_1 \), we see that \( uX_4, vX_4 \) and \( rX_4 \) belong to \( \tilde{\Gamma} G \). Then \( uX_2, vX_2 \) and \( rX_2 \) belong \( \tilde{\Gamma} G \) by \( uW_3, vW_3 \) and \( rW_3 \). Considering \( uW_5 \) and \( vW_5 \), we obtain \( \lambda u X_1, \lambda v X_1 \in \tilde{\Gamma} G \). Similarly we obtain \( \lambda u X_3, \lambda v X_3 \in \tilde{\Gamma} G \) by \( W_{10} \). By \( \lambda W_1 \) and \( \lambda W_3 \), we have \( \lambda X_2, \lambda X_4 \in \tilde{\Gamma} G \). Finally, considering \( \lambda W_5 \) and \( \lambda W_{10} \) yields \( \lambda^2 X_1, \lambda^2 X_3 \in \tilde{\Gamma} G \). \( \square \)

We now see that \( \tilde{\Gamma} G \) is equal to the E-module generated by \( M^2 X_j, rX_j, M X_k \)
\((j = 1, 2; k = 2, 4)\) and

\[ Z_1 = X_4, \quad Z_2 = v X_1 + 2 u X_3, \quad Z_3 = 2 b v X_1 + X_2 + 2 b v X_3 - X_4, \quad Z_4 = u X_1, \]

\[ Z_5 = (\varepsilon \lambda + b v) X_1 + X_2, \quad Z_6 = (\delta \lambda + \varepsilon b) X_3 + X_4 \]

Therefore \( \tilde{\Gamma} G \) is independent of \( p \) and \( q \).

THEOREM 3.1. If \( \hat{b} \neq 0, \hat{b} \neq \varepsilon \delta b \), then \( \hat{\Gamma} G \) is independent of \( p \) and \( q \). Moreover we have

\[ \Gamma G \oplus RX_1 \oplus RVX_1 = \hat{E}. \]

Proof. Since \( M^2 X_1, M^2 X_3 \subset \tilde{\Gamma} G \), we have

\[ \Gamma G = \tilde{\Gamma} G + \mathbb{R} \frac{\partial G}{\partial \lambda} + \mathbb{R} \lambda \frac{\partial G}{\partial \lambda}. \]

We have

\[ \frac{\partial G}{\partial \lambda} = ((\varepsilon + p \lambda) \xi, (\delta + q \lambda) \zeta) \]

\[ \lambda \frac{\partial G}{\partial \lambda} = (\lambda (\varepsilon + p \lambda) \xi, \lambda (\delta + q \lambda) \zeta). \]

By the assumption on \( p \) and \( q \), the functions \( p \lambda, q \lambda \) belong to \( M^2 + < r > \). therefore we have

\[ \Gamma G = \tilde{\Gamma} G + \mathbb{R} (\varepsilon X_1 + \delta X_3) + \mathbb{R} (\varepsilon \lambda X_1 + \delta \lambda X_3) \]

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which is independent of $p$ and $q$. By $Z_1 \sim Z_6$ the following elements belong to $\tilde{\Gamma}G$:

(3.13) \[ X_4, \quad uX_1, \quad vX_1 + 2uX_3, \]
\[ 2buvX_1 + X_2 + 2buvX_3, \quad (\epsilon\lambda + bv)X_1 + X_2, \quad (\delta\lambda + b\nu)X_3. \]

The last two and the fact that $\epsilon\lambda X_1 + \delta\lambda X_3 \in \Gamma G$ imply

\[ bvX_1 + X_2 + b\nu X_3 \in \Gamma G \]

This and the fact that the fourth element in (3.14) belongs to $\Gamma G$ imply that

\[ X_2, \quad bvX_1 + b\nu X_3 \in \Gamma G. \]

The remaining part of the proof is now easy. \[ \square \]

**Theorem 3.2.** The mapping (3.3,4) is $O(2)$-equivalent to

(3.15) \[ G_1 = (\epsilon\lambda + bv)\xi + \bar{\xi}\zeta \]

(3.16) \[ G_2 = (\delta\lambda + b\nu)\xi + \xi^2, \]

if $b \neq 0, b \neq \epsilon\delta$. A universal unfolding of (3.15,16) is given by

(3.17) \[ F(\alpha, \beta, \lambda, \xi, \zeta) = (\alpha + \epsilon\lambda + (b + \beta)\nu)\xi + \bar{\xi}\zeta, (\delta\lambda + b\nu)\zeta + \xi^2). \]
§4. Computation of $\Gamma G$ : Case $(II)$. In this section we consider the case $(II)$ and compute $\Gamma G$. We first transform $G$ into $(G_1, G_2/f_4)$. Note that we can write it as

\begin{equation}
G_1 = (\epsilon \lambda + p_1 + q_1)\xi + (\eta \lambda + p_2 + q_2)\xi \zeta
\end{equation}

\begin{equation}
G_2 = (\delta \lambda + p_3 + q_3)\zeta + \xi^2,
\end{equation}

where $\epsilon, \delta, \eta$ are real constants, $p_1, p_2$ and $p_3$ belong to $M^2_\lambda$ and $q_1, q_2, q_3 \in < u > + < v > + < r >$.

**Proposition 4.1.** Assume that $\epsilon \neq 0$. Then in (4.1,2) we may assume, without loss of generality, that $\eta = 0$, $\epsilon = \pm 1$, $\delta = \pm 1$.

**Proof.** By $\lambda \rightarrow \lambda/|\epsilon|$ and $(G_1, G_2) \rightarrow (G_1, \frac{\xi}{|\epsilon|}G_2)$, we can assume that $\epsilon = \pm 1$, $\delta = \pm 1$.

We use the following coordinate change:

$$(\xi, \zeta) \rightarrow (\xi + \alpha \xi \zeta, \zeta),$$

where $\alpha$ is a real parameter. Then $G$ is transformed to

$$H_1 = (\epsilon \lambda + p_1 + q_4)(\xi + \alpha \xi \zeta) + [\eta \lambda + p_2 + q_3](\xi + \phi \xi \zeta) \zeta,$$

$$H_2 = (\delta \lambda + p_3 + q_6)\zeta + (\xi + \alpha \xi \zeta)^2.$$

Here and hereafter $q_j \in < u > + < v > + < r >$. We can write these equations as follows:

$$H_1 = (\epsilon \lambda + p_1 + q_4)(\xi + \alpha \xi \zeta) + [\eta \lambda + p_2 + q_7](\xi \zeta + \alpha \nu \xi),$$

$$= (\epsilon \lambda + p_1 + q_8)\xi + [\eta \lambda + p_2 + \epsilon \alpha \lambda + \alpha p_1 + q_9] \xi \zeta,$$

$$H_2 = (\delta \lambda + p_3 + q_{10})\zeta + \xi^2 + 2\alpha \nu \xi + \alpha^2 \zeta(2r - \xi^2 \zeta)$$

$$= (\delta \lambda + p_3 + q_{11})\zeta + (1 - \nu \alpha^2) \xi^2.$$

Therefore

$$H_2/(1 - \nu \alpha^2) = (\delta \lambda + p_3 + q_{12})\zeta + \xi^2.$$

To complete the proof, it is enough to choose $\alpha$, so that

$$\eta + \epsilon \alpha = 0$$

$q$.

Henceforth we consider

\begin{equation}
G_1 = (\epsilon \lambda + au + bv + cr + f)\xi + (du + ev + kr + g)\xi \zeta,
\end{equation}

\begin{equation}
G_2 = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + h)\zeta + \xi^2,
\end{equation}

where $f, g$ and $h$ belong to $M^2$. 

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PROPOSITION 4.2. If \( ac \neq 0 \), then in (4.3,4) we may assume, without loss of generality, that \( d = \hat{c} = k = 0 \).

Proof. Consider the change of variables:

\[
H_1(\lambda, \xi, \zeta) = G_1(\lambda, \xi, \zeta) + \gamma_1 \xi (\overline{\zeta}G_2 - \zeta \overline{G_2}),
\]

\[
H_2(\lambda, \xi, \zeta) = G_2(\lambda, \xi, \zeta) + \alpha_2 \xi^2 \overline{G_2}.
\]

Then we have

\[
H_1(\lambda, \xi, \zeta) = (\epsilon \lambda + au + bv + (c + 2\gamma_1) r + g_1) \xi \\
+ ((d - 2\gamma_1) u + ev + kr + g_2) \overline{\xi} \zeta,
\]

(4.5)

\[
H_2(\lambda, \xi, \zeta)/(1 - \alpha_2 v) = (\delta \lambda + \hat{a} u + \hat{b} v + (\hat{c} + 2\alpha_2) r + g_3) \zeta + (1 - \alpha_2 v) \xi^2,
\]

where \( g_1, g_2 \) and \( g_3 \in M^2 \). The second equality is further transformed to

(4.6)

\[
H_2(\lambda, \xi, \zeta)/(1 - \alpha_2 v) = (\delta \lambda + \hat{a} u + \hat{b} v + (\hat{c} + 2\alpha_2) r + g_4) \zeta + \xi^2,
\]

with \( g_4 \in M^2 \). We transform (4.4,6) by the following coordinate change:

\[(\xi, \zeta) \rightarrow (\xi, \zeta + \alpha_0 \xi^2).\]

Note that \( v \) is replaced by \( v + 2\alpha_0 r + \alpha^2 u^2 \) and that \( r \) is replaced by \( r + \alpha_0 u^2 \). Now (4.4,6) are transformed to

\[
H_1' = (\epsilon \lambda + au + bv + (c + 2\gamma_1 + 2\alpha_0 b) r + g_5) \xi \\
+ ((d - 2\gamma_1) u + ev + (k + 2\alpha_0 e) r + \overline{g_6}) \overline{\xi} \zeta,
\]

\[
H_2' = (\delta \lambda + \hat{a} u + \hat{b} v + (\hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b}) r + g_7) \zeta + (1 + g_1) \xi^2,
\]

where \( g_j \in M^2 \) and \( g_1 \in M \). Consequently

\[
H_2'/(1 + g_1) = (\delta \lambda + \hat{a} u + \hat{b} v + (\hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b}) r + g_8) \zeta + \xi^2,
\]

where \( g_8 \in M^2 \). Choosing \( \gamma_1, \alpha_0 \) and \( \alpha_2 \) so that

\[
k + 2\alpha_0 e, \quad d - 2\gamma_1 = 0, \quad \hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b} = 0,
\]

the proof is completed. []

Hereafter we consider

(4.7) \[
G_1 = (\epsilon \lambda + au + bv + cr + f) \xi + (ev + g) \overline{\xi} \zeta,
\]

