BUCKLING FOR AN ELASTOPLASTIC PLATE WITH AN INCREMENTAL CONSTITUTIVE RELATION

By

Jean-Pierre Puel
and
Annie Raoult

IMA Preprint Series # 359
October 1987

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
# Author(s) | Title | # Author(s) | Title
---|---|---|---
Harry Kesten | Scaling Relations for 2D-Perculation | 242. A. Leizarowitz | Infinite Horizon Optimization for Markov Processes with Finite States Spaces
Louis H. Y. Chen | The Rate of Convergence in a Central Limit Theorem for Dependent Random Variables with Arbitrary Index Set | 244. G. Kallianpur | Stochastic Differential Equations in Duals of Nuclear Spaces with some Applications
Tzu-Shuh Chiung, Yunshong Chow, Yuh-Jia Lee | Evaluation of Certain Functional Integrals | 246. L. Karp, M. Pinsky | The First Eigenvalue of a Small Geodesic Ball in a Riemannian Manifold
Chi-Sing Man | Towards an Acoustoelastic Theory for Measurement of Residual Stresses | 248. Andreas Stoll | Invariance Principles for Brownian Intersection Local Time and Polymer Measures
A. Carverhill | Conditioning a `Lifted' Stochastic System in a Product Case | 256. R. J. Williams | Local Time and Excursions of Reflected Brownian Motion
H. Holmer, S. Orey | Large Deviations for the Empirical Field of a Gibbs Measure | 258. A. Leizarowitz | Characterization of Optimal Trajectories on an Infinite Horizon
V. Perez-Abreu | Decompositions of Semimartingales on Duals of Countably Nuclear Spaces | 262. J. M. Ball | Does Rank-One Convexity Imply Quasiconvexity?
P. N. Shivakumar, Chi-Sing Man, Simon W. Rabin | Modelling of the Heart and Pericardium at End-Diastole | 266. Jose-Luis Menaldi | Probabilistic View of Estimates for Finite Difference Methods
Chi-Sing Man, Quan-Xin Sun | Stress Effects in the Flow of Glaciers | 274. Omar Hijab | On Partially Observed Control of Markov Processes
Lawrence Gray | The Behavior of Processes with Statistical Mechanical Properties | 276. R. Hardt, D. Kinderlehrer, M. Lucskin | Remarks about the Mathematical Theory
C. Foias, B. Nicolaenko, G.R.Sell, R. Temam | Inertial Manifolds for the Kuramoto-Sivashinsky Equation and an Estimate of their Lowest Dimension | 280. R. Duran | On the Approximation of Mieable Displacement in Porous Media by a Method of Characteristics Combined with a Mixed Method
H. Aixiang, Zhang Bo | The Convergence for Nodal Expansion Method | 262. V. Tversky | Dispersive Bulk Parameters for Coherent Propagation in Correlated Random Distributions
Li Kaitai, Yan Ningning | The Extrapolation for Boundary Finite Elements | 288. R. Durrett, R.H. Schonmann | Stochastic Growth Models
Leo Arbogast | Analysis of the Simulation of Single Phase Flow Through a Naturally Fractured Reservoir | 296. He Yinnian, Li Kaitai | The Coupling Method of Finite Elements and Boundary Elements for Radiation Problems
Li Kaitai | Perturbation Solutions of Simple and Double Parabolic Equation with a Gradient Term | 300. Chen Zhangxin, Li Kaitai | The Convergence on the Multigrid Algorithm for Navier-Stokes Equations
Ricardo G. Duran | Error Analysis in $L^p$, $1 \leq p \leq \infty$, for Mixed Finite Element Methods for Linear and Quasi-Linear Elliptic Problems | 304. Nochetto, Ricardo H., Verdi, Claudio | An Efficient Linear Scheme to Approximate Parabolic Free Boundary Problems: Error Estimates and Implementation
BUCKLING FOR AN ELASTOPLASTIC PLATE WITH
AN INCREMENTAL CONSTITUTIVE RELATION

Jean-Pierre Puel *; Annie Raoult *

Résumé : On étudie le flambage d'une plaque élastoplastique chargée latéralement par une force $\lambda(t)F(x_1, x_2)$ dépendant du temps. Le modèle choisi est quasi-statique; la loi de comportement incrémentale permet de distinguer le chargement du déchargement. On donne une condition nécessaire pour l'existence d'une solution non triviale après un instant $t_0$: cette condition fait intervenir l'histoire du chargement sur tout l'intervalle $[0, t_0]$, et ne peut s'exprimer en fonction de la seule valeur de $\lambda$ à l'instant $t_0$. Sous une condition suffisante, portant sur la dérivée par rapport au temps du chargement à l'instant $t_0$, on obtient l'existence de branches bifurquées et leur comportement local; on montre, en particulier, que la bifurcation peut se produire alors que la charge diminue.

Abstract : We study the buckling of an elastoplastic plate loaded on its edges by a planar thrust $\lambda(t)F(x_1, x_2)$ depending on time. The incremental constitutive relation is of Prandtl-Reuss type and permits the distinction between loading and unloading. We give a necessary condition for the existence of a nontrivial solution (together with the zero one) after a time $t_0$; an important feature is that this condition takes into account the whole story of the loading and cannot be expressed in terms of its value at a single time. We prove the existence of bifurcated branches, for some sufficient condition involving the slope of $\lambda$ at time $t_0$, and describe their local behaviour; we point out the fact that buckling can actually occur, when the loading is decreasing.

*Part of this work was done while the authors were visiting the I.M.A. in Minneapolis, Minnesota, U.S.A.
1. INTRODUCTION

This paper is devoted to the study of the quasistatic equilibria of a thin elastoplastic plate with an incremental constitutive relation, loaded on its edges by a planar thrust depending on time.

The middle surface of the plate occupies a bounded regular open set $\omega$ in $\mathbb{R}^2$, with boundary $\partial \omega$ and exterior normal $n = (n_1, n_2)$. During a time interval $[0, T]$, we apply on the boundary $\partial \omega$ a planar force the density of which is taken to be $\lambda(t)F(x_1, x_2)$, $(x_1, x_2) \in \partial \omega$, $t \in [0, T]$, $\lambda(t) \in \mathbb{R}$. The function $\lambda(.)$ will be the loading parameter. In the present work, we won't consider volumic forces nor forces applied on the upper and the lower surface of the plate. In this situation and within the restrictions imposed on $\lambda(.)$ by the limit-analysis, there always exists an equilibrium corresponding to the zero deflexion that we will call trivial solution.

After having given a precise statement of the problem in section 2, we exhibit in section 3 necessary conditions on $\lambda(.)$ for the existence of non trivial solutions close to the trivial one after a time $t_0$ (bifurcation). These conditions will take into account the behaviour of $\lambda$ on the whole interval $[0, t_0]$ and not only the value $\lambda(t_0)$. In section 4, we prove the actual existence of non trivial solutions bifurcating from the trivial one at time $t_0$ (buckling phenomenon) under some sufficient conditions on $\lambda(.)$ which involve the slope of $\lambda(.)$ at time $t_0$. Finally, we make some comments upon the results in section 5.

Useful technical results are gathered in Appendices A and B.
2. STATEMENT OF THE PROBLEM

Let \( u = (u_1, u_2) \) be the horizontal displacement of a point of the plate and \( \xi \) the vertical displacement or deflexion. The planar linear strain tensor is

\[
\varepsilon_{\alpha\beta}(u) = \frac{1}{2} \left( \partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha} \right), \quad \alpha, \beta = 1, 2.
\]

We will consider an uncomplete nonlinear strain tensor defined by

\[
\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta}(u) + \delta_{\alpha\beta}(\xi),
\]

where

\[
\delta_{\alpha\beta}(\xi) = \frac{1}{2} \left( \partial_{\alpha} \xi \partial_{\beta} \xi \right).
\]

The stresses of the plate can be defined by two symmetric tensors: the tension stress tensor \( \sigma = (\sigma_{\alpha\beta}) \), and the flexion stress tensor \( M = (M_{\alpha\beta}) \).

Using the classical summation convention for repeated indices, the equilibrium equations can now be expressed as follows:

\[
\partial_{\beta} \sigma_{\alpha\beta} = 0 \quad \text{in} \quad \omega \times (0, T), \quad (2.1)
\]

\[
\sigma_{\alpha\beta} \cdot n_{\beta} = \lambda(t) F_{\alpha} \quad \text{on} \quad \partial \omega \times (0, T), \quad (2.2)
\]

\[
\partial_{\alpha\beta} M_{\alpha\beta} + h \partial_{\alpha} (\sigma_{\alpha\beta} \partial_{\beta} \xi) = 0 \quad \text{in} \quad \omega \times (0, T). \quad (2.3)
\]

Assuming, for simplicity, the plate to be clamped, we add the boundary conditions

\[
\xi = 0, \quad \partial_{\alpha} \xi = 0 \quad \text{on} \quad \partial \omega \times (0, T). \quad (2.4)
\]

Of course, (2.1), (2.2) imply the compatibility conditions

\[
\int_{\partial \omega} F_1 \, ds = \int_{\partial \omega} F_2 \, ds = 0; \int_{\partial \omega} (x_1 F_2 - x_2 F_1) \, ds = 0. \quad (2.5)
\]

We assume that, at time \( t = 0 \), the thrust is zero and that the system is at rest, which means that
\[ \lambda(0) = 0 \quad , \quad \sigma(0) = 0 \quad , \quad u(0) = 0 \quad , \quad \xi(0) = 0 . \]

In order to fully describe the model, we need to complete the system of equations, by specifying the constitutive relations between the stress and the strain tensors. First of all, we assume the existence of a linear relation between the flexion stress tensor and the curvatures, which means that there exist coefficients \( A_{\alpha\beta\gamma\delta} \) satisfying
\[ A_{\alpha\beta\gamma\delta} = A_{\beta\alpha\gamma\delta} = A_{\alpha\gamma\beta\delta} = A_{\gamma\delta\alpha\beta} . \]
\[ \exists \rho_0 > 0 \text{ such that } \forall \xi = (\xi_{\alpha\beta}) \in \mathbb{R}^4 , \quad A_{\alpha\beta\gamma\delta} \xi_{\alpha\beta} \xi_{\gamma\delta} \geq \rho_0 |\xi|^2 , \]
such that
\[ M_{\alpha\beta} = - A_{\alpha\beta\gamma\delta} \delta_{\gamma\delta} \xi , \quad \forall \alpha , \beta = 1 , 2 . \]

The second constitutive relation, between the tension stress tensor and the nonlinear planar strain tensor, is supposed to be an incremental plastic relation of Prandtl-Reuss type which can be expressed as follows: there exist a non-empty closed convex set \( C \) in \( \mathbb{R}^4 \) and coefficients \( a_{\alpha\beta\gamma\delta} \) satisfying
\[ a_{\alpha\beta\gamma\delta} = a_{\beta\alpha\gamma\delta} = a_{\alpha\gamma\beta\delta} = a_{\gamma\delta\alpha\beta} . \]
\[ \exists \rho_0 > 0 \text{ such that } \forall \zeta = (\zeta_{\alpha\beta}) \in \mathbb{R}^4 , \quad a_{\alpha\beta\gamma\delta} \zeta_{\alpha\beta} \zeta_{\gamma\delta} \geq \rho_0 |\zeta|^2 , \]
such that if \( I_C \) denotes the indicator function of \( C \), and \( a \) the linear mapping from the space of second order symmetric tensors into itself defined by
\[ (a\tau)_{\alpha\beta} = a_{\alpha\beta\gamma\delta} \tau_{\gamma\delta} , \]
then
\[ \sigma(x,t) \in C ; \quad a\lambda \sigma(x,t) + \partial I_C(\sigma(x,t)) \in \partial \gamma(x,t) . \]

Relation (2.11) can be viewed in another way. If the strain tensor \( \gamma \) is divided into its elastic
part $\gamma^p$ and its plastic part $\gamma^p$, (2.11) becomes

\begin{equation}
(2.12) \quad \partial_i \gamma^p = a \partial_i \sigma; \quad \partial_i \gamma^p \in \partial l_C(\sigma); \quad \sigma \in C.
\end{equation}

This type of relation takes into account the distinction between loading and unloading of the plate.