(4.8) \[
G_2 = (\delta \lambda + \hat{a} u + \hat{b} v + h) \zeta + \xi^2,
\]

where \( f, g \) and \( h \) belong to \( M^2 \).
PROPOSITION 4.3. If the determinant of $K$ which appears in the course of the proof is assumed to be nonzero, then we can assume in (4.7,8) that

\begin{align*}
i) & \quad f \in \mathcal{M}^3 + r\mathcal{M} \\
in other words, the coefficients of $\lambda^2, \lambda u, \lambda v, u^2, uv, v^2$ of $f$ vanish, \\
ii) & \quad the coefficients of $\lambda^2, \lambda u, \lambda v, uv, v^2$ of $h$ vanish \\
iii) & \quad the coefficients of $\lambda^2, \lambda v, v^2$ of $g$ vanish
\end{align*}

Proof. We transform $G_1$ and $G_2$ to

\begin{align*}H_1(\lambda, \xi, \zeta) &= (1 + \alpha_0 \lambda + \alpha_1 u + \alpha_2 v)G_1(\lambda, (1 + \beta_0 \lambda + \beta_1 u + \beta_2 v)\xi, (1 + \gamma_0 \lambda + \gamma_1 u + \gamma_2 v)\zeta), \\
\text{and} \\
H_2(\lambda, \xi, \zeta) &= (1 + \beta_0 \lambda + \beta_1 u + \beta_2 v)^{-2}G_2(\lambda, (1 + \beta_0 \lambda + \beta_1 u + \beta_2 v)\xi, (1 + \gamma_0 \lambda + \gamma_1 u + \gamma_2 v)\zeta),
\end{align*}

respectively. Let $f_0, g_0$ and $h_0$ denote the quadratic parts of $f, g$ and $h$, respectively such that $f_0, g_0$ and $h_0$ are independent of $r$. Therefore we can write $f = f_0 + f_1, g = g_0 + g_1, h = h_0 + h_1$, where $f_1, g_1, h_1 \in \mathcal{M}^3 + r\mathcal{M}$. From now on until the end of the proof of the present proposition, $f_j, g_j, h_j \ (1 \leq j)$ imply the elements of $\mathcal{M}^3 + r\mathcal{M}$. We now have

\begin{align*}H_1 &= \{e\lambda + au + bv + cr + f_0 + e(\alpha_0 + \beta_0)\lambda^2 + (3a\beta_0 + e\beta_1 + e\alpha_1 + a\alpha_0)\lambda u \\
&\quad + (2b\gamma_0 + b\beta_0 + e\beta_2 + e\alpha_2 + b\alpha_0)\lambda v + (3a\beta_1 + a\alpha_1)u^2 \\
&\quad + (3a\beta_2 + 2b\gamma_1 + b\beta_1 + b\alpha_1 + a\alpha_2)uv + (2b\gamma_2 + b\beta_2 + b\alpha_2)v^2 + f_2\}\xi, \\
H_2 &= \{e\lambda + \hat{u} + b\hat{v} + h_0 + \delta(\gamma_0 - 2\beta_0)\lambda^2 + (\delta\gamma_0 + \delta\gamma_1 - 2\delta\beta_1)\lambda u \\
&\quad + (3b\gamma_0 + \delta\gamma_2 - 2\delta\beta_0 - 2\delta\beta_2)\lambda v + \hat{u}\gamma_1 u^2 + (3b\gamma_1 + \hat{u}\gamma_2 - 2\hat{u}\beta_1)uv \\
&\quad + (3b\gamma_2 - 2\hat{u}\beta_2)v^2 + h_2\}\zeta + \xi^2.
\end{align*}

We change $\lambda$ to $\lambda + \eta_1 \lambda^2$, then transform $H_1$ and $H_2$ as follows:

\begin{align*}K_1 &= H_1 + \eta_2 \xi(\overline{\xi}H_2 + \overline{H_2}), \\
K_2 &= H_2 + \eta_3 \xi(\overline{\xi}H_1 + \overline{H_1}).
\end{align*}
We have
\[ K_1 = \{ \epsilon \lambda + au + bv + (c + 2\eta_2) r + f_0 + \epsilon(\alpha_0 + \beta_0 + \eta_1) \lambda^2 \]
\[ + (3a\beta_0 + \epsilon\beta_1 + \epsilon\alpha_1 + a\alpha_0) \lambda u + (2b\gamma_0 + b\beta_0 + \epsilon\beta_2 + \epsilon\alpha_2 + b\alpha_0 + 2\delta\eta_2) \lambda v \]
\[ + (3a\beta_1 + a\alpha_1) u^2 + (3a\beta_2 + 2b\beta_1 + b\beta_1 + b\alpha_1 + a\alpha_2 + 2\alpha\eta_2)uv \]
\[ + (2b\gamma_2 + b\beta_2 + b\alpha_2 + 2\beta\eta_2) v^2 + f_3 \} \xi \]
\[ + \{ ev + g_0 + \epsilon(\alpha_0 + \beta_0 + 3\gamma_0) \lambda v + \epsilon(\alpha_1 + \beta_1 + 3\gamma_1) uv \]
\[ + \epsilon(\alpha_2 + \beta_2 + 3\gamma_2) v^2 + g_3 \} \bar{\xi} \zeta, \]
and
\[ K_2 = \{ \delta \lambda + \hat{a} u + \hat{b} v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1) \lambda^2 \]
\[ + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1 + 2\epsilon\eta_3) \lambda u + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\delta\beta_0 - 2\delta\beta_2) \lambda v \]
\[ + (\hat{a}\gamma_1 + 2\epsilon\eta_3) u^2 + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{a}\beta_1 + 2\epsilon\eta_3)uv \]
\[ + (3\beta\gamma_2 - 2\beta\beta_2) v^2 + h_3 \} \zeta + \xi^2. \]

Finally we change $\xi$ to $\xi + \eta_4 \lambda \bar{\xi} \zeta + \eta_5 v \bar{\xi} \zeta$. Then $K_1$ and $K_2$ are transformed to the following $K'_1$ and $K'_2$, respectively.

\[ K'_1 = \{ \epsilon \lambda + au + bv + (c + 2\eta_2) r + f_0 + \epsilon(\alpha_0 + \beta_0 + \eta_1) \lambda^2 \]
\[ + (3a\beta_0 + \epsilon\beta_1 + \epsilon\alpha_1 + a\alpha_0) \lambda u + (2b\gamma_0 + b\beta_0 + \epsilon\beta_2 + \epsilon\alpha_2 + b\alpha_0 + 2\delta\eta_2) \lambda v \]
\[ + (3a\beta_1 + a\alpha_1) u^2 + (3a\beta_2 + 2b\beta_1 + b\beta_1 + b\alpha_1 + a\alpha_2 + 2\alpha\eta_2)uv \]
\[ + (2b\gamma_2 + b\beta_2 + b\alpha_2 + 2\beta\eta_2) v^2 + f_4 \} \xi \]
\[ + \{ ev + g_0 + \epsilon\eta_4 \lambda^2 + a\eta_4 \lambda u + (e(\alpha_0 + \beta_0 + 3\gamma_0) + b\eta_4 + \epsilon\eta_5) \lambda v + (e(\alpha_1 + \beta_1 + 3\gamma_1) + a\eta_5) uv \]
\[ + (e(\alpha_2 + \beta_2 + 3\gamma_2) + b\eta_5) v^2 + g_4 \} \bar{\xi} \zeta, \]
and
\[ K'_2 = \{ \delta \lambda + \hat{a} u + \hat{b} v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1) \lambda^2 \]
\[ + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1 + 2\epsilon\eta_3 + 2\eta_4) \lambda u + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\hat{a}\beta_0 - 2\delta\beta_2) \lambda v \]
\[ + (\hat{a}\gamma_1 + 2\epsilon\eta_3) u^2 + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{a}\beta_1 + 2\epsilon\eta_3 + 2\eta_5) uv \]
\[ + (3\beta\gamma_2 - 2\beta\beta_2) v^2 + h_4 \} \zeta + (1 - v(\eta_4 \lambda + \eta_5 v)^2) \xi^2. \]
Dividing $K'_2$ by $1 - v(\eta_4 \lambda + \eta_5 v)^2$, we have

$$K'_2/(1 - v(\eta_4 \lambda + \eta_5 v)^2) = \{\delta \lambda + \hat{\alpha} u + \hat{\beta} v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1)\lambda^2
+ (\hat{\alpha} \gamma_0 + \delta \gamma_1 - 2\delta \beta_1 + 2\epsilon \eta_3 + 2\eta_4) \lambda u + (3\hat{\beta} \gamma_0 + \delta \gamma_2 - 2\hat{\beta} \beta_0 - 2\delta \beta_2) \lambda v
+ (\hat{\alpha} \gamma_1 + 2\epsilon \eta_3) u^2 + (3\hat{\beta} \gamma_1 + \hat{\alpha} \gamma_2 - 2\hat{\beta} \beta_1 + 2\beta \eta_3 + 2\eta_5) uv
+ (3\hat{\beta} \gamma_2 - 2\hat{\beta} \beta_2) v^2 + h_5\} \zeta + \xi^2.$$}

From this computation we see that we can make the coefficients listed in the proposition vanish, if the following $14 \times 14$ matrix is nonsingular.

$$
\begin{pmatrix}
\varepsilon & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\
\alpha & \varepsilon & 0 & 3\alpha & \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & \varepsilon & b & 0 & \varepsilon & 2b & 0 & 0 & 0 & 2\delta & 0 & 0 & 0 \\
0 & a & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & a & 0 & b & 3\alpha & 0 & 2b & 0 & 2\delta & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & b & 0 & 0 & 2b & 0 & 2b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\
0 & 0 & \varepsilon & 0 & \varepsilon & 0 & 0 & 3\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\delta & 0 & \hat{\alpha} & \delta & 0 & 0 & 0 & 2\varepsilon & 2 \varepsilon \\
0 & 0 & 0 & -2\hat{\beta} & 0 & -2\delta & 3\hat{\beta} & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\hat{\beta} & 0 & 0 & 3\hat{\beta} & \hat{\alpha} & 0 & 0 & 2b & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & -2\hat{\beta} & 0 & 0 & 3\hat{\beta} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Let $K$ denote this matrix. Then we have completed the proof. □

From now on we consider

$$G_1 = (\varepsilon \lambda + au + bv + cr + f)\xi + (ev + g)\bar{\xi}\zeta,$$

$$G_2 = (\delta \lambda + \hat{\alpha} u + \hat{\beta} v + h)\zeta + \xi^2,$$

where $f, g$ and $h$ have the following form:

$$f \in \mathcal{M}^3 + Mr$$

$$h = d_1 u^2 + h_1, \quad (d_1 \in \mathbb{R}, \quad h_1 \in \mathcal{M}^3 + Mr)$$
\[ g = d_2 \lambda u + d_3 u^2 + d_4 uv + g_1, \quad (d_2, d_3, d_4 \in \mathbb{R}, \quad g_1 \in \mathcal{M}^3 + \mathcal{M} \tau). \]

**Proposition 4.4.** Assume that \( \epsilon \neq 0 \) and \( \delta \neq 0 \). Then in (4.1,2) we may assume, without loss of generality, that

\[ f(\lambda; 0, 0, 0) \equiv 0, g(\lambda; 0, 0, 0) \equiv 0, h(\lambda; 0, 0, 0) \equiv 0 \]

**Proof.** We write \( f = p_1 + f_1, g = p_2 + g_1 h = p_3 + h_1 \), where

\[ p_1 = f(\lambda; 0, 0, 0), p_2 = g(\lambda; 0, 0, 0), p_3 = h(\lambda; 0, 0, 0). \]

Therefore we have \( p_j \in \mathcal{M}^3_\lambda (j = 1, 2, 3) \). We use the following coordinate change:

\[ (\xi, \zeta) \to (\xi + \phi \xi \zeta_1, (1 + \psi(\lambda)) \zeta), \]

where \( \phi, \psi \in \mathcal{M}_\lambda^3 \). Then \( G \) is transformed to

\[ H_1 = (\epsilon \lambda + p_1 + au + bu + cr + f_2)(\xi + \phi \xi \zeta_1) + [p_2 + ev + g_2](\bar{\xi} + \phi \bar{\xi} \zeta)(1 + \psi(\lambda)), \]

\[ H_2 = (\delta \lambda + p_3 + \hat{a} u + \hat{b} + h_2) \zeta (1 + \psi(\lambda)) + (\xi + \phi \bar{\xi} \zeta)^2. \]

Here and hereafter

\[ f_j \in (\mathcal{M}^3 + r \mathcal{M}) \cap (< u > + < v > + < r >), \]

\[ g_j - (d_2 \lambda u + d_3 u^2 + d_4 uv) \in (\mathcal{M}^3 + r \mathcal{M}) \cap (< u > + < v > + < r >), \]

\[ h_j - d_1 u^2 \in (\mathcal{M}^3 + r \mathcal{M}) \cap (< u > + < v > + < r >). \]

We can write these equations as follows:

\[ H_1 = (\epsilon \lambda + p_1 + au + bu + cr + f_3) \xi + (p_2 + p_2 \psi + \epsilon \phi \lambda + \phi p_1 + ev + g_3) \bar{\xi} \zeta, \]

\[ H_2 = (\delta \lambda + p_3 + \delta \psi \lambda + p_3 \psi + \hat{a} u + \hat{b} v + h_3) \zeta + (1 - \nu \phi^2) \xi^2 \]

Therefore

\[ H_2 / (1 - \nu \phi^2) = (\delta \lambda + p_3 + \delta \psi \lambda + p_3 \psi + \hat{a} u + \hat{b} v + h_3) \zeta + \xi^2. \]

To complete the proof, it is enough to show that we can choose \( \phi \) and \( \psi \), so that

\[ \lambda + \epsilon p_1 = \lambda + \delta p_3 + \psi \lambda + \delta p_3 \psi, \]

and

\[ p_2 + p_2 \psi + \epsilon \phi \lambda + \phi p_1 \equiv 0 \]

The former yields

\[ \psi = \frac{\epsilon p_1 - \delta p_3}{\lambda + \delta p_3}, \]

which certainly belong to \( \mathcal{M}_\lambda^3 \) since \( p_j \in \mathcal{M}^3 \lambda \). The latter yields

\[ \phi = \frac{-p_2 - p_2 \psi}{\epsilon \lambda + p_1} \]

which belong to \( \mathcal{M}_\lambda^3 \), since \( p_j \in \mathcal{M}_\lambda^3, \psi \in \mathcal{M}_\lambda^2 \) and \( \epsilon \neq 0 \). We now have only to put \( \Lambda = \lambda + \epsilon p_1 \). \( \square \)
**Proposition 4.5.** If $\epsilon \delta \hat{a} \hat{b} \neq 0$, then in (4.3,4) we may assume, without loss of generality, that

$$\epsilon, \delta, \hat{a}, \hat{b} = \pm 1.$$  

**Proof.** Let $\alpha, \beta, \gamma$ be positive parameters and change variables in the following way:

$$(\lambda, \xi, \zeta) \rightarrow (\alpha \lambda, \beta \xi, \gamma \zeta)$$

Then $G_1$ and $G_2$ are transformed to

$$H_1 = (\epsilon \alpha \beta \lambda + a \beta^3 u + b \beta \gamma^2 v + c \beta^3 \gamma r + f')\xi + (e \gamma^2 v + g') \beta \gamma \bar{\zeta} \zeta,$$

$$H_2 = (\delta \alpha \gamma \lambda + \hat{a} \beta^2 \gamma u + \hat{b} \gamma^3 v + h') \xi + \beta \gamma \bar{\xi} \xi^2,$$

Then we consider $H_1 \alpha^{-1} \beta^{-1} / |\epsilon|$ and $H_2 \beta^{-2}$. The proof is completed by choosing $\alpha, \beta, \gamma$ so that

$$\gamma = |\hat{a}|^{-1}, \gamma^3 \beta^{-2} = |\hat{b}|^{-1}, \alpha \beta^{-2} \gamma = 1 / |\delta|$$

□

We have transformed $G$ to (4.9,10) where $\epsilon, \delta, \hat{a}, \hat{b} = \pm 1$ and $f, g$ and $h$ have the following form:

$$f \in \mathcal{M}^3 + Mr$$

$$h = d_1 u^2 + h_1, \quad (d_1 \in \mathbb{R}, \quad h_1 \in \mathcal{M}^3 + Mr)$$

$$g = d_2 \lambda u + d_3 u^2 + d_4 uv + g_1, \quad (d_2, d_3, d_4 \in \mathbb{R}, \quad g_1 \in \mathcal{M}^3 + Mr).$$

and satisfy

$$f(\lambda; 0, 0, 0) \equiv 0, g(\lambda; 0, 0, 0) \equiv 0, h(\lambda; 0, 0, 0) \equiv 0.$$

We now compute the coefficients of $dG$ as follows:

$$\frac{\partial G_1}{\partial \xi} = \epsilon \lambda + 2au + bv + cr + f + uf_u + (c + fr) \xi \frac{\partial r}{\partial \xi} + gu \bar{\zeta} \xi + gr \bar{\zeta} \xi \frac{\partial r}{\partial \xi},$$

$$\frac{\partial G_1}{\partial \xi} = (a + fu) \xi^2 + (c + fr) \xi \frac{\partial r}{\partial \xi} + gu \bar{\zeta} \xi \frac{\partial r}{\partial \xi} + (ev + g + ug_u) \xi,$$

$$\frac{\partial G_1}{\partial \zeta} = (b \bar{\zeta} + f_u \xi + (c + fr) \frac{\partial r}{\partial \zeta}) \xi + (2ev + g + vg_v + gr \frac{\partial r}{\partial \zeta}) \bar{\xi},$$

$$\frac{\partial G_1}{\partial \zeta} = (b \bar{\zeta} + f_u \xi + (c + fr) \frac{\partial r}{\partial \zeta}) \xi + (e \zeta + g_v \xi + gr \frac{\partial r}{\partial \zeta}) \bar{\zeta}.$$
\[
\frac{\partial G_2}{\partial \xi} = 2\xi + (\hat{\lambda} \xi + h_u \xi + h_r \frac{\partial r}{\partial \xi}) \xi, \quad \frac{\partial G_2}{\partial \zeta} = (\hat{\lambda} \xi + h_u \xi + h_r \frac{\partial r}{\partial \zeta}) \xi,
\]
\[
\frac{\partial G_2}{\partial \lambda} = \hat{\lambda} u + 2b v + h + \nu v + h_r \xi \frac{\partial r}{\partial \zeta},
\]
\[
\frac{\partial G_2}{\partial \zeta} = (\hat{b} \xi + h_v \xi + h_r \frac{\partial r}{\partial \zeta}) \xi,
\]
where the subscript means differentiation. By these equalities we obtain
\[
dG(X_1) = (\epsilon \lambda + 3au + bv + cr + f + 2uf_u + 2r(c + f_r))X_1
\]
\[
+ (ev + g + 2ug_u + 2rg_r)X_2 + (2\hat{\lambda}u + 2uh_u + 2rh_r)X_3 + 2X_4,
\]
\[
dG(X_2) = [2r(a + f_u) + 2uv(c + f_r) + ev^2 + vg]X_1
\]
\[
+ [\epsilon \lambda + au + bv + cr + f + 2uvg_r + 2rg_u]X_2 + (2u + 2\hat{\lambda}r + 2rh_u + 2uvh_r)X_3,
\]
\[
dG(X_3) = [2bv + 2vf_v + r(c + f_r)]X_1 + [3ev + g + 2vg_v + rg_r]X_2
\]
\[
+ [\hat{\delta} \lambda + \hat{\lambda} u + 3b v + h + 2vh_v + rh_v]X_3,
\]
\[
dG(X_4) = [2br + 2rf_v + u^2(c + f_r) + euv + u g]X_1 + [2r(e + g_v) + u^2g_r]X_2
\]
\[
+ [2br + 2rh_v + u^2h_r]X_3 + [\hat{\delta} \lambda + \hat{\lambda} u + \hat{b} v + h]X_4.
\]

Our next task is to compute \( T_j G \) \( (j = 1, 2, \cdots, 8) \), where \( T_j \) is given in Proposition 2.3.

\[
T_1 G = (\epsilon \lambda + au + bv + cr + f)X_1 + (ev + g)X_2
\]
\[
T_2 G = [u(\epsilon \lambda + au + bv + cr + f) + 2r(ev + g)]X_1 - u(ev + g)X_2
\]
\[
T_3 G + T_4 G = [2v(\delta \lambda + \hat{\lambda} u + \hat{b} v + h) + 2r]X_1
\]
\[
T_3 G - T_4 G = 2rX_1 - 2uX_2
\]
\[
T_5 G + T_6 G = [2u(\epsilon \lambda + au + bv + cr + f) + 2r(ev + g)]X_3
\]
\[
T_5 G - T_6 G = (ev + g)(2rX_3 - 2vX_4)
\]
\[
T_7 G = [v(\delta \lambda + \hat{\lambda} u + \hat{b} v + h) + 2r]X_3 - vX_4
\]
\[
T_8 G = (\delta \lambda + \hat{\lambda} u + \hat{b} v + h)X_3 + X_4
\]

Note that \( T_2 G \) and \( T_5 G - T_6 G \) are \( E \)-combination of the remaining 6 elements. In fact \( T_2 = uT_1 G + (ev + g)(T_3 G - T_4 G), T_5 - T_6 G = (ev + g)(T_7 G - vT_8 G) \). We have to compute \( is \)-multiple of these elements. But this is easily done by (3.13). In conclusion we have
Proposition 4.6. The E-module $\mathring{\Gamma}G$ is generated by the following 13 elements.