For purely static models (which do not permit this distinction), similar constitutive relations have already been considered in [Demengel-Hadri], [Do], [Hadri], [Mignot-Puel 1].

As we have not yet specified the correct spaces for $\sigma$ and $\gamma$, (2.11) has to be taken in a formal sense. In fact, as will be seen in the sequel, we would like to obtain a solution $\sigma$ of (2.11) in a weak sense when $\gamma$ is given only continuous in time. This is still an open problem.

This and other technical difficulties lead us to the consideration of an approximate model. Let $\Phi$ be a mapping from $R_s^4$ into $R_s^4$ of class $C^1$ with Lipschitz derivative which "approaches" $\partial l_C$. For example, $\Phi$ could be a penalization or could correspond to adding a viscosity parameter in the constitutive relation. We denote by $\psi$ the associate Nemytsky operator defined on a function $\tau$ from $\omega$ into $R_s^4$ by

\begin{equation}
(2.13) \quad \psi(\tau)(x) = \Phi(\tau(x)) \quad \text{a.e. in } \omega \times (0, T).
\end{equation}

Properties of the operator $\psi$ are studied in Appendix A. We now consider the following equation which will be taken as an "approximate constitutive relation", and will henceforth be substituted to (2.11):

\begin{equation}
(2.14) \quad a \partial_i \sigma(., t) + \psi(\sigma(., t)) = \partial_i (\epsilon(u) + \delta(\xi))(., t), \quad t \in (0, T).
\end{equation}

Again we intend to give a weak sense to (2.14) and this is done by taking the associate integral formulation. More precisely, let us set

\begin{equation}
(2.15) \quad \Sigma_0 = \{ \tau \in [L^2(\omega)]_S^4, \partial_{n_\psi} \tau = 0, \tau_{n_\psi} n_\psi = 0 \}.
\end{equation}
and let $P$ denote the a-projection on $\Sigma_0$ in $[L^2(\omega)]^d_\delta$ defined for $g \in [L^2(\omega)]^d_\delta$ by

$$(2.16) \quad P(g) \in \Sigma_0 \ ; \ (aP(g), \tau) = (g, \tau) \ \forall \tau \in \Sigma_0,$$

where $(\ , \ , \ )$ is the scalar product in $[L^2(\omega)]^d_\delta$. Moreover, let us write $C_0([T_1, T_2]; X)$ for the space $\{ \nu \in C([T_1, T_2]; X); \nu(T_1) = 0 \}$.

If $\sigma^* \in C_0([0, T]; [L^2(\omega)]^d_\delta)$ satisfies $(2.1), (2.2)$, we give sense to $(2.14)$ together to $(2.1), (2.2)$ as follows:

$$(2.17) \quad \sigma \in C_0([0, T]; [L^2(\omega)]^d_\delta); \, \sigma(t) - \sigma^*(t) = P[\delta(\xi(t)) - a\sigma^*(t) - \int_0^t \phi(\sigma(s))ds].$$

Notice that $(2.17)$ makes sense for $\xi \in C_0([0, T]; H^2_0(\omega))$ because $\delta$ maps continuously $H^2_0(\omega)$ into $[L^2(\omega)]_\delta^d$ and $\phi$ maps continuously $[L^2(\omega)]_\delta^d$ into itself. Equation $(2.17)$ contains the following informations:

$$(\sigma(t) - \sigma^*(t) \in \Sigma_0), \text{ which means that } \sigma \text{ satisfies (2.1) and (2.2);}$$

$$(a\sigma(t) + \int_0^t \phi(\sigma(s))ds , \tau) = (\delta(\xi(t)), \tau), \ \forall \tau \in \Sigma_0), \text{ which implies the existence of } u(t) \in [H^1(\omega)]^2, \text{ such that } a\sigma(t) + \int_0^t \phi(\sigma(s))ds = \xi(u(t)) + \delta(\xi(t)); \text{ in other words, an integral formulation of (2.14) is satisfied.}$$

Therefore, a complete mathematical formulation of the model is the following:

The functions $\lambda \in C_0([0, T]), (F_\omega) \in [L^2(\omega)]^2_\delta$ satisfying $(2.5)$ and $\sigma^* \in C_0([0, T]; [L^2(\omega)]^d_\delta)$ satisfying $(2.1)$ and $(2.2)$ are given data; the stress tensor $\sigma \in C_0([0, T]; [L^2(\omega)]^d_\delta)$ and the vertical displacement $\xi \in C_0([0, T]; H_0^2(\omega))$ satisfy, for $t \in [0, T]$

$$(2.17) \quad \sigma(t) - \sigma^*(t) = P[\delta(\xi(t)) - a\sigma^*(t) - \int_0^t \phi(\sigma(s))ds],$$

$$(2.18) \quad \delta_{\alpha\beta}(A_{\alpha\beta\mu\nu}\partial_{\nu\mu}\xi(t)) - h\delta_{\alpha\beta}(\sigma_{\alpha\beta}(t)\partial_{\beta}\xi(t)) = 0.$$
A complete study of the equation

\[ \sigma(t) - \sigma^*(t) = P[g(t) - a\sigma^*(t) - \int_0^t \psi(\sigma(s))ds] , \]

where \( g \) is a given function, is performed in Appendix B. Notice that formulation (2.17), (2.18) does not involve the unknown function \( u \).

3. NECESSARY CONDITIONS FOR BIFURCATION

From Appendix B, Theorem B2, we know that there exists a unique solution \( \sigma^0 \) to the following problem

\[ \sigma^0 \in C_0([0, T] ; [L^2(\omega)]^4) , \quad \sigma^0(t) - \sigma^*(t) = P[-a\sigma^*(t) - \int_0^t \psi(\sigma^0(s))ds] . \]

Then, \((\sigma, \xi) = (\sigma^0, 0)\) is always solution of (2.17),(2.18). This is the only solution with zero deflexion and we will call it the trivial solution.

In this chapter, we are interested in conditions which are necessary for the existence, after a time \( t_0 \) and in a neighborhood of the trivial solution, of another solution of (2.17),(2.18) corresponding to a non zero deflexion.

DEFINITION 3.1:

A time \( t_0 \in \mathbb{R}^+ \) is called a bifurcation time if there exists \( T > t_0 \) and a solution \((\sigma, \xi)\) of (2.17),(2.18) on \([0, T]\) such that

\[ \forall t \in [0, t_0] , \quad \xi(t) = 0 , \]

\[ \forall t_1 \in [t_0, T] , \quad \exists t \in [t_0, t_1] , \quad \xi(t) \neq 0 . \]

Such a solution will be called a bifurcated solution.
REMARK 3.1:

1) If \((\sigma, \xi)\) is a bifurcated solution, then for \(t \in [0, t_0]\), \(\sigma(t) = \sigma^0(t)\).

2) As \(\sigma\) and \(\xi\) must be continuous in time, for \(t\) in a right neighborhood of \(t_0\),

\((\sigma(t), \xi(t))\) must be close to \((\sigma^0(t), 0)\).

For \(\sigma\) in \([L^2(\omega)]^d\), we define the operators \(A\) and \(B(\sigma)\) from \(H^2_0(\omega)\) to \(H^{-2}(\omega)\) by

\[(3.2)\quad A\theta = \partial_{\alpha} (A_{\alpha\beta\gamma} \partial_{\beta} \theta) ,\]

\[(3.3)\quad B(\sigma) \theta = \hbar \partial_\alpha [\sigma_{\alpha\beta} \partial_\beta \theta] .\]

Then, we can write the following result.

THEOREM 3.1:

A necessary condition for a time \(t_0\) to be a bifurcation time is that \(0\) is an eigenvalue of

the operator \(A - B(\sigma^0(t_0))\), or equivalently that there exists a non-zero function \(\theta\) such that

\[(3.4)\quad \theta \in H^2_0(\omega), \quad \|\theta\|_{H^2_0(\omega)} = 1, \quad A\theta - B(\sigma^0(t_0))\theta = 0 .\]

REMARK 3.2:

The possibility of bifurcation at time \(t_0\) is related to the value of \(\sigma^0(t_0)\) and therefore,

from (3.1), to the whole history of the loading parameter \(\lambda\) on the interval \([0, t_0]\). Then, the

actual bifurcation parameter is not, as usual, a real number but a function.

It is straightforward to check that the initial time \(0\) cannot be a bifurcation time.
Proof of Theorem 3.1.

Let us assume that $t_0$ is a bifurcation time in the sense of Definition 3.1, and let $(\sigma, \xi)$ be the associated bifurcated solution. We know that

$$\sigma \in C_0([0, T); [L^2(\omega)]_d^4), \ \xi \in C_0([0, T); H_0^2(\omega)).$$

We can choose a sequence $(t^n)_{n \in \mathbb{N}}$ such that

$$t^n > t_0 \ \forall \ n \in \mathbb{N}, \ \ t^n \to t_0 \ \text{if} \ n \to +\infty,$$

$$\xi(t^n) \to 0, \ \ \xi(t^n) \to 0 \ \text{in} \ H_0^2(\omega) \ \text{if} \ n \to +\infty,$$

$$\sigma(t^n) \to \sigma^0(t_0) \ \text{in} \ [L^2(\omega)]_d^4 \ \text{if} \ n \to +\infty.$$

From (2.18), we have

$$A\xi(t^n) - B(\sigma(t^n))\xi(t^n) = 0.$$

If we set

$$\xi^n = \xi(t^n)/\|\xi(t^n)\|_{H_0^2(\omega)},$$

$\xi^n$ satisfies

$$\|\xi^n\|_{H_0^2(\omega)} = 1, \ \ A\xi^n - B(\sigma(t^n))\xi^n = 0.$$

From the sequence $\xi^n$ we can extract a subsequence (again denoted by $\xi^n$), such that

$$\xi^n \to \theta \ \text{in} \ H_0^2(\omega) \ \text{weakly}.$$

As $\sigma(t^n)$ converges strongly to $\sigma^0(t_0)$ in $[L^2(\omega)]_d^4$, using the compactness of the embedding of $H_0^2(\omega)$ into $W^{1,p}(\omega)$ (for any finite $p$), we obtain

$$B(\sigma(t^n))\xi^n \to B(\sigma^0(t_0))\theta \ \text{in} \ H^{-2}(\omega) \ \text{strongly}.$$

Therefore

$$A\xi^n \to A\theta \ \text{in} \ H^{-2}(\omega) \ \text{strongly}.$$

which means that

\[ \xi^n \to 0 \text{ in } H^2_0(\omega) \text{ strongly}. \]

Passing to the limit, we obtain (3.4), which proves Theorem 3.1.

Theorem 3.1 provides a result which is useful for applications because it gives sufficient conditions to avoid bifurcation, and this is an important problem for engineers, for example. This suggests an open question in optimal control theory: given the system at rest at time \( t = 0 \) (in particular \( \lambda(0) = 0 \)) and given a prescribed loading intensity \( \lambda_0 \) that we want to reach at time \( t_0 \), can we choose a function \( \lambda \) from \([0, t_0] \) to \( \mathbb{R} \) with \( \lambda(0) = 0 \) and \( \lambda(t_0) = \lambda_0 \) (with possibly some other constraints) such that for all \( t \in [0, t_0] \), \( 0 \) is not an eigenvalue of the operator \( A - B(\sigma^0(t_0)) \) (which implies that \( t \) is not a bifurcation time)?

4. SUFFICIENT CONDITIONS FOR BIFURCATION

From now on, we assume that the necessary condition given in Theorem 3.1 is fulfilled at time \( t_0 \), and in addition, we assume that \( 0 \) is a simple eigenvalue of \( A - B(\sigma^0(t_0)) \).