\[ W_1 = [2au + 2uf_u + 2r(c + fr)]X_1 + [2ug_u + 2rhr]X_2 + [2au + 2uh_u + 2rh_r]X_3 + 2X_4, \]
\[ W_2 = [2r(a + fr) + 2uv(c + fr) + euv + vg]X_1 + [e\lambda + au + bv + cr + f + 2uvg_r + 2rg_u]X_2 \]
\[ + [2u + 2a\lambda + 2rh_u + 2uvh_r]X_3, \]
\[ W_3 = [2bv + 2vf_v + r(c + fr)]X_1 + [3ev + g + 2vg_v + rhr]X_2 + [2a\lambda + 2uvh_v + rhr]X_3 - X_4, \]
\[ W_4 = [2br + 2rf_v + u^2(c + fr) + euv + ug]X_1 + [2r(e + g_v) + u^2g_r]X_2 \]
\[ + [2uv + 2rh_v + u^2h_r]X_3 + [\delta\lambda + au + bv + h]X_4, \]
\[ W_5 = (e\lambda + au + bv + cr + f)X_1 + (ev + g)X_2, \]
\[ W_6 = [v((\hat{a} - e\delta a)u + (\hat{b} - e\delta b)v + h - e\delta f) + r - e\delta curv]X_1 - e\delta v(ev + g)X_2 \]
\[ W_7 = rX_1 - uX_2, \]
\[ W_8 = [u((a - e\delta a)u + (b - e\delta b)v + f - e\delta h) + r(cu + ev + g)]X_3 - e\delta uX_4 \]
\[ W_9 = rX_3 - vX_4 \]
\[ W_{10} = (\delta\lambda + au + bv + h)X_3 + X_4, \]
\[ W_{11} = (ev + g)(uvX_1 - rX_2), \]
\[ W_{12} = u(uvX_1 - rX_2), \]
\[ W_{13} = u^2X_3 - rX_4. \]

Proof is similar to that of Proposition 3.4. From these 13 elements we now make elements which contain $X_1$ and $X_3$ only:

\[ Y_1 = \frac{u}{2}W_1 + (ug_u + rgr)W_7 + e\delta W_6 = [au^2 + ru(c + fr) + u^2f_u + r(ug_u + rgr)]X_1 \]
\[ + [\hat{au}^2 + u^2h_u + ruh_r + (ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (e\delta f - h)u + e\delta rv(cu + ev + g)]X_3, \]
\[ Y_2 = \frac{1}{2}v(ev + g)W_1 + e\delta(ug_u + rgr)W_6 + (ev + g)W_9 = [v(ev + g)(au + r(c + fr) + uf_u) \]
\[ + (ug_u + rgr)((\hat{a}e\delta - a)uv + (\hat{b}e\delta - b)v^2 + v(e\delta h - f) + e\delta r - crv)]X_1 \]
\[ + (ev + g)[\hat{au} + uvh_u + ruh_r + r]X_3, \]
\[ Y_3 = uW_3 + (3ev + g + 2vg_v + rgr)W_7 - e\delta W_8 \]
\[
= [2bu + 2uf + ru(c + f_r) + r(3ev + g + 2vg + rg_r)]X_1 \\
\quad + [(\hat{a} e \delta a)u^2 + (3\hat{b} e \delta b)uv + 2uvh + ruh_r - (e\delta f - h)u - e\delta r(cu + ev + g)]X_3,
\]
\[Y_4 = v(ev + gl)W_3 + e\delta(3ev + g + 2vg + rg_r)W_6 - (ev + g)W_9 = [v(ev + g)(2u + 2u + r(c + f_r))
\quad + (3ev + g + 2vg + rg_r)(\hat{a} e \delta a)u^2 + (\hat{b} e \delta b)uv + u(e\delta h - f) + e\delta r - crv)]X_1
\]
\[+ (ev + g)(2u^2 + 2u^2h + ruh_r - r)X_3
\]
\[Y_5 = r(ev + g)W_3 + (3ev + g + 2vg + rg_r)W_11 - (ev + g)W_13
\]
\[= (ev + g)[2bru + 2ruf + r^2(c + f_r) + uv(3ev + g + 2vg + rg_r)]X_1
\]
\[+ (ev + g)[2bru + 2ruh + r^2h_r - u^2]X_3
\]
\[Y_6 = \frac{ru}{2}W_1 + r(ug + rg_r)W_7 + uW_13
\]
\[= [ru(u + r(c + f_r) + u^2u + r^2u + r^3u)]X_1 + [ru(\hat{a}u + uh + rh) + u^3]X_3
\]
\[Y_7 = ruW_3 + r(3ev + g + 2vg + rg_r)W_7 - uW_13
\]
\[= [ru(2bu + 2uf + r(c + f_r)) + r^2(3ev + g + 2vg + rg_r)]X_1 + [ru(\hat{a}u + uh + rh) - u^3]X_3,
\]
\[Y_8 = uW_2 + (e\lambda + au + bu + cr + f + 2uv + 2rg_r)rW_7 - rW_5 - W_11
\]
\[= [2ru(a + f_u) + 2u^2v(c + f_r) + 2ruvgr + 2r^2g_r]X_1 + [2u^2 + 2ar + ruh_u + u^2r_h_r]X_3
\]
\[Y_9 = v(ev + g)W_2 + e\delta(e\lambda + au + bu + cr + f + 2uv + 2rg_r)W_6 - (e\delta - cv)W_11
\]
\[= (e\delta a - a)u + (e\delta b - b)uv + (e\delta h - f)uv + e\delta r - crv]W_5 - [(e\delta a - a)u + (e\delta b - b)uv + e\delta h - f\delta]W_6
\]
\[= (2ru + 2uv(c + f_r) + 2uvr + 2v)(uv + g)](ev + g) - (e\delta - cv)uv(ev + g) + (e\delta a - a)uv
\]
\[+ (e\delta b - b)uv + (e\delta h - f)uv + e\delta r - crv) - (e\delta a - a)u + (e\delta b - b)uv + e\delta h + f + 2uvr + 2rg_r]X_1
\]
\[+ [2u + 2ar + 2rh_u + 2uhr]u(e^2 + vg)X_3,
\]
\[Y_{10} = uW_4 + (2ru + e + g_r + u^2g_r)W_7 + e\delta(\delta + a + bu + h)W_8 - e\delta(cu + ev + g)W_{13}
\]
\[-[(e\delta a - a)u + (e\delta b - b)uv + (e\delta f - h)u + e\delta r(cu + ev + g)]W_10
\]
\[-[(e\delta a - a)u + (e\delta b - b)uv + e\delta f - h]e\delta W_8
\]
\[= [(2ru + e + g_v) + u^3(c + f_r) + eu^2v + u^2g + 2r^2(e + g_v) + 2u^2g_r]X_1
\]
\[+ [2ru(a + f_u) + u^3h_r - e\delta u^2(cu + ev + g)]
\]

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\[-((\epsilon \delta a - \hat{a})u + (\epsilon \delta b - \hat{b})v + \epsilon \delta f - h)((\epsilon \delta a - \hat{a})u^2 + (\epsilon \delta b - \hat{b})uv + (\epsilon \delta f - h)u + \epsilon \delta r(ev + g))]X_3\]

**Proposition 4.7.** If $\epsilon \delta a \hat{a} \hat{b}(b \epsilon \delta - b)(a \hat{b} - \hat{b}) \neq 0$, then the following elements belongs to $\tilde{\Gamma}G$:

$$u^3X_j, \ u^2vX_j, \ uv^2X_j, \ v^4X_j, \ ru^2X_j, \ ruvX_j, \ rv^2X_j \ (j = 1, 3)$$

**Proof.** Let us define $V_j \ (j = 1, 2, \ldots, 14)$ by

$$V_1 = u^3X_1, \ V_2 = u^2vX_1, \ V_3 = uv^2X_1, \ V_4 = v^4X_1, \ V_5 = ru^2X_1, \ V_6 = ruvX_1, \ V_7 = rv^2X_1,$$

$$V_8 = u^3X_3, \ V_9 = u^2vX_3, \ V_{10} = uv^2X_3, \ V_{11} = v^4X_3, \ V_{12} = ru^2X_3, \ V_{13} = ruvX_3,$$

$$V_{14} = rv^2X_3.$$

Then we express the fact that $uY_k, vY_k \ (k = 1, 2, 3), \ vY_4, Y_i \ (i = 5, 6, 7), \ 2Y_8, vY_9, uY_{10}, vY_{10}$ belong to $\tilde{\Gamma}G$ in the following way:

$$AV \in (\tilde{\Gamma}G)^{14},$$

where $A$ is a matrix whose entries are in $E$. At $(\lambda, u, v, r) = (0, 0, 0, 0)$ it is expressed as

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \epsilon \delta a & -\epsilon \delta q & 0 & 0 & \epsilon \delta c & \epsilon \delta e & 0 \\
0 & a & 0 & 0 & 0 & \epsilon \delta a & -\epsilon \delta q & 0 & 0 & \epsilon \delta c & \epsilon \delta e \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & 3e & 0 & \epsilon \delta p & 2\hat{b} + \epsilon \delta q & 0 & 0 & -\epsilon \delta c & -\epsilon \delta e & 0 \\
0 & 0 & 2b & 0 & 0 & 0 & 3e & 0 & \epsilon \delta p & 2\hat{b} + \epsilon \delta q & 0 & 0 & -\epsilon \delta c & -\epsilon \delta e \\
0 & 0 & 0 & * & 0 & 0 & 3e \epsilon \delta & 0 & 0 & 0 & 2\hat{b}e & 0 & 0 & -e \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\hat{b}e \\
0 & 0 & 0 & 0 & a & 0 & 0 & 1 & 0 & 0 & 0 & \hat{a} & 0 & 0 \\
0 & 0 & 0 & 0 & 2b & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2\hat{b} & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 1 & 0 & 0 & 0 & \hat{a} & 0 \\
0 & 0 & 0 & q^2 & 0 & \epsilon \delta p & \epsilon \delta q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & \hat{2b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & \hat{2b} & 0 & 0
\end{pmatrix}
$$

where $p = \hat{\epsilon \delta} - a, q = \hat{b \epsilon \delta} - b$ and $* = e(3q + 2b)$. In fact, let us consider $uY_1$, for example. We can write it as

$$uY_1 = (a + \mu_1)V_1 + (c + \mu_2)V_5 + (a \epsilon \delta + \mu_3)V_8 + ((b \epsilon \delta - b) + \mu_4)V_9 + (\epsilon \delta c + \mu_5)V_{12}$$

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\[ + (\epsilon \delta e + \mu_6) V_{13} + \mu_7 r^2 X_1 + \mu_8 r^2 X_3, \]

Where \(\mu_j \in \mathbb{M} \quad (1 \leq j \leq 8)\). On the other hand, we have

\[ (4.9) \quad r W_7 - W_{12} = (r^2 - u^2 v) X_1, \quad r W_9 - v W_{13} = (r^2 - u^2 v) X_3 \]

Therefore we have

\[ u Y_1 - \mu_7 (r W_7 - W_{12}) - \mu_8 (r W_9 - v W_{13}) = (a + \mu_1) V_1 + \mu_7 V_2 + (c + \mu_2) V_5 \]

\[ + (a \epsilon \delta + \mu_3) V_8 + ((b \epsilon \delta - \hat{b}) + \mu_4 + \mu_5) V_9 + (\epsilon \delta c + \mu_6) V_{12} + (\epsilon \delta e + \mu_6) V_{13}. \]

This leads to the first line of the matrix above. Other lines are similarly obtained. It is elementary to compute the determinant of this matrix and we have

\[ \det A(0, 0, 0, 0) = 64 \epsilon \delta a b^4 \epsilon \hat{a} b^4 (b \epsilon \delta - b)^3 (a \hat{b} - b \hat{a}). \]

Therefore we have the conclusion. \[ \square \]

From this proposition we further conclude that some elements which are of lower order than those in this proposition. Below we give a list, which indicates that if we see the elements in the left hand side of the arrow, we have the right hand side as elements in \( \tilde{\Gamma} \mathcal{G} \).

\[ (4.9) \rightarrow r^2 X_1, r^2 X_3 \]

\[ u W_8, v W_8, r W_8 \rightarrow u^2 X_4, u v X_4, r u X_4, \]

\[ v^2 W_9, r W_9 \rightarrow v^2 X_4, r v X_4, \quad u W_9 \rightarrow r u X_3, \]

\[ u^2 W_7, u v W_7, v^2 W_7, r W_7 \rightarrow u^3 X_2, u^2 v X_2, u v^2 X_2, r u X_2 \]

\[ r W_{13} \rightarrow r^2 X_4, \quad r v W_4 \rightarrow r^2 v X_2 \]

\[ r v W_3 \rightarrow r^2 X_2, \quad v^3 W_3 \rightarrow v^4 X_2, \quad v^2 W_1 \rightarrow v^2 X_4 \]

\[ v W_9 \rightarrow r v X_3, \quad u W_6 \rightarrow r u X_1 \]

\[ u W_7 \rightarrow u^2 X_2, \quad Y_8 \rightarrow u^2 X_3, \quad W_{13} \rightarrow r X_4 \]

\[ r W_4 \rightarrow r^2 X_2, \quad W_{11} \rightarrow r v X_2, \quad r W_{10} \rightarrow \lambda r X_3 \]

\[ \lambda W_9 \rightarrow \lambda v X_4, \quad v W_4 \rightarrow r v X_1, \]

\[ Y_9 \rightarrow v^3 X_1, \quad v W_6 \rightarrow v^3 X_2, \quad v^2 W_3 \rightarrow v^3 X_3 \]

\[ r W_5 \rightarrow \lambda r X_1, \quad \lambda W_7 \rightarrow \lambda u X_2, \quad u W_2 \rightarrow u v X_2, \]

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\[ u^2 W_5, uv W_5 v^2 W_5, r W_5 \to \lambda u^2 X_1 \lambda uv X_1, \lambda v^2 X_1, \lambda r X_1 \]
\[ u^2 W_{10}, uv W_{10}, v^2 W_{10} \to \lambda u^2 X_3, \lambda uv X_3, \lambda v^2 X_3, \]
\[ uv W_7 \to uv X_2 \]
\[ u W_2, v^2 W_2, r W_2 \to \lambda u X_2, \lambda v^2 X_2, \lambda r X_2, \]
\[ \lambda u W_1, \lambda v W_1 \to \lambda u X_4, \lambda v X_4, \]
\[ W_5 \to \lambda^2 u X_1, \lambda^2 v X_1, \]
\[ W_{10} \to \lambda^2 u X_3, \lambda^2 v X_3, \]
\[ W_2 \to \lambda^2 u X_2, \]
\[ W_4 \to \lambda^2 X_4, \]
\[ W_5 \to \lambda^3 X_1, \]
\[ W_{10} \to \lambda^3 X_3. \]

From these considerations we have proved

**Proposition 4.8.**

\[ M^3 X_1, M^3 X_3, M^3 X_2, M^2 X_4, r MX_1, r MX_2, r MX_3, \]
\[ r X_4, u^2 X_2, uv X_2, \lambda u X_2, u^2 X_3 \]

belong to \( \tilde{G}. \)

Propositions 4.6 and 4.8 ensure that the elements in (4.10) and the following elements generate \( \tilde{G}: \)

\[ Z_1 = [2au + 2cr]X_1 + 2\hat{a}u X_3 + 2X_4, \]
\[ Z_2 = [2ar + 2cuv + ev^2]X_1 + [\epsilon \lambda + au + bv + cr]X_2 + [2u + 2\hat{a}r]X_3, \]
\[ Z_3 = [2bv + cr]X_1 + 3ev X_2 + 2b\hat{v} X_3 - X_4, \]
\[ Z_4 = [2br + cu^2 + euv]X_1 + 2er X_2 + 2b\hat{r} X_3 + [\delta \lambda + \hat{a}u + \hat{b}v]X_4, \]
\[ Z_5 = (\epsilon \lambda + au + bv + cr)X_1 + ev X_2, \]
\[ Z_6 = [(\hat{a} - \epsilon \delta a)uv + (\hat{b} - \epsilon \delta b)v^2 + r]X_1 - \epsilon \delta ev^2 X_2, \]
\[ Z_7 = r X_1 - u X_2, \]
\[ Z_8 = (b - \epsilon \delta b)uv X_3 - \epsilon \delta u X_4, \]
\[ Z_9 = r X_3 - vX_4, \]
\[ Z_{10} = (\delta \lambda + \hat{a}u + \hat{b}v)X_3 + X_4, \]

Therefore \( \tilde{G} \) is independent of \( f, g, h. \) Now we have
THEOREM 4.1. If \( e\delta = a\hat{b}e(\hat{a}b - \hat{a}b)(\hat{b}\delta - b) \neq 0 \) and \( detK \neq 0 \), then (4.7,8) is \( O(2) \)-equivalent to

\[
G_1 = (e\lambda + au + bv + cr)\xi + e\zeta, \tag{4.21}
\]

\[
G_2 = (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2, \tag{4.22}
\]

Proof. Since we know that \( \tilde{\Gamma}G \) is independent of \( f, g, h \), this theorem follows from Theorem 2.1.

By this theorem we have only to consider (4.11,12). From now on \( G \) means what is given by (4.11,12).

THEOREM 4.2. Let \( G \) be given by (4.21,22). In addition to the assumption in the preceding theorem, we assume that \( detA_1 \neq 0 \). Then \( F(\alpha, \beta, s_1, s_2, s_3, s_4, \lambda; \xi, \zeta) \) is a universal unfolding of \( G \), where \( F = (F_1, F_2) \) is given by

\[
F_1 = (e\lambda + \alpha + (a + s_1)u + (b + s_2)v + (c + s_3)r)\xi + (\beta + (e + s_4)v)\zeta, \tag{4.23}
\]

\[
F_2 = (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2, \tag{4.24}
\]

Therefore \( O(2) \)-codimension of \( G \) is 6.

Proof. By Proposition 4.5 and 7 we know that \( \tilde{\Gamma}G \) is generated by (4.10-20). By Theorem 2.1, we have only to prove

\[
\Gamma G \oplus RX_1 \oplus RX_2 \oplus RuX_1 \oplus RuX_1 \oplus RxX_1 \oplus RuX_2 = \hat{E}.
\]

It is clear that \( \Gamma G = \hat{\Gamma}G \oplus R \frac{\partial G}{\partial \alpha} \oplus R\lambda \frac{\partial G}{\partial \lambda} \oplus R\lambda^2 \frac{\partial G}{\partial \lambda} \). Therefore

\[
\Gamma G = \hat{\Gamma}G \oplus R(\epsilon X_1 + \delta X_3) \oplus R(\epsilon\lambda X_1 + \delta\lambda X_3) \oplus R(\epsilon\lambda^2 X_1 + \delta\lambda^2 X_3). \tag{4.25}
\]

Let \( N \) denote the \( R \)-linear space

\[
\Gamma G + RX_1 + RX_2 + RuX_1 + RuX_1 + RxX_1 + RuX_2. \tag{4.26}
\]

By (4.24) and (4.25) we easily see that

\[
X_1, \lambda X_1, uX_1, vX_1, rX_1, X_3, \lambda X_3
\]
belong to $N$. By $Z_1, Z_3$ and $Z_{10}$, the elements

$$\hat{a}uX_3 + X_4, 2\hat{b}vX_3 - X_4, (\hat{a}u + \hat{b}v)X_3 + X_4$$

belong to $N$. Therefore $uX_3, vX_3, X_4$ belongs to $N$ if $\hat{a}\hat{b} \neq 0$. If we consoder

$$\lambda Z_1, uZ_1, vZ_1, \lambda Z_2, vZ_2, \lambda Z_3, uZ_3, vZ_3, Z_4, \lambda Z_5, uZ_5, vZ_5, Z_6, -\epsilon \delta Z_8, Z_9, \lambda Z_{10}, uZ_{10}, vZ_{10}$$

then the following 18 elements belongs to $\tilde{G}$.

$$a\lambda uX_1 + \hat{a}\lambda uX_3 + \lambda X_4$$

$$au^2X_1 + uX_4$$

$$auvX_1 + \hat{a}uvX_3 + vX_4$$

$$(\epsilon \lambda^2 + b\lambda v)X_2 + 2\lambda uX_3$$

$$(\epsilon \lambda v + bv^2)X_2 + 2uvX_3$$

$$2b\lambda vX_1 + 3e\lambda vX_2 + 2\hat{b}\lambda vX_3 - \lambda X_4$$

$$2buvX_1 + 2buvX_3 - uX_4$$

$$2bv^2X_1 + 3ev^2X_2 + 2bv^2X_3 - vX_4$$

$$(2br + euv)X_1 + 2evX_2 + 2brX_3 + (\delta \lambda + \hat{a}u + \hat{b}v)X_4$$

$$(\epsilon \lambda^2 + a\lambda u + b\lambda v)X_1 + e\lambda vX_2$$

$$(\epsilon \lambda u + au^2 + buv)X_1$$

$$(\epsilon \lambda v + auv + bv^2)X_1 + ev^2X_2$$

$$[(\hat{a} - \epsilon \delta a)uv + (\hat{b} - \epsilon \delta b)v^2 + \epsilon]X_1 - \epsilon \delta ev^2X_2$$

$$(\hat{b} - \epsilon \delta b)uvX_3 + vX_4$$

$$rX_3 - vX_4$$

$$(\delta \lambda^2 + \hat{a}\lambda u + \hat{b}\lambda v)X_3 + \lambda X_4$$

$$(\delta \lambda u + buv)X_3 + uX_4$$

$$(\delta \lambda v + \hat{a}uv + \hat{b}v^2)X_3 + vX_4$$

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In addition to this we know that $\epsilon \lambda^2 X_1 + \delta \lambda^2 X_3$ belongs to $\Gamma G$. We now define

$$V = (\lambda^2 X_1, \lambda u X_1, \lambda v X_1, u^2 X_1, u v X_1, v^2 X_1, \lambda^2 X_3, \lambda u X_3,$$

$$\lambda v X_3, u v X_3, v^2 X_3, r X_3, \lambda v X_2, v^2 X_2, \lambda X_4, u X_4, v X_4)$$