We want to find conditions which ensure the existence of a solution \((\sigma, \xi)\) of (2.17), (2.18), bifurcating at time \( t_0 \) from the trivial solution \((\sigma^0, 0)\), and to give some properties of this solution in a neighborhood of \( t_0 \). For this purpose, we will use the Lyapounov-Schmidt procedure, but it will turn out to be much more difficult to apply in our context than it usually is, because, here, the effective unknowns are functions and not real numbers.
We denote by \( < , > \) the duality pairing between \( H^{-2}(\omega) \) and \( H^2_0(\omega) \), and by \(( , )_A\) and \( \| \|_A\) the scalar product associated with the operator \( A \) and the corresponding norm.

\[
(4.1) \quad (\psi, \psi)_A = < A \psi, \psi > = \int_\omega A_{\nu\lambda\mu} \partial_{\nu\lambda} \psi \partial_{\nu\lambda} \psi \, dx.
\]

We write \( L(t) \) for the operator \( A - B(\sigma^0(t)) \) and we take \( \theta \) satisfying

\[
(4.2) \quad \theta \in H^2_0(\omega), \quad \| \theta \|_A = 1, \quad L(t_0)\theta = 0.
\]

We define the sets

\[
(4.3) \quad \theta^\perp = \{ z \in H^2_0(\omega), \ (\theta, z)_A = 0 \}, \quad \theta^o = \{ g \in H^{-2}(\omega), \ < g, \theta > = 0 \},
\]

and we denote by \( Q \) the projection in \( H^{-2}(\omega) \) on \( \theta^o \). We can decompose \( \xi(t) \) on \( \theta \) and \( \theta^\perp \):

\[
\xi(t) = k(t)\theta + z(t), \quad k(t) \in \mathbb{R}, \quad z(t) \in \theta^\perp.
\]

Let us state an equivalent formulation for system (2.17)(2.18) taking into account the informations known about \( t_0 \). First of all, due to (4.2), we can rewrite equation (2.18) as

\[
(4.4) \quad L(t_0)z(t) = B(\sigma(t) - \sigma^0(t))\xi(t).
\]

Before time \( t_0 \), equations (2.17)(2.18) are satisfied by \( (\sigma^0, 0) \); then, we can focalize on what happens after \( t_0 \). First of all, from (3.1), we get

\[
(4.5) \quad \forall \ t > 0, \quad P \left( \int_0^t \psi(\sigma^0(s)) \, ds \right) = - P \left( a\sigma^0(t) \right);
\]

then, by substracting (3.1) from (2.17), using (4.5) and projecting (4.4) on \( \theta \) and \( \theta^\perp \) system (2.17), (2.18) becomes:

\[
(4.6) \quad \sigma(t) - \sigma^0(t) = P[\delta(k(t)\theta + z(t)) - a(\sigma^0(t) - \sigma^0(t_0)) - \int_0^t \psi(\sigma(s)) ds],
\]

\[
(4.7) \quad L(t_0)z(t) = Q \left[ B(\sigma(t) - \sigma^0(t_0))(k(t)\theta + z(t)) \right],
\]

\[
(4.8) \quad < B(\sigma(t) - \sigma^0(t_0))(k(t)\theta + z(t)), \theta > = 0.
\]
Our problem is now to find \( t_1 > t_0 \) and continuous functions \( \sigma, z, k \) defined on the interval \([t_0, t_1]\) such that equations (4.6),(4.7),(4.8) are satisfied for any \( t \) in \([t_0, t_1]\). Of course, there always exists the trivial solution \( \sigma = \sigma^0, \quad k = 0, \quad z = 0 \). Then, our purpose is to exhibit sufficient conditions which ensure the existence of \( t_1 \) (close enough to \( t_0 \)) and \((\sigma, k, z)\) solution of (4.6),(4.7),(4.8) on interval \([t_0, t_1]\) with \((k, z)\) different from zero. For the convenience of the reader, let us write the problem at length:

Find \( t_1 > t_0, \quad \sigma \in C([t_0, t_1]; [L^2(\omega)]^4) \), \( \kappa \in C_0([t_0, t_1]) \), \( z \in C([t_0, t_1]; \theta^4) \), with \((k, z) \neq (0, 0)\) such that

\[
(4.9) \quad \forall t \in [t_0, t_1], \quad \sigma(t) - \sigma^0(t) = \int_0^t \varphi(\sigma(s))ds, \\
(4.10) \quad \forall t \in [t_0, t_1], \quad L(t) z(t) = \int_0^t \omega(t) \kappa(t) \varphi(\sigma(s))ds, \\
(4.11) \quad \forall t \in [t_0, t_1], \quad < \omega(t) \kappa(t) \varphi(\sigma(s))ds > = 0.
\]

Notice that (4.11) can be written as

\[
(4.11') \quad \forall t \in [t_0, t_1], \quad \omega(t) \int_\omega [\sigma_{\alpha\beta}(t) - \sigma^0_{\alpha\beta}(t_0)] \partial_\beta \vartheta \partial_\alpha \vartheta dx + \int_\omega [\sigma_{\alpha\beta}(t) - \sigma^0_{\alpha\beta}(t_0)] \partial_\beta \vartheta dx = 0.
\]

In order to state the main results, we need to introduce an auxiliary problem. We shall denote by \((\sigma^0, v^+\)) the solution (stress and displacement) of the following stationary problem

\[
(4.12) \quad a \sigma^0 + \varphi(\sigma^0(t_0)) = \epsilon(v^+), \quad \partial_\alpha \sigma^0_{\alpha\beta} = 0 \text{ in } \omega, \quad \sigma^0_{\alpha\beta} \cdot n_{\beta} = \partial_\alpha ^{+} \lambda(t_0) F_{\alpha} \text{ on } \partial \omega.
\]

Notice that (4.12) is a standard two dimensional linear elasticity problem where \( \varphi(\sigma^0(t_0)) \) and \( \partial_\alpha ^{+} \lambda(t_0) \) (right derivative of \( \lambda \) at time \( t_0 \)) are given data. Due to (2.5), (4.12) has a solution and the stress \( \sigma^0 \) is uniquely defined, with \( \sigma^0 \in [L^2(\omega)]^4 \). We set
\begin{equation}
(4.13) \quad r = 2(\sigma^0_+, \delta(\theta)) = \int_{\omega} \sigma^0_+ \partial_\alpha \theta \partial_\beta \theta \, dx,
\end{equation}

\begin{equation}
(4.14) \quad s = 2( P(\delta(\theta)), \delta(\theta) ).
\end{equation}

\textbf{REMARK 4.1}

As $P$ is a projection, we see that $s \geq 0$. In fact, it is easily checked that $s = 0$ if and only if $\theta$ is a solution of the Monge-Ampère equation

\begin{equation}
(4.15) \quad \det (\partial_\alpha \theta) = 0.
\end{equation}

We know from [Potier-Ferry] that with the boundary conditions chosen here ($\theta = \partial_\nu \theta = 0$ on $\partial \omega$), the only solution of (4.15) is 0. As $\theta$ is non-zero, we have $s > 0$.

Let us mention that this possibility of degeneracy involving the Monge-Ampère equation appears in all the phenomena related to the buckling of Von Kármán plates (c.f. [Berger], [Berger-Fife], [Potier-Ferry], [Mignot-Puel 1], [Clairlet-Rabler], ...)

\textbf{THEOREM 4.1}

If the parameter $r$ is strictly negative, then $t_0$ is a bifurcation time. Problem (2.17),(2.18) admits in a right neighborhood of $t_0$ at least two bifurcated solutions $(\sigma, \xi)$ and $(\sigma, -\xi)$. The $k$-component of $\xi$ on $\theta$ satisfies

\begin{equation}
(4.16) \quad k(t) = \left( -r/s \right)^{1/2} (t - t_0)^{1/2} [1 + O(t - t_0)],
\end{equation}

and the $z$-component of $\xi$ on $\theta^\perp$ satisfies

\begin{equation}
(4.17) \quad z(t) = O((t - t_0)^{3/2}).
\end{equation}
Before proving Theorem 4.1, we want to comment upon the condition $r < 0$. Let us define $(\sigma^0_-, \psi^-)$ as the solution of the following two dimensional linear elasticity problem

$$a \sigma^0_+ + \varphi(\sigma^0(t_0)) = \epsilon(\psi^-), \quad \partial_\nu \sigma^0_+ = 0 \text{ in } \omega, \quad \sigma^0_+ \cdot n_\nu = \partial^-_1 \lambda(t_0) F_\alpha \text{ on } \partial \omega.$$  

and

$$r^- = 2(\sigma^0_+, \delta(\theta)) = \int_\omega \sigma^0_+ \partial_\alpha \theta \cdot \partial_\beta \theta \, dx.$$  

Then

$$r = r^- + 2(\sigma^0_+, \delta(\theta)).$$

Finally, denoting by $(\tau, w)$ the solution of

$$a \tau = \epsilon(w), \quad \partial_\nu \tau = 0 \text{ in } \omega, \quad \tau \cdot n_\nu = F_\alpha \text{ on } \partial \omega,$$

we set

$$\rho = 2(\tau, \delta(\theta)).$$

The parameter $\rho$ depends only on $F$ and $\theta$ and the sign of $\rho$ expresses whether $F$ globally acts as a compression or as a traction. Thus, the constants $r$ and $r^-$ are related by

$$r = r^- + (\partial^+_1 \lambda(t_0) - \partial^-_1 \lambda(t_0)) \rho.$$  

Information upon the sign of $r^-$ is given by the following result.

**Proposition 4.1**

If $t_0$ is the first time when the necessary condition (3.4) for bifurcation is fulfilled, then

$r^- < 0.$
Proof

First of all, one can easily prove that when condition (3.4) is satisfied, there actually exists a first time \( t_0 \), where 0 is an eigenvalue of \( L(t_0) \). Now, using classical methods (see, for instance, [Mignot-Puel-Suquet]) and the continuity of \( \sigma^0 \), one can show that 1 is the largest eigenvalue of \( A^{-1}B(\sigma^0(t_0)) \) and furthermore that

\[
(4.23) \quad \forall \ t < t_0, \ \forall \ \zeta \neq 0, \ \langle A \zeta, \zeta \rangle - \langle B(\sigma^0(t)) \zeta, \zeta \rangle > 0
\]

Let

\[
j(t) = h \int_0^t \sigma^0_s(t) \partial^\theta \partial^\theta dx = - \langle B(\sigma^0(t)) \theta, \theta \rangle .
\]

By means of (4.2), we know that

\[
j(t_0) = -1;
\]

on the other hand, due to (4.23)

\[
\forall \ t \in [0, t_0[, \ \langle A \theta, \theta \rangle - \langle B(\sigma^0(t)) \theta, \theta \rangle > 0 ,
\]

which means that

\[
\forall \ t \in [0, t_0[, \ j(t) > -1 .
\]

Using the fact that \( hr^- = \delta^- j(t_0) \), we obtain \( r^- < 0 \).

REMARK 4.2

The case \( r^- = 0 \) appears as a degeneracy condition which is not easy to interpret in terms of the original problem.

Equation (4.22) and Theorem 4.1 say that bifurcation will occur when

\[
(4.24) \quad [\delta^\min(t_0) - \delta^\max(t_0)] \rho > r^- .
\]
in the case when \( r^- < 0 \), which we can assume to be the usual case when \( t_0 \) is the first possible bifurcation time, (4.24) will be satisfied for example when \( \lambda \) is differentiable at \( t_0 \) or, when \( \rho < 0 \), even when the slope of \( \lambda \) strictly decreases from the left to the right (up to some extent).

We now come back to the proof of Theorem 4.1.

**Proof of Theorem 4.1.**

In order to solve system (4.9), (4.10), (4.11') in \((\sigma, z, k)\), we would like to apply first the implicit function theorem to (4.9), (4.10) to obtain \( \sigma \) and \( z \) as functions of \( k \) and then to plug this result in (4.11') which would become a nonlinear equation in the only unknown \( k \). This classical idea is impossible to develop here because, as it is explained in Theorem A1, Appendix A, the mapping \( \sigma \rightarrow \psi(\sigma) \) is not sufficiently regular, and in particular is not differentiable from \([L^p(\omega)]^\mathbb{R}_s\) into itself when \( \psi \) is not affine, but only from \([L^q(\omega)]^\mathbb{R}_s\) into \([L^p(\omega)]^\mathbb{R}_s\) with \( q > p \).

Nevertheless, the properties proved in Appendices A and B will enable us to overcome the difficulty and to obtain the same result as the one which would derive from a formal use of the implicit function theorem.

The proof is long and technical and will be divided into three steps. Some lemmas with their proofs will be included therein.

**First step.**

First of all, for every \( t_1 > t_0 \) and for \((k, z)\) fixed in \( C_0([t_0, t_1]) \times C_0([t_0, t_1]; \mathbb{R}^d) \), we consider equation (4.9). For every finite \( q \), the inclusion \( H^2(\omega) \subset W^{1,q}(\omega) \) holds true and
therefore
\[ \delta(k\theta + z) \in C([t_0, t_1]; L^2(\omega)^d). \]

We then obtain from Theorem B2, Appendix B, the existence of a unique \( \sigma \in C([t_0, t_1]; [L^2(\omega)]^d) \) satisfying (4.9). This defines a mapping
\[ (k, z) \in C_0([t_0, t_1]) \times C_0([t_0, t_1]; \Theta^d) \rightarrow \tilde{\sigma}(k, z) \in C([t_0, t_1]; [L^2(\omega)]^d), \]
where, using the notations of Appendix B,
\[ (4.25) \quad \tilde{\sigma}(k, z) = \hat{\sigma}[\delta(k\theta + z) + \alpha^0(t_0)]. \]

We can derive the properties of \( \tilde{\sigma} \) from the properties of \( \hat{\sigma} \) given in Theorem B2. In particular, \( \tilde{\sigma} \) is a \( C^1 \) mapping. Replacing now \( \sigma \) by \( \tilde{\sigma}(k, z) \) in (4.10), we obtain the following equation, where only two unknowns remain left
\[ (4.26) \quad L(t_0) \ z(t) = Q[\{ \tilde{\sigma}(k, z)(t) - \sigma^0(t_0)\}(k(t)\theta + z(t))]. \]

Equation (4.26) is studied hereafter.

Let \( B_{t_1}(M_1) \) be the ball centered at 0, of radius \( M_1 \) in \( C_0([t_0, t_1]). \)

**LEMMA 4.1**

There exist \( T_1 > t_0 \) and a real number \( M_1 > 0 \) such that, for every \( t_1 \) with \( t_0 < t_1 \leq T_1 \), there exists a unique \( C^1 \) mapping \( \tilde{Z}_{t_1} \) from \( B_{t_1}(M_1) \) to \( C_0([t_0, t_1]; \Theta^d) \) such that \( \tilde{Z}_{t_1}(0) = 0 \) and for every \( k \) in \( B_{t_1}(M_1) \), \( (k, \tilde{Z}_{t_1}(k)) \) is a solution of equation (4.26). Moreover, for every \( t_1 \) and \( t_2 \) such that \( t_0 < t_1 < t_2 < T_1 \), and every \( k \) in \( B_{t_2}(M_1) \),
\[ (4.27) \quad \tilde{Z}_{t_2}(k)(t_0, t_1) = \tilde{Z}_{t_1}(k)(t_0, t_1). \]
Proof.

Equation (4.26) can be written as $F(k, z) = 0$, where

$$F : (k, z) \rightarrow L(t_0)z - Q[B(\tilde{o}(k, z) - \sigma^0(t_0))(k\theta + z)]$$

is a $C^1$ mapping from $C_0([t_0, t_1]) \times C_0([t_0, t_1]; \theta^1)$ into $C_0([t_0, t_1]; \theta^0)$. One can easily check that

$$D_2F(0, 0) \tilde{z} = L(t_0)\tilde{z} - Q[B(\sigma^0(. - \sigma^0(t_0)) \tilde{z}].$$

As $L(t_0)$ is invertible from $C_0([t_0, t_1]; \theta^1)$ to $C_0([t_0, t_1]; \theta^0)$, and as for $t$ close to $t_0$, $\sigma^0(t) - \sigma^0(t_0)$ is small, $\tilde{z} \rightarrow Q[B(\sigma^0(. - \sigma^0(t_0)) \tilde{z}$] is a small perturbation of an invertible operator when $t_1$ is sufficiently close to $t_0$. Hence, there exists $T_1 > t_0$, such that, for every $t_1$ with $t_0 < t_1 < T_1$, $D_2F(0, 0)$ is an isomorphism and we can apply the implicit function theorem which shows that there exists a ball $B_{\tilde{z}_1}(M(t_1))$ in $C_0([t_0, t_1])$ such that equation (4.26) defines a $C^1$ mapping $\tilde{z}_1$ from $B_{\tilde{z}_1}(M(t_1))$ into $C_0([t_0, t_1]; \theta^1)$ satisfying $\tilde{z}_1(0) = 0$.

By a careful writing of the standard proof of the implicit function theorem in this particular case (see, for instance, [Dieudonné]), one can make sure that the radii $M(t_1)$ can be chosen independent of $t_1$; then, proving (4.27) is straightforward.

Due to relation (4.27), we shall henceforward write $z$ for any of the mappings $\tilde{z}_1$, where $t_0 < t_1 < T_1$; similarly, we define $\sigma$ by

$$\sigma(k) = \tilde{\sigma}(k, \tilde{z}(k)).$$

Plugging this into (4.11), we see that the problem reduces now to find $t_1$, with $t_0 < t_1 < T_1$ and
\( k \in B_{1}(M_{1}) \) \( k \neq 0 \), such that

\[
\langle B(\sigma(k)(t) - \sigma^{0}(t_{0}))(k(t)\theta + z(k(t))), \theta \rangle = 0
\]

or equivalently

\[
k(t) \int_{\omega} [\sigma_{\theta\theta}(k(t)) - \sigma^{0}_{\theta\theta}(t_{0})] \partial_{\theta} \theta . \partial_{\theta} \theta \, dx + \int_{\omega} [\sigma_{\phi\phi}(k(t)) - \sigma^{0}_{\phi\phi}(t_{0})] \partial_{\phi} z(k(t)) . \partial_{\phi} \theta \, dx = 0.
\]

Notice that problem \((4.29')\) contains only one unknown, namely \( k \).

**Second step.**

Studying \((4.29')\) requires determining the behaviour of \( \sigma(k) \) and \( z(k) \) when \( k \) is "small". From now on, we shall write \( \| k \| \_k \) for \( \| k \|_{C_{c}([t_{0}, t])} \) and \( C \) for various constants; we make use of the notation \((A.3)\) of Appendix A is the following lemma.

**Lemma 4.2.**

When \( k \) lies in a sufficiently small neighborhood of \( 0 \) in \( C_{0}(t_{0}, t_{1}) \), the functions \( \sigma(k) \) and \( z(k) \) admit the following expansions in \( k \)

\[
\sigma(k) = \sigma^{0} + \sigma^{2}(k) + \Sigma(k),
\]

\[
z(k) = k.z^{1} + Z(k),
\]

where \( \sigma^{0} \) is given by \((3.1)\), \( \sigma^{2}(k) \) (which is an element of \( C([t_{0}, t_{1}); [L^{2}(\omega)]_{s}^{4} \)) quadratic in \( k \) ) and \( z^{1} \) (which is an element of \( C_{0}([t_{0}, t_{1}); \theta^{1}] \) independent of \( k \) ) are solutions of

\[
\sigma^{2}(k)(t) = P[ k^{2}(t) \delta(\theta + z^{1}(t)) - \int_{0}^{1} \psi_{\theta^{0}}(s) [\sigma^{2}(k)(s)] \, ds ]
\]

\[
L(t_{0}) z^{1}(t) - Q[ B(\sigma^{0}(t) - \sigma^{0}(t_{0}))[z^{1}(t)] = Q[ B(\sigma^{0}(t) - \sigma^{0}(t_{0})) \theta ]
\]
and where $\Sigma$ and $Z$ satisfy

(4.34) $\forall \, t \in [t_0, t_1], \, \| \Sigma(k) \|_{C([L_0, t]); [L^2(\omega)]_q} \leq C \| k \|_q^4,$

(4.35) $\forall \, t \in [t_0, t_1], \, \| \Sigma(k_1) - \Sigma(k_2) \|_{C([L_0, t]); [L^2(\omega)]_q} \leq C \| k_2 - k_1 \|_q \| k_1 \|_q + \| k_2 \|_q \|^3,$

(4.36) $\forall \, t \in [t_0, t_1], \, \| Z(k) \|_{C([L_0, t]); H_0^2(\omega))} \leq C \| k \|_q^3,$

(4.37) $\forall \, t \in [t_0, t_1], \, \| Z(k_2) - Z(k_1) \|_{C([L_0, t]); H_0^2(\omega))} \leq C \| k_2 - k_1 \|_q \| k_1 \|_q + \| k_2 \|_q \|^2.$

Proof.

We know that $g \to \sigma(g)$ is differentiable from $C([t_0, t]; [L^q(\omega)]_q)$ to $C([t_0, t]; [L^q(\omega)]_q)$ if $q > 2,$ and $(k, z) \to \delta(k \theta + z)$ is differentiable from $C([t_0, t]) \times C([t_0, t]; H_0^2(\omega))$ to $C([t_0, t]; [L^q(\omega)]_q)$ for any $q.$ Therefore, the mapping $(k, z) \to \sigma(k, z),$ defined in (4.25), is differentiable from $C([t_0, t]) \times C([t_0, t]; H_0^2(\omega))$ to $C([t_0, t]; [L^q(\omega)]_q).$ Now, $k \to Z(k)$ is a $C^1$ mapping from $B_1(\mathcal{M})$ to $C([t_0, t]; H_0^2(\omega)).$ Then, the mappings $k \to Z(k)$ and $k \to \sigma(k)$ are differentiable and we can easily compute their derivatives at $k = 0.$ Firstly, we obtain

$$D_k Z(O)[k] = k \cdot z^1,$$

where $z^1$ is independent of $k$ and satisfies (4.33), and we can write $z(k)$ as

$$z(k) = k \cdot z^1 + Z(k), \text{ where } \| Z(k) \|_{C([L_0, t]); H_0^2(\omega))} = o(\| k \|_q).$$

Therefore,

$$\delta(k \theta + z(k)) = k^2 \delta(\theta + z^1) + \delta(k), \text{ with } \| \delta(k) \|_{C([L_0, t]); [L^q(\omega)]_q} = o(\| k \|_q^2);$$

then, writing $\sigma(k)$ as
(4.38) \( \sigma(k) = \hat{\sigma}[\delta(k\theta + z(k)) + a\sigma(t_0)] \)

and using the derivative of \( g \rightarrow \hat{\sigma}(g) \) computed in Theorem B.2, we obtain

\[
\sigma(k) = \sigma^0 + \delta_2 \left( k^2 \delta(\theta + z^1) \right) + o(\|k\|^2) = \sigma^0 + \sigma^2(k) + \Sigma(k),
\]

where \( \sigma^2(k) \) satisfies (4.32) and

\[
\| \Sigma(k) \|_{C([t_0,1];L^2(\omega))} \leq o(\|k\|^2).
\]

Let us now improve those estimates by proving (4.34) and (4.36). Replacing \( \sigma(k) \) by its above decomposition in (4.26), we obtain

\[
L(t_0)Z(k)(t) - Q[B(\sigma^0(t) - \sigma^0(t_0))(z(k)(t) + k(t)\theta)] = Q[B(\sigma^2(k)(t) + \Sigma(k)(t))(k(t)\theta + z(k)(t))],
\]

and subtracting (4.33) multiplied by \( k(t) \), we get

(4.39) \( L(t_0)Z(k)(t) - Q[B(\sigma^0(t) - \sigma^0(t_0))Z(k)(t)] = Q[B(\sigma^2(k)(t) + \Sigma(k)(t))(k(t)\theta + z(k)(t))]. \)

For \( t \in [t_0, t_1] \), the linear operator in the left hand side of (4.39) is continuously invertible and the right hand side is \( O(\|k\|^3) \). Then

\[
\| Z(k) \|_{C([t_0,1];H^q(\omega))} \leq C \|k\|^3,
\]

and this proves (4.36). Now, let us use again the form (4.38) of \( \sigma(k) \). Because of (4.36),

\[
\delta(k\theta + z(k)) = k^2 \delta(\theta + z^1) + \delta(k),
\]

with

\[
\| \delta(k) \|_{C([t_0,1];L^q(\omega))} \leq C \|k\|^q, \quad q < +\infty.
\]

Taking the expansion (B.4) of \( \hat{\sigma} \) near \( g_0 = a\sigma(t_0) \), we get

\[
\sigma(k) = \sigma^0 + \sigma^2(k) + \hat{\sigma}'(a\sigma(t_0))[\delta(k)] + \Sigma(\delta(k\theta + z(k)))
\]

then,

\[
\Sigma(k) = \hat{\sigma}'(a\sigma(t_0))[\delta(k)] + \Sigma(\delta(k\theta + z(k))),
\]
which, due to (B.5), implies (4.34).