Then these 19 relations are written as follows:

$$A_1v = (0, 0, 0, 0, 0, 0, 0, 0, 2br X_1, 0, 0, 0, r X_1, 0, 0, 0, 0, 0, 0)$$

where $A_1$ is $19 \times 19$ real matrix given by

$$
\begin{pmatrix}
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}

p and q are given by $p = \hat{a} - \epsilon \delta a$, $q = \hat{b} - \epsilon \delta b$. Therefore, if $\text{det} A_1 \neq 0$, then all the components of $V$ belong to $N$. By $Z_7$, $u X_2$ belongs to $N$, then $Z_7$ ensures that $\lambda X_2$ belongs to $N$. Thus we have proved that $N = \hat{E}$. There remains to show that the sum in (4.26) is a direct sum. Let $W$ denote the $E$-module generated by (4.10). Then we can easily see that $\mathbb{R}$-linear space $\hat{E}/W$ has dimension 33. On the other hand, we have only 24 relations which is nontrivial modulo $W$. These and $\epsilon X_1 + \delta X_3, \epsilon \lambda X_1 + \delta \lambda X_3, \epsilon \lambda^2 X_1 + \delta \lambda^2 X_3 \in N$ are the only nontrivial relations in $\hat{E}/W$. Thus we see that $\hat{E}/W$ is of dimension 6, which completes the proof. □
§5. Computation of $\Gamma G$ : Case (III). In this section we consider the case (III) and compute $\Gamma G$. We first transform $G$ into $(G_1/f_2, G_2)$. We can write

\begin{equation}
G_1 = (\epsilon \lambda + au + bv + cr + f)\xi + \bar{\xi}\zeta,
\end{equation}

\begin{equation}
G_2 = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)\zeta + (\eta \lambda + du + ev + kr + h)\xi^2,
\end{equation}

where $f, g$ and $h$ belong to $M^2$ and $\epsilon, \delta, \eta$ are real constants.

**Proposition 5.1.** If $\delta \neq 0, d \neq 0$, then $G = (G_1, G_2)$ is $O(2)$-equivalent to the equation (5.1, 2) with $c = 0, e = 0, \eta = 0, k = 0$ and $h(\lambda, 0, 0, 0) \equiv 0$, although numerical values of the other coefficients may change.

**Proof.** Consider the coordinates change:

\[(\xi, \zeta) \rightarrow (\xi + \alpha \bar{\xi}\zeta, \zeta + \beta \xi^2),\]

where $\alpha$ is a real constant and $\beta \in \mathcal{E}_\lambda$. Choosing $\alpha$ and $\beta$ so that $k + 2d\alpha + 2e\beta(0) = 0$ and $(\delta \lambda + g(\lambda, 0, 0, 0))\beta + \eta \lambda + h(\lambda, 0, 0, 0) = 0$, we can make $\eta$ and $k$ be zero. This choice is possible, since $g(\lambda, 0, 0, 0), h(\lambda, 0, 0, 0) \in M^2_\lambda$ and $\delta \neq 0$. Consider next the following transformations.

\[
H_1(\lambda, \xi, \zeta) = G_1(\lambda, \xi, \zeta) + \gamma_1 \xi^2 \overline{G_1},
\]

\[
H_2(\lambda, \xi, \zeta) = G_2(\lambda, \xi, \zeta) + \gamma_2 \zeta(\bar{\xi}G_1 - \xi \overline{G_1}).
\]

Choosing $\gamma_1$ and $\gamma_2$ so that $e - 2\gamma_2 = 0, c + 2\gamma_1 = 0$ and dividing $H_1$ by $1 - \gamma_1 u$, the proof is completed. \(\square\)

From now on we consider

\begin{equation}
G_1 = (\epsilon \lambda + au + bv + f)\xi + \bar{\xi}\zeta,
\end{equation}

\begin{equation}
G_2 = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)\zeta + (du + h)\xi^2,
\end{equation}

where $f, g$ and $h$ belong to $M^2$ and $h(\lambda, 0, 0, 0) \equiv 0$. 

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Proposition 5.2. If $\epsilon \delta d \neq 0$ and $2d(\hat{a} - 2e\delta a) + a(\hat{a} - e\delta a)(e\delta a - 2\hat{a}) \neq 0$, then $G = (G_1, G_2)$ is $O(2)$-equivalent to the equation (5.3,4) in which the coefficients of $\lambda^2$, $\lambda u$ and $u^2$ in $f, g$ and $h$ vanish and

$$f(\lambda, 0, 0, 0) \equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0.$$  

Proof. We transform $G_1$ and $G_2$ to

$$H_1(\lambda, \xi, \zeta) = (1 + \beta_0 \lambda + \beta_1 u)^{-1}(1 + \gamma_1 u)^{-1}G_1(\lambda, (1 + \beta_0 \lambda + \beta_1 u)\xi, (1 + \gamma_1 u)\zeta),$$

and

$$H_2(\lambda, \xi, \zeta) = (1 + \alpha_0 \lambda + \alpha_1 u)G_2(\lambda, (1 + \beta_0 \lambda + \beta_1 u)\xi, (1 + \gamma_1 u)\zeta),$$

respectively. Let $f_0, g_0$ and $h_0$ denote the quadratic parts of $f, g$ and $h$, respectively such that $f_0, g_0$ and $h_0$ are independent of $r$. Therefore we can write $f = f_0 + f_1, g = g_0 + g_1, h = h_0 + h_1$, where $f_1, g_1, h_1 \in \mathcal{M}^3 + \mathcal{M}r$. From now on until the end of the proof of the present proposition, $f_j, g_j, h_j$ (1 ≤ $j$) imply the elements of $\mathcal{M}^3 + \mathcal{M}r$. We now have

$$H_1 = \{e\lambda + au + bv + f_0 + (2a\beta_0 - e\gamma_1)\lambda u$$

$$+(2a\beta_1 - a\gamma_1)u^2 + b\gamma_1 uv + f_2\} \xi + \overline{\xi}$$

and

$$H_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g_0 + \delta\alpha_0 \lambda^2 + (\hat{a}(\alpha_0 + 2\beta_0) + \delta(\alpha_1 + \gamma_1))\lambda u$$

$$+b\alpha_0 \lambda v + \hat{a}(\alpha_1 + 2\beta_1 + \gamma_1)u^2 + \hat{b}(\alpha_1 + 3\gamma_1)uv + g_2\} \xi$$

$$+[du + h_0 + d(\alpha_0 + 4\beta_0)\lambda u + d(\alpha_1 + 4\beta_1)u^2 + h_2] \xi^2.$$  

We change $\lambda$ to $\lambda + \eta_0 \lambda^2$. Then $H_1$ and $H_2$ are transformed to the following $H'_1$ and $H'_2$, respectively.

$$H'_1 = \{e\lambda + au + bv + f_0 + e\eta_0 \lambda^2 + (2a\beta_0 - e\gamma_1)\lambda u$$

$$+(2a\beta_1 - a\gamma_1)u^2 + b\gamma_1 uv + f_2\} \xi + \overline{\xi}$$

and

$$H'_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g_0 + \delta(\alpha_0 + \eta_0)\lambda^2 + (\hat{a}(\alpha_0 + 2\beta_0) + \delta(\alpha_1 + \gamma_1))\lambda u$$

$$+\hat{b}\alpha_0 \lambda v + \hat{a}(\alpha_1 + 2\beta_1 + \gamma_1)u^2 + \hat{b}(\alpha_1 + 3\gamma_1)uv + g_2\} \xi$$

$$+[du + h_0 + d(\alpha_0 + 4\beta_0)\lambda u + d(\alpha_1 + 4\beta_1)u^2 + h_2] \xi^2.$$  

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Note that \(h_0\) and \(h_3\) satisfy \(h_0(\lambda, 0, 0, 0) \equiv h_3(\lambda, 0, 0, 0) \equiv 0\). We then transform \(H'_2\) in to \(K_2 = H'_2 + \eta_1\zeta(\overline{\xi}H'_1 + \xi H'_1)\) and change \(\zeta\) into \(\zeta + \eta_2u\zeta^2\). Then the equations are transformed to

\[
K_1 = \{e\lambda + au + bv + f_0 + e\eta_0\lambda^2 + (2a\beta_0 - e\gamma_1)\lambda u \\
+ (2a\beta_1 - a\gamma_1 + \eta_2)u^2 + b\gamma_1uv + f_2\} \xi + \overline{\xi}\zeta
\]

and

\[
K_2 = (\delta \lambda + \hat{\lambda}u + \hat{\lambda}v + (\hat{\lambda} + 2\eta_1)r + g_0 + \delta(\alpha_0 + \eta_0)\lambda^2 + (\hat{\alpha}(\alpha_0 + 2\beta_0) + \delta(\alpha_1 + \gamma_1) + 2\eta_1)\lambda u \\
+ b\alpha_0\lambda v + (\hat{\alpha}(\alpha_1 + 2\beta_1 + \gamma_1) + 2\alpha\eta_1)u^2 + (\hat{b}(\alpha_1 + 3\gamma_1) + 2b\eta_1 + \hat{b}\eta_2)uv + g_2)\zeta \\
+ [du + h_0 + (d(\alpha_0 + 4\beta_0) + \delta\eta_2)\lambda u + (d(\alpha_1 + 4\beta_1) + \hat{\alpha}\eta_2)u^2 + \hat{b}\eta_2uv + h_2]\xi^2.
\]

Note that \(h_3\) satisfy \(h_3(\lambda, 0, 0, 0) \equiv 0\). In order to show that the coefficients listed in the proposition can be put to zero, we have only to prove that the following \(8 \times 8\) matrix is nonsingular under the conditions stated in the proposition.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & e & 0 & 0 \\
0 & 0 & 2a & 0 & -e & 0 & 0 & 0 \\
0 & 0 & 0 & 2a & -a & 0 & 0 & 1 \\
\delta & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\
\hat{\lambda} & \delta & 0 & 2\hat{\alpha} & 0 & \delta & 0 & 2\epsilon \\
0 & \hat{\lambda} & 0 & 2\hat{\alpha} & \hat{\alpha} & 0 & 2a & 0 \\
d & 0 & 4d & 0 & 0 & 0 & 0 & \delta \\
0 & d & 0 & 4d & 0 & 0 & 0 & \hat{\alpha}
\end{pmatrix}
\]