Next, we want to prove (4.35) and (4.37). By definition, we have

\[
\Sigma(k) = \sigma(k) - \sigma^0 - \sigma^2(k)
\]

\[
= \hat{\sigma}'(\sigma^0(t_0) + \delta(k\theta + z(k))) - \hat{\sigma}'(\sigma^0(t_0)) - \hat{\sigma}'(\sigma^0(t_0))[k^2 \delta(\theta + z^1)],
\]

\[
Z(k) = z(k) - k_z^1 .
\]

Then, \( \Sigma \) and \( Z \) are differentiable and writing \( \delta^1(\xi, \zeta) \) for \( \delta^r(\xi)[\zeta] \), we obtain

\[
(4.40) \quad \Sigma'(k)[h] = [\hat{\sigma}'(\sigma^0(t_0) + \delta(k\theta + z(k))) - \hat{\sigma}'(\sigma^0(t_0))][\delta^1(k\theta + kz^1, h\theta + hz^1)]
\]

\[
+ \sigma'(\sigma^0(t_0) + \delta(k\theta + z(k)))[\delta^1(Z(k), h\theta + hz^1)]
\]

\[
+ \hat{\sigma}'(\sigma^0(t_0) + \delta(k\theta + z(k)))[\delta^1(k\theta + z(k), Z'(k)[h])] ,
\]

and differentiating (4.39),

\[
(4.41) \quad L(t_0)Z'(k)[h](t) - Q[B(\sigma^0(t) - \sigma^0(t_0)Z'(k)[h](t)] = Q[B(\sigma'(k)[h](t)(k(t)\theta + z(k)(t))]
\]

\[
+ Q[B(\sigma^2(k)(t) + \Sigma(k)(t))(h(t)\theta + z'(k)[h](t)) .
\]

Using Theorem B1 (iii), we obtain from (4.40)

\[
\|
\Sigma'(k)[h] \|
C[\|l\|t^3_l \|h\|_t + C \|k\|_t \|Z'(k)[h] \|
C[\|l\|t^2_l \|h\|_t + C \|k\|_t \|l\|_t \|Z'(k)[h] \|
C[\|l\|t^2_l \|h\|_t + C \|k\|_t \|l\|_t \|\Sigma'(k)[h] \|
C[\|l\|t^3_l \|h\|_t , \quad \Sigma'(k)[h] \|
C[\|l\|t^2_l \|h\|_t + C \|k\|_t \|l\|_t \|\Sigma'(k)[h] \|
C[\|l\|t^2_l \|h\|_t + C \|k\|_t \|l\|_t \|\Sigma'(k)[h] \|
C[\|l\|t^3_l \|h\|_t .
\]

Therefore, if \( k \) is chosen sufficiently small, we have

\[
(4.42) \quad \|
\Sigma'(k)[h] \| \leq C \|k\|_t^3 \|h\|_t , \quad \|
\Sigma'(k)[h] \| \leq C \|k\|_t^3 \|h\|_t .
\]

We now write
\[ \Sigma(k_2) - \Sigma(k_1) = \int_0^1 \Sigma'(k_1 + t(k_2 - k_1))(k_2 - k_1) \, dt , \]

and the analogous equality for \( Z \), and we get (4.35) and (4.37) for \( k_1 \) and \( k_2 \) chosen in a suitable neighborhood of \( 0 \) in \( C_0([t_0, t_1]) \). This completes the proof of Lemma 4.2.

Solving problem (4.29') by means of Lemma 4.2 will now ask for the behaviors in \( t \) (close to \( t_0 \)) of \( \sigma^0, \sigma^2(k) \) and \( Z^1 \); this is the purpose of the following lemma.

**Lemma 4.3**

For \((t - t_0)\) small enough, we have

\[ \sigma^0(t) = \sigma^0(t_0) + \sigma^0+(t - t_0) + \Sigma^0(t) , \]

where \( \sigma^0+ \) is defined by (4.12) and \( \Sigma^0(t) = O((t - t_0)^2) \),

\[ z^1(t) = O(t - t_0) , \]

\[ \sigma^2(k)(t) = k^2(t)P(\delta(\theta)) + \Sigma^2(k)(t) , \]

where

\[ \| \Sigma^2(k)(t) \|_{L^2(\omega)} \leq C(t - t_0) \| k \|_k^2 , \]

\[ \| \Sigma^2(k_2) - \Sigma^2(k_1) \|_{C([t_0,1];L^2(\omega))} \leq C(t - t_0) \| k_2 - k_1 \|_k \| k_1 \|_k + \| k_2 \|_k \] .

**Proof:**

Let us recall that \( \sigma^0 \) is defined by

\[ \sigma^0(t) - \sigma^*(t) = P \left[ -a^*(t) - \int_0^t \psi(\sigma^0(s)) \, ds \right] , \]
where

\[ \sigma^*(t) = \lambda(t)\sigma^*_F, \]

with

\[ \sigma^*_F \in [L^p(\omega)]^d; \quad \partial_\theta \sigma^*_F = 0; \quad \sigma^*_F \ n_\theta = F_\alpha. \]

If \( \lambda \) is \( C^1 \) on \( [t_0, t_1] \), with Lipschitz derivative, so is \( \sigma^* \). From (3.1) it is easy to check that \( \sigma^0 \) is \( C^1 \) on \( [t_0, t_1] \) with Lipschitz derivative and has a right derivative at \( t_0 \), noted \( \sigma^0^+ \), which satisfies

\[ \sigma^0^+ - \sigma^*(t_0) = P[ - a_0 \sigma^*(t_0) - \varphi(\sigma^0(t_0))], \]

which is equivalent to (4.12).

Then, for \( t \geq t_0 \), we have

\[ \sigma^0(t) = \sigma^0(t_0) + \sigma^0^+(t - t_0) + O((t - t_0)^2), \]

which is (4.43).

The definition of \( z^1 \) by (4.33) implies

\[ \| z^1 \|_{L^2(\omega)} \leq C \| \sigma^0(t) - \sigma^0(t_0) \|_{L^2(\omega)}^d, \]

which, in turn, implies (4.44). We know by (4.32) that

\[ \sigma^2(k)(t) = P[k^2(t)\delta(\theta + z^1(t)) - \int_{t_0}^{t} \psi_0(\omega)[\sigma^2(k)(s)] ds] \]

\[ = P[k^2(t)\delta(\theta)] + \Sigma^2(k)(t), \]

where

\[ \Sigma^2(k)(t) = P[k^2(t)\delta^1(\theta, z^1(t)) + k^2(t) \delta(z^1(t)) - \int_{t_0}^{t} \psi_0(\omega)[\sigma^2(k)(s)] ds]. \]

Therefore
\[ \Sigma^2(k_2)(t) - \Sigma^2(k_1)(t) = P[(k_2(t)-k_1(t))(k_2(t)+k_1(t))[\delta^1(\theta, z^1(t)) + \delta(z^1(t))] \\
- \int_0^1 \psi_{\omega}(s)[\delta^2(a\sigma^0(t_0))[k_2-k_1](k_2+k_1)\delta(\theta+z^1(s))[s] ds}. \]

Then, using (4.44), we obtain
\[ \|\Sigma^2(k_2) - \Sigma^2(k_1)\|_{c([l_0, l_1]; l_2(\omega)\omega)} \leq C(t - t_0)\|k_2 - k_1\|_{l_2} \left[ \|k_1\|_{l_2} + \|k_2\|_{l_2} \right], \]
which proves (4.46) and (4.47).

**Third step.**

We are now ready to study (4.29) or (4.29'), that is to say (4.11) or (4.11') with \( \sigma \) replaced by \( \sigma(k) \) and \( z \) is replaced by \( z(k) \). Our original problem is now reset as finding a non-zero solution \( k \) of the following so-called "bifurcation equation"

\[ k(t)\int_\omega [\sigma_{\alpha\beta}(k)(t) - \sigma_{\alpha\beta}^0(t_0)] \delta_\alpha \theta \delta_\theta dx + k^3(t) \int_\omega P[\delta(\theta)] \delta_\theta \delta_\theta dx + R(k, t) = 0. \]

By use of (4.30), (4.43) and (4.45), (4.48) writes as

\[ k(t)(t - t_0)\int_\omega \sigma_{\alpha\beta}^0 \delta_\alpha \theta \delta_\alpha \theta dx + k^3(t) \int_\omega P[\delta(\theta)] \delta_\theta \delta_\theta dx + R(k, t) = 0, \]

where

\[ R(k, t) = k(t)\int_\omega [\Sigma_{\alpha\beta}(k)(t) + \Sigma_{\alpha\beta}^2(k)(t) + \Sigma_{\alpha\beta}^0(t)] \delta_\alpha \theta \delta_\alpha \theta dx \\
+ \int_\omega [\sigma_{\alpha\beta}(k)(t) - \sigma_{\alpha\beta}^0(t_0)] \delta_\alpha z(k)(t) \delta_\alpha \theta dx. \]

Then, with the definitions of \( r \) and \( s \) given in (4.13) and (4.14), equation (4.49) becomes

\[ E(k, t) = rk(t)(t - t_0) + sk^3(t) + R(k, t) = 0. \]

We shall need the principal part of equation (4.51) which will turn out to be

\[ \tilde{E}(k, t) = rk(t)(t - t_0) + sk^3(t) = 0. \]
LEMMA 4.4

The remainder term $R$ satisfies

$$R(k, t) = \sum_{i=1}^{3} R_i(k, t),$$

with

$$|R_1(k, t)| \leq C(t - t_0)^2 \|k\|_l; \quad |R_1(k_1, t) - R_1(k_2, t)| \leq C(t - t_0)^2 \|k_1 - k_2\|_l;$$

$$|R_2(k, t)| \leq C(t - t_0)\|k\|_l^3; \quad |R_2(k_1, t) - R_2(k_2, t)| \leq C(t - t_0)\|k_1 - k_2\|_l \left(\|k_1\|_l + \|k_2\|_l\right)^2;$$

$$|R_3(k, t)| \leq C\|k\|_l^5; \quad |R_3(k_1, t) - R_3(k_2, t)| \leq C\|k_1 - k_2\|_l \left(\|k_1\|_l + \|k_2\|_l\right)^4.$$

Proof.