We omit the details of the computation. Utilizing the same technique in Proposition 4.4, we can make \(f(\lambda, 0, 0, 0)\) and \(g(\lambda, 0, 0, 0)\) identically zero. \(\square\)

From now on we consider \((5.3, 4)\) in which the coefficients of \(\lambda^2, \lambda u, u^2\) in the Taylor expansion of \(f, g, h\) vanish and \(f(\lambda, 0, 0, 0) \equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0\). We compute \(dG\) as follows:

\[
\frac{\partial G_1}{\partial \xi} = e\lambda + 2au + bv + f + uf + f_r\frac{\partial r}{\partial \xi},
\]

\[
\frac{\partial G_1}{\partial \zeta} = (a + f_u)\xi^2 + f_r\xi\frac{\partial r}{\partial \xi} + \zeta,
\]

\[
\frac{\partial G_1}{\partial \overline{\xi}} = (b\overline{\xi} + f_u\overline{\xi} + f_r\frac{\partial r}{\partial \overline{\xi}})\xi + \overline{\xi},
\]

\[
\frac{\partial G_1}{\partial \overline{\zeta}} = (b\zeta + f_u\zeta + f_r\frac{\partial r}{\partial \zeta})\xi,
\]

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\[
\frac{\partial G_2}{\partial \xi} = 2\xi(du + h) + \xi^2(d\xi + h_u\xi + h_r\frac{\partial r}{\partial \xi}) + (\hat{d}\xi + g_u\xi + (\hat{c} + g_r)\frac{\partial r}{\partial \xi})\xi,
\]
\[
\frac{\partial G_2}{\partial \xi} = (\hat{d}\xi + g_u\xi + (\hat{c} + g_r)\frac{\partial r}{\partial \xi})\xi + \xi^2(d\xi + h_u\xi + h_r\frac{\partial r}{\partial \xi}),
\]
\[
\frac{\partial G_2}{\partial \xi} = \delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + \nu \nu u + (\hat{c} + g_r)\frac{\partial r}{\partial \xi} + \xi^2(h_v\xi + h_r\frac{\partial r}{\partial \xi}),
\]
\[
\frac{\partial G_2}{\partial \xi} = (\hat{b}\xi + g\xi + (\hat{c} + g_r)\frac{\partial r}{\partial \xi})\xi + \xi^2(h_v\xi + h_r\frac{\partial r}{\partial \xi}).
\]

where the subscript means differentiation. By these formulas we obtain
\[
dG(X_1) = (e\lambda + 3au + bv + f + 2uf_r + 2rf_r)X_1
\]
\[
+ X_2 + (2\hat{a}u + 2ug_v + 2r(\hat{c} + g_r))X_3 + [4du + 2h + 2uh_u + 2rh_r]X_4,
\]
\[
dG(X_2) = [v + 2r(a + f_u) + 2uvf_r]X_1 + [e\lambda + au + bv + f]X_2
\]
\[
+ [2u(du + h) + 2\hat{a}r + 2rg_u + 2uv(\hat{c} + g_r)]X_3 + [2r(d + u) + 2uvh_r]X_4,
\]
\[
dG(X_3) = [2bv + 2vf_v + rf_r]X_1 + X_2
\]
\[
+ [\delta \lambda + \hat{a}u + 3bv + \hat{c}r + g + 2uv + r(\hat{c} + g_r)]X_3 + [rh_r + 2vh_v]X_4,
\]
\[
dG(X_4) = [2br + 2rf_v + uf_r + u]X_1 + [\hat{b}r + 2rg_v + u^2(\hat{c} + g_r)]X_3
\]
\[
+ [\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + 2rh_v + u^2h_r]X_4.
\]

Our next task is to compute $T_jG$ \quad $j = 1, 2, \ldots, 8$, where $T_j$ is given in Proposition 2.3.
\[
T_1G = (e\lambda + au + bv + f)X_1 + X_2
\]
\[
T_2G = [u(e\lambda + au + bv + f) + 2r]X_1 - uX_2
\]
\[
T_3G + T_4G = [2v(\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g) + 2r(du + h)]X_1
\]
\[
T_3G - T_4G = 2(du + h)(rX_1 - uX_2)
\]
\[
T_5G + T_6G = [2u(e\lambda + au + bv + f) + 2r]X_3
\]
\[
T_5G - T_6G = 2rX_3 - 2uvX_4
\]
\[
T_7G = [v(\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g) + 2r(du + h)]X_3 - v(du + h)X_4
\]
\[
T_8G = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)X_3 + (du + h)X_4
\]

Note that $T_3G - T_4G$ and $T_7G$ are $E$-combination of the remaining 6 elements. We have to compute $is$-multiple of these elements. But this is easily done by (3.13). In conclusion we have
PROPOSITION 5.3.  \( \hat{G} \) is generated by the following 13 elements:

\[
W_1 = [2au + 2uf_u + 2rf_r]X_1 + [2\hat{a}u + 2ug_u + 2r(\hat{c} + g_r)]X_3 + [4du + 2h + 2uh_u + 2rh_r]X_4
\]

\[
W_2 = [v + 2r(a + f_u) + 2uvf_r]X_1 + [\varepsilon\lambda + au + bv + f]X_2
\]

\[+ [2u(du + h) + \hat{a}r + 2rg_u + 2uv(\hat{c} + g_r)]X_3 + [(d + h_u)2r + 2uvh_r]X_4,\]

\[
W_3 = [2bv + 2vf_v + rf_r]X_1 + X_2 + [2\hat{b}v + 2vg_v + r(\hat{c} + g_r)]X_3 + [-du + rh_r + 2vh_v - h]X_4,
\]

\[
W_4 = [2br + 2rf_v + u^2f_r + u]X_1 + [2\hat{b}r + 2rg_v + u^2(\hat{c} + g_r)]X_3
\]

\[+ [\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + u^2h_r + 2rh_v]X_4.
\]

\[
W_5 = (\varepsilon\lambda + au + bv + f)X_1 + X_2,
\]

\[
W_6 = [v((\hat{a} - \varepsilon\delta a)u + (\hat{b} - \varepsilon\delta b)v + g - \varepsilon\delta f) + r(du + \hat{c}v + h)]X_1 - \varepsilon\delta vX_2
\]

\[
W_7 = rX_1 - uX_2,
\]

\[
W_8 = [u((a - \varepsilon\delta a)u + (b - \varepsilon\delta b)v + f - \varepsilon\delta g) + r - \varepsilon\delta \hat{c}ru]X_3 - \varepsilon\delta u(du + h)X_4
\]

\[
W_9 = rX_3 - vX_4
\]

\[
W_{10} = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)X_3 + (du + h)X_4,
\]

\[
W_{11} = uvX_1 - rX_2,
\]

\[
W_{12} = v(u^2X_3 - rX_4),
\]

\[
W_{13} = (du + h)(u^2X_3 - rX_4).
\]

From these 13 elements we now make elements which contain \( X_1 \) and \( X_3 \) only:

\[
Y_1 = \frac{1}{2}u(du + h)W_1 + \varepsilon\delta(2du + h + uh_u + rh_r)W_8
\]

\[
= u(du + h)(au + uf_u + rf_r)X_1 + [u(du + h)(\hat{a}u + ug_u + r(\hat{c} + g_r))
\]

\[+ (2du + h + uh_u + rh_r)((ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (fe\delta - g)v + \varepsilon\delta r + \hat{c}ru)]X_3
\]

\[
Y_2 = \frac{v}{2}W_1 + (2du + h + uh_u + rh_r)W_9
\]

\[
= (auv + uvf_u + rf_v)X_1 + [\hat{a}uv + uvg_u + rv(\hat{c} + g_r) + r(2du + h + uh_u + rh_r)]X_3
\]

\[
Y_3 = u(du + h)W_3 + (rh_r + 2vh_v - du - h)\varepsilon\delta W_8 + (du + h)W_7
\]
\[= [u(du + h)(2bv + 2vf_v + rf_r) + r(du + h)]X_1 + [u(du + h)(2bv + 2vg_v + r(\hat{\delta} + g_r))] \]
\[+(rh_r + 2vh_v - du - h)((ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (fe\delta - g)u + e\delta r - \hat{c}ru)]X_3 \]
\[Y_4 = vW_3 + e\delta W_6 + (rh_r + 2vh_v - du - h)W_9 \]
\[= [v(2bv + 2vf_v + rf_r) + (\hat{a}e\delta - a)uv + (\hat{b}e\delta - b)v^2 + (g\delta - f)v + e\delta r(du + \hat{c}v + h)]X_1 \]
\[+[2bv^2 + 2v^2 g_v + rv(\hat{c} + g_r) + r(rh_r + 2vh_v - du - h)]X_3 \]
\[Y_5 = \frac{1}{2}r(du + h)W_1 + (2du + h + uh_h + rh_r)W_{13} \]
\[= r(du + h)(au + uf_u + rf_r)X_1 + (du + h)[\hat{a}ru + rug_u + r^2(\hat{c} + g_r) + u^2(2du + h + uh_u + rh_r)]X_3 \]
\[Y_6 = r(du + h)W_3 + (du + h)W_{11} + (rh_r + 2vh_v - du - h)W_{13} \]
\[= (du + h)[2brv + 2rvf_v + r^2 f_r + uv]X_1 \]
\[+(du + h)[2brv + 2rvu + r^2(\hat{c} + g_r) + u^2(rh_r + 2vh_v - du - h)]X_3 \]
\[Y_7 = rvW_3 + vW_{11} + (rh_r + 2vh_v - du - h)W_{12} \]
\[= v[2brv + 2rvf_v + r^2 f_r + uv]X_1 + v[2brv + 2rvu + r^2(\hat{c} + g_r) + u^2(rh_r + 2vh_v - du - h)]X_3 \]
\[Y_8 = (du + h)(uW_2 + (e\lambda + au + bv + f)W_7 - rW_5 - W_{11}) + e\delta(2r(d + h_u) + 2uvh_r)W_8 \]
\[= (du + h)(2ru(a + f_u) + 2u^2 v f_r)X_1 + [(du + h)(2u^2(du + h) + 2\hat{a}ru + rug_u + 2u^2 v(\hat{c} + g_r)) \]
\[+(2r(d + h_u) + 2uvh_r)((ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (fe\delta - g)u + re\delta - \hat{c}ru)]X_3 \]
\[Y_9 = vW_2 + e\delta(e\lambda + au + bv + f)W_6 - [(\hat{a}e\delta - a)uv + (\hat{b}e\delta - b)v^2 + v(g\delta - f) + e\delta r(du + \hat{c}v + h)]W_5 \]
\[+(2r(d + h_u) + 2uvh_r)W_9 - e\delta(du + \hat{c}v + h)W_{11} - ((\hat{a} - e\delta a)u + (\hat{b} - e\delta b)v + g - e\delta f)W_6 \]
\[= [v^2 + 2rv(a + f_u) + 2uv^2 f_r - e\delta(du + \hat{c}v + h)uv \]
\[-(\hat{a} - e\delta a)u + (\hat{b} - e\delta b)v + g - e\delta f)((\hat{a} - e\delta a)uv + (\hat{b} - e\delta b)v^2 + (g - e\delta f)v + r(du + \hat{c}v + h))X_1 \]
\[+[2uv(du + h) + 2\hat{a}rv + 2rvu + 2uv^2(\hat{c} + g_r) + 2r^2(d + h_u) + 2uvh_r]X_3 \]
\[Y_{10} = u(du + h)W_4 + e\delta(\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + 2rh_v + u^2 h_r)W_8 \]
\[= -[((ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (fe\delta - g)u + re\delta - \hat{c}ru)]W_{10} - (e\delta - \hat{c}u)W_{13} \]
\[= -[(ae\delta - \hat{a})u + (be\delta - \hat{b})v + f{\delta - g}]e\delta W_8 \]