We define

$$R_1(k, t) = k(t) \int_{\omega} \Sigma_0^0 k(t) \partial_{\theta} \theta \partial_{\phi} \theta \ dx + k(t) \int_{\omega} [\varphi^0_0(t - t_0) + \Sigma_0^0(t)] \partial_{\phi} Z^1(t) \partial_{\theta} \theta \ dx,$$

$$R_2(k, t) = k(t) \int_{\omega} \Sigma_2^2(k(t) \partial_{\theta} \theta \partial_{\phi} \theta \ dx + \int_{\omega} [\varphi^0_0(t - t_0) + \Sigma_0^0(t)] \partial_{\phi} Z(k(t) \partial_{\theta} \theta \ dx$$

$$+ k^3(t) \int_{\omega} p_{\theta}([\delta(\theta)]) \partial_{\phi}^2 \partial Z^1(t) \partial_{\phi} \theta \ dx + k(t) \int_{\omega} \Sigma_2^2(k(t) \partial_{\phi}^2 \partial Z^1(t) \partial_{\phi} \theta \ dx,$$

$$R_3(k, t) = k(t) \int_{\omega} \Sigma_0^0(k(t) \partial_{\theta} \theta \partial_{\phi} \theta \ dx + \int_{\omega} [\varphi^0_0(k(t) + \Sigma_0^0(k(t)] \partial_{\phi} Z(k(t) \partial_{\theta} \theta \ dx$$

$$+ k^2(t) \int_{\omega} p_{\theta}([\delta(\theta)]) \partial_{\phi} Z(k(t) \partial_{\phi} \theta \ dx + k(t) \int_{\omega} \Sigma_0^0(k(t) \partial_{\phi} Z^1(t) \partial_{\phi} \theta \ dx.$$

Then

$$R(k, t) = \sum_{i=1}^{3} R_i(k, t),$$

and we can easily prove the estimates of Lemma 4.4 by means of (4.34), (4.35), (4.36),

(4.44), (4.46) and (4.47).
We now want to solve equations (4.52) and (4.51) with a method analogous to the one developed in [Mignot-Puel 2]. We know from Remark 4.1 that $s > 0$. If $r > 0$, the only solution of (4.52) is $k = 0$. If $r < 0$, besides the zero solution, there exist exactly two solutions $\bar{k}$ and $-\bar{k}$ to equation (4.52) with

$$
\bar{k}(t) = (-r/s)^{1/2} (t - t_0)^{1/2}.
$$

For $r < 0$, we shall look for solutions of (4.51) of the form

$$
k = \bar{k} + d,
$$

with $d$ "small" with respect to $\bar{k}$, i.e., we look for $d$ such that

$$
E(\bar{k} + d, t) = 0.
$$

Writing $E$ as $\bar{E} + R$ and using the explicit formulation (4.52) of $\bar{E}$, one can easily write (4.54) as a fixed point problem; namely, find the fixed points of $S$, where $S$ is defined by

$$
S(d)(t) = [3(-rs)^{1/2}(t - t_0)^{1/2} d^2(t) + sd^3(t) + R(\bar{k} + d, t)]/[2r(t - t_0)].
$$

Of course, $d = -\bar{k}$ is a solution of (4.55) (corresponding to the zero solution of (4.51)), and we are interested in the other fixed points of $S$.

**Lemma 4.5**

There exists $t_1 > t_0$ and there exists $M_1 > 0$, such that $S$ maps $B_{t_1}^{1/2}(M_1)$ into itself, where

$$
B_{t_1}^{1/2}(M_1) = \{ d \in C_0([t_0, t_1]) ; \forall t \in [t_0, t_1], \| d \| _t \leq M_1 (t - t_0)^{1/2} \}.
$$
Proof.

If \( d \in B_{l_1}^{1/2}(M_1) \), using Lemma 4.4 and the expression of \( S \), we have the following estimates:

\[
|S(d(t))| \leq CM_1^2 (t - t_0)^{1/2} + CM_1^3 (t - t_0)^{1/2} + C(t - t_0)^{3/2} [1 + M_1] + C(t - t_0)^{3/2} [1 + M_1]^3 + C(t - t_0)^{3/2} [1 + M_1]^5.
\]

We can easily check that, by choosing \((t_1 - t_0)\) and \(M_1\) sufficiently small, we have

\[
|S(d(t))| \leq M_1 (t - t_0)^{1/2}.
\]

The conclusion follows.

**LEMMA 4.6**

There exist \( t_1 > t_0 \) and \( M_1 > 0 \) such that \( S \) is a strict contraction on \( B_{l_1}^{1/2}(M_1) \).

Proof.

If \( d_1 \) and \( d_2 \) are elements of \( B_{l_1}^{1/2}(M_1) \), we have

\[
|S(d_1(t)) - S(d_2(t))| \leq CM_1 |d_1(t) - d_2(t)| + CM_1^2 |d_1(t) - d_2(t)| + C(t - t_0) \|d_1 - d_2\| + (1 + (1 + M_1)^2 + (1 + M_1)^3).
\]

Then, if \((t_1 - t_0)\) and \(M_1\) are chosen sufficiently small, \( S \) is a strict contraction on \( B_{l_1}^{1/2}(M_1) \).

Lemma 4.6 implies the existence (and uniqueness) of a fixed point \( d \) for \( S \in B_{l_1}^{1/2}(M_1) \); we can take \( M_1 < (-r/s)^{1/2} \) and, as \( d \in B_{l_1}^{1/2}(M_1) \), \( d \) cannot be \(-\bar{k}\). Then, \( k = \bar{k} + d \) is a
non zero solution of equation (4.51) and \((k, z(k), s(k))\) is a non trivial solution of (4.9),(4.10),(4.11).

In order to complete the proof of Theorem 4.1, it remains to show that
\[d(t) = O((t - t_0)^{3/2})\]. This will be a consequence of the following result.

**Lemma 4.7**

There exists \(M_2 > 0\) such that, if \((t_1 - t_0)\) is chosen sufficiently small, \(S\) is a strict contraction on \(B_{t_1}^{3/2}(M_2)\), where

\[B_{t_1}^{3/2}(M_2) = \{ d \in C_0([t_0, t_1]); \forall t \in [t_0, t_1], \|d\|_{t_1} \leq M_2(t - t_0)^{3/2}\}.

**Proof.**

Let \(d \in B_{t_1}^{3/2}(M_2)\); we have from (4.55) and Lemma 4.4:

\[|S(d)(t)| \leq C M_2^2 (t - t_0)^{5/2} + C \frac{M_2^3}{2} (t - t_0)^{7/2} + C \left(1 + (t - t_0)\right)^{1/2} + M_2 (t - t_0)^{3/2}\]

\[+ C \left[(t - t_0)^{1/2} + M_2 (t - t_0)^{3/2}\right]^3 + \frac{C}{(t - t_0)}[(t - t_0)^{1/2} + M_2 (t - t_0)^{3/2}]^5,
\]

which shows that for \(M_2\) chosen sufficiently large and \((t - t_0)\) sufficiently small,

\[|S(d)(t)| \leq M_2 (t - t_0)^{3/2},
\]

and \(S\) maps \(B_{t_1}^{3/2}(M_2)\) into itself. Now, if \(d_1\) and \(d_2\) belong to \(B_{t_1}^{3/2}(M_2)\), we have

\[|S(d_1)(t) - S(d_2)(t)| \leq \|d_1 - d_2\|_{t_1} \left[ CM_2 (t - t_0) + C M_2^2 (t - t_0)^2 + C \left(1 + (t - t_0)\right)^{1/2} + M_2 (t - t_0)^{3/2}\right]
\]

\[+ C (t - t_0)(1 + M_2(t - t_0))^2 + C(t - t_0)(1 + M_2(t - t_0))^4],
\]

and for \((t - t_0)\) small enough, \(S\) is a strict contraction on \(B_{t_1}^{3/2}(M_2)\). For \((t_1 - t_0)\) sufficiently
small, $B_{t_1}^{3/2}(M_2) \subseteq B_{t_1}^{1/2}(M_1)$ and the unique fixed point of $S$ in $B_{t_1}^{1/2}(M_1)$ is then, in fact, in $B_{t_1}^{3/2}(M_2)$. This proves that $d = O((t - t_0)^{3/2})$.

5. CONCLUSION

In the present work, we have considered a quasistatic model of elastoplastic material with memory.

This model seems to be more realistic than the purely static ones governed by the Hencky law, and is currently used and studied by engineers. In particular, it describes a kind of buckling phenomena which cannot be reached by von Karman or Hencky models. In these ones, the bifurcation parameter (loading intensity) can only be a real number, and buckling may only take place when the loading intensity is above a critical value. Here, buckling is governed by a time dependent function and depends on the whole history of the loading and not only on its value at a single time. For example, there may be situations in which buckling arises while the loading is decreasing.

The corresponding mathematical model leads to solving nonlinear integro-differential equations instead of scalar nonlinear equations and this brings in technical difficulties.

The results on necessary conditions, sufficient conditions and the description of the local behaviour which we obtain give a framework and answers to problems arising in the engineering field and whose mathematical formulation was somewhat unprecise.
APPENDIX A

SOME PROPERTIES OF THE NEMYTSKY OPERATOR

Let \( \Phi \) be a \( C^1 \) mapping from \( \mathbb{R}^d_s \) into itself such that

\[
\forall \gamma \in \mathbb{R}^d_s, \quad \| \Phi'(\gamma) \|_{L(\mathbb{R}^d_s)} < M;
\]

we associate to \( \Phi \) the Nemytsky operator \( \psi \) such that, if \( \sigma \) is a function defined on \( \omega \) with values in \( \mathbb{R}^d_s \), we have

\[
(\text{A.2}) \quad \psi(\sigma)(x) = \Phi(\sigma(x)), \quad \forall x \in \omega.
\]

We summarize in the sequel some properties (and their proofs) of the Nemytsky operator which are useful in our context and are well known of specialists (for more details, see, for instance [Valent-Zampieri], and the bibliography therein).

Before stating the following theorem which describes the main results on the operator \( \psi \) which are useful in our problem, let us define, for any \( \sigma \in [L^1(\omega)]^d_s \), the mapping \( \psi_\sigma \) by

\[
(\text{A.3}) \quad \text{for } \tau : \omega \to \mathbb{R}^d_s, \quad \psi_\sigma(\tau)(x) = \Phi'(\sigma(x))(\tau(x)) \quad \text{a.e. in } \omega.
\]

THEOREM A 1:

(i) For every \( p > 1 \), the operator \( \psi \) is Lipschitz continuous from \( [L^p(\omega)]^d_s \) into itself, with Lipschitz constant \( M \).

(ii) For every \( r > 1 \), \( \psi_\sigma \) is a continuous linear map from \( [L^r(\omega)]^d_s \) into itself, and

\[
(\text{A.4}) \quad \| \psi_\sigma \|_{L([L^r(\omega)]^d_s)} < M.
\]
Moreover, if \( q > p > 1 \), the mapping \( \sigma \to \psi_{\sigma} \) is continuous from \( [L^1(\omega)]^4 \) to \( L([L^q(\omega)]^4, [L^p(\omega)]^4) \).

(iii) For every real numbers \( p \) and \( q \) such that \( q > p > 1 \), \( \psi \) is a \( C^1 \) mapping from \( [L^q(\omega)]^4 \) into \( [L^p(\omega)]^4 \) and

\[ \forall \sigma \in [L^q(\omega)]^4, \forall \tau \in [L^q(\omega)]^4, \quad \varphi(\sigma)[\tau] = \psi(\tau). \]

(A.5) In particular, \( \varphi(\sigma) \) which is a priori a continuous linear mapping from \( [L^q(\omega)]^4 \) into \( [L^p(\omega)]^4 \) is in fact a continuous linear mapping from \( [L^q(\omega)]^4 \) into itself.

(iv) If \( \Phi \) is Lipschitz continuous from \( \mathcal{R}^4 \) to \( L(\mathcal{R}^4) \), then for every real numbers \( p \) and \( q \) such that \( q > p > 1 \), the mapping \( \sigma \to \psi_{\sigma} \) is Lipschitz continuous from \( [L^{pq/(q-p)}(\omega)]^4 \) into \( L([L^q(\omega)]^4, [L^p(\omega)]^4) \).

Proof.

(1) From (A.1), we have

\[ \forall y \in \mathcal{R}^4, \| \Phi(y) \|_{\mathcal{R}^4} \leq M \| y \|_{\mathcal{R}^4} + C, \]

and

\[ \forall y, z \in \mathcal{R}^4, \| \Phi(y) - \Phi(z) \|_{\mathcal{R}^4} \leq M \| y - z \|_{\mathcal{R}^4}. \]

Therefore, if \( \sigma \in [L^p(\omega)]^4 \), \( \| \Phi(\sigma(x)) \|_{\mathcal{R}^4} \leq M \| \sigma(x) \|_{\mathcal{R}^4} + C \), and \( \psi(\sigma) \in [L^p(\omega)]^4 \).

Moreover, if \( \sigma, \tau \in [L^p(\omega)]^4 \),

\[ \| \varphi(\sigma) - \varphi(\tau) \|_{[L^p(\omega)]^4} = (\int_\omega \| \Phi(\sigma(x)) - \Phi(\tau(x)) \|_{\mathcal{R}^4}^p \, dx)^{1/p} \leq M \| \sigma - \tau \|_{[L^p(\omega)]^4}. \]
(ii) For any \( \sigma \) and \( \tau \) from \( \omega \) to \( \mathbb{R}^d \),

\[
\| \psi_\varepsilon (\tau(x)) \|_{\mathbb{R}^d} \leq \| \Phi'(\sigma(x))(\tau(x)) \|_{\mathbb{R}^d} \leq M \| \tau(x) \|_{\mathbb{R}^d}.
\]

Therefore, for any \( \sigma \) in \( [L^1(\omega)]^d \), \( \psi_\varepsilon \) maps \( [L^r(\omega)]^d \) into itself and

\[
\| \psi_\varepsilon (\tau) \|_{[L^r(\omega)]^d} \leq M \| \tau \|_{[L^r(\omega)]^d}.
\]

Let us now prove that \( \sigma \to \psi_\varepsilon \) is continuous from \( [L^1(\omega)]^d \) into \( L([L^q(\omega)]^d, [L^p(\omega)]^d) \)

when \( q > p \geq 1 \). If \( (\sigma_n) \), \( \sigma_n \in [L^1(\omega)]^d \), is such that \( \sigma_n \to 0 \) in \( [L^1(\omega)]^d \), we define \( g_{\sigma_n} \) by

\[
g_{\sigma_n}(x) = \| \Phi'(\sigma(x) + \sigma_n(x)) - \Phi'(\sigma(x)) \|_L([L^1(\omega)]^d),
\]

and we write the following estimate

\[
\| \psi_\varepsilon \sigma_n - \psi_\varepsilon \|_{L([L^q(\omega)]^d, [L^p(\omega)]^d)} = \int \| \Phi'(\sigma(x) + \sigma_n(x)) - \Phi'(\sigma(x))(\tau(x)) \|_{\mathbb{R}^d} \, dx
\]

\[
\leq \int [g_{\sigma_n}(x)]^p \| \tau(x) \|_{\mathbb{R}^d} \, dx \leq \| g_{\sigma_n} \|_{L^{pq/(q-p)}(\omega)} \| \tau \|_{[L^q(\omega)]^d},
\]

or

\[
\| \psi_\varepsilon \sigma_n - \psi_\varepsilon \|_{L([L^q(\omega)]^d, [L^p(\omega)]^d)} \leq \| g_{\sigma_n} \|_{L^{pq/(q-p)}(\omega)}.
\]

The problem is now reduced to showing that \( g_{\sigma_n} \to 0 \) in \( L^{pq/(q-p)}(\omega) \). First of all, from (A.1),

we get

\[ |g_{\sigma_n}(x)| \leq 2M, \text{ a.e. in } \omega. \]

On the other hand, since \( \sigma_n \to 0 \) in \( [L^1(\omega)]^d \), we can extract from any subsequence \( (\sigma'_{n'}) \) of \( (\sigma_n) \), a subsequence \( (\sigma''_{n''}) \) which converges almost everywhere to 0. Using the continuity of \( \Phi' \),
we observe that on this subsequence, \((g_{\alpha_n})\) converges almost everywhere to 0. Therefore, from Lebesgue's dominated convergence theorem,

\[
g_{\alpha_n} \rightarrow 0 \text{ in } L^{\frac{pq}{q-p}}(\omega)
\]

As this is valid for every subsequence \((n')\) extracted from \((n)\), it is easy to show that the whole sequence \((g_{\alpha_n})\) converges to 0 in \(L^{\frac{pq}{q-p}}(\omega)\).

(iii) Let \(p, q \in \mathbb{R}\) satisfy \(q > p > 1\). For \(\sigma, \tau \in [L^q(\omega)]_S^4\), we want to show that

\[
\|\psi(\sigma + \tau) - \psi(\sigma) - \psi_\sigma(\tau)\|_{L^p(\omega)} \leq o(\|\tau\|_{L^q(\omega)})
\]

We have, for almost every \(x \in \omega\),

\[
\Phi(\sigma(x) + \tau(x)) - \Phi(\sigma(x)) - \Phi'(\sigma(x))(\tau(x)) = \int_0^1 [\Phi'(\sigma(x) + st(x)) - \Phi'(\sigma(x))](\tau(x)) \, ds.
\]

Therefore,

\[
\|\psi(\sigma + \tau) - \psi(\sigma) - \psi_\sigma(\tau)\|_{L^p(\omega)} \leq \int_\omega [h_t(x)]^p \, dx
\]

where, with obvious notations,

\[
h_t(x) = \int_0^1 \|\Phi'(\sigma(x) + st(x)) - \Phi'(\sigma(x))\|_{L^p(\omega)} \, ds = \int_0^1 g_{st}(x) \, ds.
\]

This implies

\[
\|\psi(\sigma + \tau) - \psi(\sigma) - \psi_\sigma(\tau)\|_{L^p(\omega)} \leq \|h_t\|_{L^{\frac{pq}{q-p}}(\omega)} \|\tau\|_{L^q(\omega)}
\]

and it remains to show that, when \(\tau \rightarrow 0\) in \([L^q(\omega)]_S^4\), \(h_t\rightarrow 0\) in \(L^{\frac{pq}{q-p}}(\omega)\); this is done exactly as in (ii) above for \(g_{\alpha_n}\).

(iv) Observe that, for \(\Phi'\) is Lipschitz continuous from \(\mathbb{R}^d_S\) into \(L(\mathbb{R}^d_S)\) with Lipschitz constant \(M'\),
\[ |g_t(x)| \leq M' \|\tau(x)\|_{g^0_a}. \]

Then
\[ \|\psi_{\sigma t} - \psi_{\sigma}\|_{L^2_\omega(L^q(\omega))^{g^0_a},(L^p(\omega))^{g^0_a}} \leq \|g_t\|_{L^{p,q/(a-p)}(\omega)} < M' \|\tau\|_{L^{p,q/(a-p)}(\omega)}^{g^0_a}. \]

This completes the proof of Theorem A1.
APPENDIX B

EXISTENCE AND UNIQUENESS RESULTS FOR EQUATION (2.19)

Let us define

\[ \Sigma_0 = \{ \tau \in [L^2(\omega)]^4_s, \partial_0 \tau = 0, \tau \cdot n = 0 \} \]

and the projection \( P \) from \([L^2(\omega)]^4_s\) onto \( \Sigma_0 \) by

\[
\begin{cases}
( a P \sigma, \tau ) = ( \sigma, \tau ), & \forall \tau \in \Sigma_0, \\
P \sigma \in \Sigma_0.
\end{cases}
\]

Now, for \( p > 2 \), and a time interval \([T_1, T_2]\), let \( \sigma^* \) be a function in \( C([T_1, T_2]; [L^p(\omega)]^4_s) \), such that, if \( \lambda \in C([T_1, T_2]) \),

\[ \partial_0 \sigma^*_\alpha(t) = 0; \quad \sigma^*_\alpha(t) \cdot n = \lambda(t) F_\alpha. \]

Our purpose is to study for a given \( g \) in \( C([T_1, T_2]; [L^p(\omega)]^4_s) \), the following nonlinear integro-differential equation

\[(B.1) \quad (\sigma(t) - \sigma^*(t) = P[ g(t) - a\sigma^*(t) - \int_{T_1}^t \varphi(s) ds ] , \forall t \in [T_1, T_2]. \]

To aid us in this study, we will consider the corresponding linearized equation, for \( \sigma \in C([T_1, T_2]; [L^4(\omega)]^4_s) \) and \( h \in C([T_1, T_2]; [L^q(\omega)]^4_s) \), \( q > 2 \),

\[(B.2) \quad \tau(t) = P[ h(t) - \int_{T_1}^t \psi(s) \tau(s) ds ] , \forall t \in [T_1, T_2], \]

where \( \psi_\sigma \) is defined in Appendix A. Results are the following.
THEOREM B 1:

(i) For every $\sigma \in C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$, for every real number $q > 2$, for every $h \in C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$, equation (B.2) has a unique solution $\tau \in C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$. This defines a linear mapping $h \rightarrow \tau = \xi_q(h)$ from $C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$ into itself which is continuous and bounded by $C_q e^{C_q M(T_2 - T_1)}$, where $C_q$ is a constant.

(ii) For every real numbers $p, q$ with $q > p > 2$, the mapping $\sigma \rightarrow \xi_q$ is continuous from $C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$ into $C([T_1, T_2]; L(\mathbb{L}^q(\omega)_{\frac{4}{s}}, \mathbb{L}^p(\omega)_{\frac{4}{s}}))$.

(iii) If $\Phi$ is Lipschitz continuous from $\mathbb{R}^d_\omega$ into $L(\mathbb{R}^d_\omega)$, then $\sigma \rightarrow \xi_q$ is Lipschitz continuous from $C([T_1, T_2]; \mathbb{L}^{pq/(q-p)}(\omega)_{\frac{4}{s}})$ into $C([T_1, T_2]; L(\mathbb{L}^q(\omega)_{\frac{4}{s}}, \mathbb{L}^p(\omega)_{\frac{4}{s}}))$.

THEOREM B 2:

(i) For every $p > 2$, for every $g \in C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}})$, equation (B.1) has a unique solution $\sigma \in C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}})$. This defines a mapping $g \rightarrow \hat{\sigma}(g) = \sigma$ which is Lipschitz continuous from $C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}})$ into itself and satisfies

$$
\forall \, t \in [T_1, T_2], \|\hat{\sigma}(g_1) - \hat{\sigma}(g_2)(t)\|_{\mathbb{L}^p(\omega)_{\frac{4}{s}}} \leq C_p \|g_1 - g_2\|_{C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}})} e^{C_p M(T_2 - T_1)}.
$$

(ii) For every real numbers $p, q$ with $q > p > 2$, $\hat{\sigma}$ is continuously differentiable from $C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}})$ into $C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}})$, and

$$
\forall \, g \in C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}}), \quad \hat{\sigma}'(g) = \xi_{\hat{\sigma}(g)}
$$

Then, $\hat{\sigma}'(g)$ which belongs a priori to $L(C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}}); C([T_1, T_2]; \mathbb{L}^p(\omega)_{\frac{4}{s}}))$ is in fact in $L(C([T_1, T_2]; \mathbb{L}^q(\omega)_{\frac{4}{s}}))$. 