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\[
= u(du + h)(2br + 2rf_r + u^2f_r + u)X_1 + [u(du + h)(2b\hat{r} + 2rg_u + u^2(\hat{c} + g_r)) \\
+ ((ae\delta - \hat{a})u^2 + (be\delta - \hat{b})uv + (fe\delta - g)u + re\delta - \hat{c}r)u) \\
x(-a - e\delta\hat{a}u - (b - e\delta\hat{b})v - (f - e\delta g) + 2rh_v + u^2h_r)]X_3
\]

**Proposition 5.4.** If \(e\delta b\hat{a}d(b - e\delta b)(-4e\delta d - (a - e\delta\hat{a})^2)(d - a\hat{a}) \neq 0\), then the following elements belong to \(\tilde{\Gamma}G\).

\[u^4X_j, \ u^2vX_j, \ uv^2X_j, \ v^3X_j, \ ,ru^2X_j, \ ruvX_j, \ rv^2X_j \quad (j = 1, 3)\]

**Proof.** Let us define \(V_j \quad (j = 1, 2, \ldots, 14)\) by

\[V_1 = u^4X_1, V_2 = u^2vX_1, V_3 = uv^2X_1, V_4 = v^3X_1, V_5 = ru^2X_1, V_6 = ruvX_1, V_7 = rv^2X_1, V_8 = u^4X_3, V_9 = u^2vX_3, V_{10} = uv^2X_3, V_{11} = v^3X_3, V_{12} = ru^2X_3, V_{13} = ruvX_3, V_{14} = rv^2X_3.\]

The remaining part of the proof is similar to that of Proposition 4.6. We consider \(vY_1, uvY_2, uY_3, vY_3, vY_4, Y_5, Y_6, Y_7, Y_8, uY_9, vY_9, uY_{10}, vY_{10}\). We have the following \(14 \times 14\) matrix and the proof is completed if we show that the determinant of this matrix is nonzero under the conditions given above.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2e\delta d & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & 0 & 2d \\
0 & 0 & a & 0 & 0 & 0 & 0 & \hat{a} & 0 & 0 & \hat{c} \\
0 & 0 & 0 & d & 0 & 0 & \epsilon\delta\epsilon\delta p & 0 & 0 & -\epsilon\delta d & 0 \\
0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & -\epsilon\delta d & 0 \\
0 & 0 & p & q + 2b & 0 & \epsilon\delta d & \epsilon\delta \hat{c} & 0 & 0 & 0 & 2b \\
0 & 0 & 0 & ad & 0 & 0 & 2d^2 & 0 & 0 & \hat{a}d & 0 \\
0 & d & 0 & 0 & 2bd & 0 & -d^2 & 0 & 0 & 0 & 2bd \\
0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 2b \\
0 & 0 & 0 & 2ad & 0 & 2d^2 & 2\epsilon\delta e & 0 & 0 & 2\epsilon\delta ad & -2\epsilon\delta d & 0 \\
0 & 0 & 1 & 0 & 2a & 0 & 0 & 0 & 0 & 0 & 2\hat{a} \\
d & 0 & 0 & 0 & 0 & -\epsilon\delta p^2 & 0 & 0 & \epsilon\delta p & \epsilon\delta q & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 0 & 0 & 2\hat{a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon\delta p & \epsilon\delta q & 0
\end{pmatrix}
\]

where \(p = \hat{a}e\delta - a, q = b\epsilon\delta - b\). We can see that under the assumption stated in the proposition, this is nonsingular. We omit the details of the computation. \(\Box\)
As in section 4, we deduce in the following way. These diagram means that if we consider the left hand side multiplied by \( \lambda, u, v \) or \( r \), the right hand side is proved to belong to \( \tilde{\Gamma}G \).

\[
\begin{align*}
W_7 & \rightarrow u^3X_2, & W_6 & \rightarrow uvX_2, v^2X_2, \\
W_9 & \rightarrow u^2vX_4, uv^2X_4, v^3X_4, & W_{12} & \rightarrow rvX_4 \\
W_{11} & \rightarrow ruX_2, rvX_2, r^2X_2, & W_7 & \rightarrow rvX_1, & u^3W_1 & \rightarrow u^4X_4 \\
W_8 & \rightarrow rvX_3, & W_9 & \rightarrow v^2X_4, & Y_9 & \rightarrow v^2X_1 \\
uvW_5 & \rightarrow \lambda uvX_1, & \lambda W_{11} & \rightarrow \lambda rX_2, & rW_2 & \rightarrow r^2X_4, \\
uW_{13} & \rightarrow ru^2X_4, & uvW_{10} & \rightarrow \lambda uvX_3 \\
u^3W_{10} & \rightarrow \lambda u^3X_3, & \lambda W_{13} & \rightarrow \lambda ruX_4.
\end{align*}
\]

Now we consider \( u^2W_1, rW_1, u^2W_3, rW_3, u^2W_4, u^2W_{10}, \frac{1}{2} \lambda uW_1, u^2W_5, \lambda W_8, rW_{10} \), \( uW_7 \) and \( uW_8 \). These 12 formulas are represented in the way that \( BV \) belongs to \( \tilde{\Gamma}G \), where \( V \) is a vector given by

\[
V = (u^3X_1, \lambda u^2X_1, u^3X_3, \lambda u^2X_3, u^3X_4, \lambda u^2X_4, u^2X_2, \lambda rX_3, rX_2, ruX_1, ruX_3, ruX_4)
\]

and \( B \) is a matrix whose entries are in \( E \) and reduces to the following when \( (\lambda, u, v, r) = (0, 0, 0, 0) \).

\[
\begin{pmatrix}
    a & 0 & \hat{a} & 0 & 2d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & \hat{a} & 2d \\
    0 & 0 & 0 & 0 & -d & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
    0 & \hat{a} & \delta & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & a & 0 & \hat{a} & 0 & 2d & 0 & 0 & 0 & 0 & 0 \\
    0 & \epsilon & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a - \epsilon \delta \hat{a} & 0 & -\epsilon \delta d & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -d \\
    0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & a - \epsilon \delta \hat{a} & 0 & -\epsilon \delta d & 0 & 0 & 0 & 0 & 1 & 0 & 0 
\end{pmatrix}
\]

We can show that this matrix is nonsingular if \( \epsilon \delta \hat{a} \neq 0 \). Therfore we have proved that \( u^3X_1, \lambda u^2X_1, u^3X_3, \lambda u^2X_3, u^3X_4, \lambda u^2X_4, u^2X_2, \lambda rX_3, rX_2, ruX_1, ruX_3, ruX_4 \) belong to \( \tilde{\Gamma}G \).

Consider next \( vW_4 + (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + 2rh_v + u^2h_r)W_9 - rW_{10} - W_{13} \). Then we see that \( uvX_1 \) belongs to \( \tilde{\Gamma}G \). By \( Y_2, uvX_3 \) belongs to \( \tilde{\Gamma}G \). \( Y_4 \), then, assures that \( v^2X_3 \) belongs to \( \tilde{\Gamma}G \). \( W_{11} \) implies that \( rX_2 \) belongs to \( \tilde{\Gamma}G \). We have now
PROPOSITION 5.5. If \( \epsilon \delta b \hat{a} b d (b - \epsilon \delta b)(-4 \epsilon \delta d -(a - \epsilon \delta \hat{a})^2)(d - a \hat{a}) \neq 0 \), then the following elements belong to \( \tilde{\Gamma} G \).

\[
(5.5) \quad M^3 X_j, rMX_j, \lambda uX_j, \nu vX_j, v^2 X_j (j = 1, 3, 4), \quad M^2 X_2, \quad rX_2, \quad vX_2,
\]

Proof is very easy. By this proposition, the \( E \)-module \( \tilde{\Gamma} G \) is generated by (5.5) and

\[
Z_1 = [2au + 2uf_n]X_1 + [2\hat{a}u + 2ug_u + 2r\hat{c}]X_3 + [4du + 2h + 2uh_u]X_4
\]

\[
Z_2 = [v + 2ar]X_1 + [\epsilon \lambda + au + bv + f]X_2 + [2du^2 + 2ar]X_3 + 2drX_4,
\]

\[
Z_3 = 2bvX_1 + X_2 + [2b\hat{v} + \hat{c}r]X_3 + [-du - h]X_4,
\]

\[
Z_4 = [2br + u^2 f_r + u]X_1 + [2br + u \hat{c} g_r]X_3 + [\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + u^2 h_r]X_4.
\]

\[
Z_5 = (\epsilon \lambda + au + bv + f)X_1 + X_2,
\]

\[
Z_6 = rX_1 - uX_2,
\]

\[
Z_7 = [(a - \epsilon \delta \hat{a})u^2 + (f - \epsilon \delta g)u + r]X_3 - \epsilon \delta du^2 X_4
\]

\[
Z_8 = rX_3 - vX_4
\]

\[
Z_9 = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)X_3 + (du + h)X_4,
\]

By the conditions imposed on \( f, g, h \), these genetors are independent of \( f, g, h \). Therefore we have proved

THEOREM 5.1. If we assume the conditions in Propositions 5.2 and 5.5, the bifurcation equations (5.3,4) are \( O(2) \)-equivalent to

\[
(5.6) \quad G_1 = (\epsilon \lambda + au + bv)\xi + \bar{\xi} \zeta,
\]

\[
(5.7) \quad G_2 = (\delta \lambda + \hat{a}u + \hat{b}v + \hat{c}r)\zeta + du \xi^2,
\]

The following \( F \) is a universal unfolding of this \( G \).

\[
(5.8) \quad F_1(\alpha, \beta, s_1, s_2, s_3, s_4, \lambda, \xi, \zeta) = (\epsilon \lambda + \alpha + au + bv)\xi + \bar{\xi} \zeta,
\]

\[
(5.9) \quad G_2 = (\delta \lambda + (\hat{a} + s_1)u + (\hat{b} + s_2)v + (\hat{c} + s_3)r + g)\zeta + (\beta + (d + s_4)u)\xi^2,
\]
REFERENCES