[(11)] Let \( g_0 \) be an element of \( C([-T_1, T_2]; [L^q(\omega)]_s^4) \); for every \( h \in C([-T_1, T_2]; [L^q(\omega)]_s^4) \), let \( \Phi(h) \) be defined by

\[
\hat{\Phi}(g_0 + h) = \hat{\Phi}(g_0) + \hat{\Phi}'(g_0)[h] + \Phi(h).
\]

If \( \Phi' \) is Lipschitz continuous from \( \mathbb{R}_s^4 \) into \( L(\mathbb{R}_s^4) \), then for every real numbers \( p, q \) with \( q > p \geq 2 \), for every \( h \in C([-T_1, T_2]; [L^{pq/(q-p)}(\omega)]_s^4) \cap [L^q(\omega)]_s^4 \),

\[
\| \Phi(h) \|_{C([-T_1, T_2]; [L^p(\omega)]_s^4)} \leq C \| h \|_{C([-T_1, T_2]; [L^{pq/(q-p)}(\omega)]_s^4)} \| h \|_{C([-T_1, T_2]; [L^q(\omega)]_s^4)}.
\]

Before proving Theorem B1, let us state the following lemma, the proof of which is analogous to the one given in [Fujiwara-Morimoto] for the \( L_p \)-decomposition of vector fields.

**Lemma B1:**

For every \( q \geq 2 \), the projection operator \( P \) maps continuously \( [L^q(\omega)]_s^4 \) into itself.

Note that, as a consequence, if \( h \in C([-T_1, T_2]; [L^q(\omega)]_s^4) \), \( P(h) \in C([-T_1, T_2]; [L^q(\omega)]_s^4) \).

**Proof of Theorem B1.**

(i) As \( \sigma \in C([-T_1, T_2]; [L^1(\omega)]_s^4) \), we know from (A.4) that, for \( \tau \in C([-T_1, T_2]; [L^q(\omega)]_s^4) \)

\[
\| \psi_{\sigma}(\tau(s)) \|_{[L^q(\omega)]_s^4} \leq M \| \tau(s) \|_{[L^q(\omega)]_s^4} ;
\]

hence, \( t \rightarrow \int_{-T_1}^{t} \psi_{\sigma}(\tau(s)) \, ds \) is an element of \( C([-T_1, T_2]; [L^q(\omega)]_s^4) \) and due to Lemma B1,

\( t \rightarrow P[h(t) - \int_{-T_1}^{t} \psi_{\sigma}(\tau(s)) \, ds] \) belongs to \( C([-T_1, T_2]; [L^q(\omega)]_s^4) \).
Let $A$ be the operator from $C([T_1, T_2]; [L^q(\omega)]_s^4)$ into itself defined by

$$A(\tau)(t) = P[h(t) - \int_{T_1}^{\tau} \psi_\sigma(s) \, ds] .$$

We have

$$\| (A(\tau_1) - A(\tau_2))(t) \|_{[L^q(\omega)]_s^4} \leq C_q M \int_{T_1}^{\tau_1} \| \psi_\sigma(s) \|_{[L^q(\omega)]_s^4} \, ds \leq C_q M(t - T_1) \| \tau_1 - \tau_2 \|_{C([T_1, T_2]; [L^q(\omega)]_s^4)} .$$

Then

$$\| (A^2(\tau_1) - A^2(\tau_2))(t) \|_{[L^q(\omega)]_s^4} \leq C_q M \int_{T_1}^{\tau_1} \| (A(\tau_1) - A(\tau_2))(s) \|_{[L^q(\omega)]_s^4} \, ds \leq 1/2 (C_q M(t - T_1))^2 \| \tau_1 - \tau_2 \|_{C([T_1, T_2]; [L^q(\omega)]_s^4)} ,$$

and by induction,

$$\| (A^k(\tau_1) - A^k(\tau_2))(t) \|_{[L^q(\omega)]_s^4} \leq (C_q M(t - T_1))^k/(k!) \| \tau_1 - \tau_2 \|_{C([T_1, T_2]; [L^q(\omega)]_s^4)} \leq (C_q M(T_2 - T_1))^k/(k!) \| \tau_1 - \tau_2 \|_{C([T_1, T_2]; [L^q(\omega)]_s^4)} .$$

There exists $k$ such that $(C_q M(T_2 - T_1))^k/(k!) < 1$, and $A^k$ is then a strict contraction.

Therefore, $A$ has a unique fixed point $\tau$ which is the unique solution of (B.2). Furthermore,

$$\| \tau(t) \|_{[L^q(\omega)]_s^4} \leq C_q \| h(t) \|_{[L^q(\omega)]_s^4} + C_q M \int_{T_1}^{\tau} \| \tau(s) \|_{[L^q(\omega)]_s^4} \, ds ,$$

and using Gronwall's lemma, we obtain

$$\| \tau(t) \|_{[L^q(\omega)]_s^4} \leq C_q \| h \|_{C([T_1, T_2]; [L^q(\omega)]_s^4)} e^{C_q M(t - T_1)} ,$$

which proves (i).

(ii) Let $\sigma_n$ be a sequence converging to $\sigma$ in $C([T_1, T_2]; [L^1(\omega)]_s^4)$ and let us show that $\ell_{\sigma_n}$ converges to $\ell_{\sigma}$ in $C([T_1, T_2]; L([L^q(\omega)]_s^4, [L^p(\omega)]_s^4))$, when $q > p > 2$. 


For \( h \in C([T_1, T_2]; [L^q(\omega)]^d) \) and \( t \in [T_1, T_2] \) we have

\[
(l_{\sigma_n} - l_\sigma)(h)(t) = P[\int_{T_1}^t (\psi_{\sigma_n}(s) - \psi_{\sigma}(s)) (l_{\sigma_n}(h)(s)) \, ds - \int_{T_1}^t \psi_{\sigma}(s) (l_{\sigma_n} - l_{\sigma})(h)(s) \, ds].
\]

If we set

\[
\tau(t) = (l_{\sigma_n} - l_\sigma)(h)(t), \quad k_n(t) = \int_{T_1}^t (\psi_{\sigma}(s) - \psi_{\sigma_n}(s)) (l_{\sigma_n}(h)(s)) \, ds,
\]

then \( k_n \in C([T_1, T_2]; [L^q(\omega)]^d) \), \( \tau \in C([T_1, T_2]; [L^q(\omega)]^d) \), and the above equation writes as

\[
\tau(t) = P[k_n(t) - \int_{T_1}^t \psi_{\sigma}(s) (\tau(s)) \, ds].
\]

It follows from (1) that

\[
\| \tau \|_{C([T_1, T_2]; [L^p(\omega)]^d)} \leq C_p e^{C_p M(T_2 - T_1)} \| k_n \|_{C([T_1, T_2]; [L^p(\omega)]^d)},
\]

and that

\[
\| k_n(t) \|_{L^p(\omega)}^d \leq \int_{T_1}^t \| \psi(s) - \psi_{\sigma_n}(s) \|_L ([L^q(\omega)]^d, [L^p(\omega)]^d) \| l_{\sigma_n}(h)(s) \|_{L^q(\omega)} \, ds,
\]

\[
\leq C_q e^{C_q M(T_2 - T_1)} \| h \|_{C([T_1, T_2]; [L^q(\omega)]^d)} \int_{T_1}^t \| \psi(s) - \psi_{\sigma_n}(s) \|_L ([L^q(\omega)]^d, [L^p(\omega)]^d) \, ds.
\]

Thus, in order to prove that \( l_{\sigma_n} \) tends to \( l_\sigma \) in \( C([T_1, T_2]; L ([L^q(\omega)]^d, [L^p(\omega)]^d)) \), it remains to show that

\[
\int_{T_1}^{T_2} \| \psi(s) - \psi_{\sigma_n}(s) \|_L ([L^q(\omega)]^d, [L^p(\omega)]^d) \, ds \to 0.
\]

As for every \( s \in [T_1, T_2] \), \( \sigma_n(s) \to \sigma(s) \) in \( [L^1(\omega)]^d \), we know from Theorem A1 that

\[
\| \psi(s) - \psi_{\sigma_n}(s) \|_L ([L^q(\omega)]^d, [L^p(\omega)]^d) \to 0.
\]
moreover,
\[ \| \psi_{\sigma}(s) - \psi_{\sigma_{0}}(s) \|_{L((L^q(\omega))_{2}^{5}, (L^p(\omega))_{2}^{5})} \leq 2M, \quad \forall \, s \in [T_1, T_2). \]

The proof of (11) is completed by use of Lebesgue's dominated convergence theorem.

(ii) If \( \sigma_1, \sigma_2 \in C([T_1, T_2]; [L^{pq/(q-p)}(\omega)]_{2}^{4}) \) and \( h \in C([T_1, T_2]; [L^{q}(\omega)]_{2}^{4}) \), we obtain as above, by setting
\[ \tau(t) = (\xi_{\sigma_1} - \xi_{\sigma_2})(h)(t), \]
\[ \| \tau \|_{C([T_1, T_2]; (L^p(\omega))_{2}^{5})} \leq C_p C_q \| h \|_{C([T_1, T_2]; (L^q(\omega))_{2}^{5})} \int_{T_1}^{T_2} \| \psi_{\sigma_1}(s) - \psi_{\sigma_2}(s) \|_{L((L^q(\omega))_{2}^{5}, (L^p(\omega))_{2}^{5})} \, ds. \]

If \( \Phi' \) is Lipschitz continuous from \( (R_{2}^{4}) \) into \( (R_{2}^{4}) \) with Lipschitz constant \( M' \), we know from Theorem A1(iv) that
\[ \| \psi_{\sigma_1}(s) - \psi_{\sigma_2}(s) \|_{L((L^q(\omega))_{2}^{5}, (L^p(\omega))_{2}^{5})} \leq M' \| \sigma_1(s) - \sigma_2(s) \|_{[L^{pq/(q-p)}(\omega)]_{2}^{4}}. \]

Then
\[ \| \tau \|_{C([T_1, T_2]; (L^p(\omega))_{2}^{5})} \leq C_p C_q M'(T_2 - T_1) \| h \|_{C([T_1, T_2]; (L^q(\omega))_{2}^{5})} \| \sigma_1 - \sigma_2 \|_{C([T_1, T_2]; [L^{pq/(q-p)}(\omega)]_{2}^{4})} \]
and this proves that \( \sigma \rightarrow \xi_{\sigma} \) is Lipschitz continuous from \( C([T_1, T_2]; [L^{pq/(q-p)}(\omega)]_{2}^{4}) \) into \( C([T_1, T_2]; L((L^q(\omega))_{2}^{4}, (L^p(\omega))_{2}^{4})) \), and completes the proof of Theorem B1.

Proof of Theorem B2.

(i) Let \( g \) and \( \sigma \) belong to \( C([T_1, T_2]; [L^{p}(\omega)]_{2}^{4}) \); we define
\[ S(\sigma)(t) = \sigma^*(t) + \int_{t}^{1} \psi(\sigma(s)) \, ds, \quad \forall \, t \in [T_1, T_2]. \]

From Theorem A1 (i) and Lemma B1, we see that \( S \) maps \( C([T_1, T_2]; [L^{p}(\omega)]_{2}^{4}) \) into itself.

Moreover, if \( \sigma_1, \sigma_2 \in C([T_1, T_2]; [L^{p}(\omega)]_{2}^{4}) \), we have, using Theorem A1 (i):
\[
\|S(\sigma_1) - S(\sigma_2)\|_{L^p(\omega)} \leq C_p \int_{T_1}^t \|\varphi(\sigma_1(s)) - \varphi(\sigma_2(s))\|_{L^p(\omega)} ds \\
\leq C_p M \int_{T_1}^t \|\sigma_1(s) - \sigma_2(s)\|_{L^p(\omega)} ds \leq C_p M (t - T_1) \|\sigma_1 - \sigma_2\|_{C([T_1, T_2]; L^p(\omega))}.
\]

Using the same argument as in Theorem B1 (1), we establish that an iterate of \( S \) is a strict contraction and consequently that \( S \) has a unique fixed point. This proves the existence of a unique solution \( \sigma \in C([T_1, T_2]; [L^p(\omega)]^d) \) of equation (B.1).

If \( g_1, g_2 \in C([T_1, T_2]; [L^p(\omega)]^d) \), we have

\[
\|\hat{\sigma}(g_1) - \hat{\sigma}(g_2)(t)\|_{L^p(\omega)} \leq C_p \left( \|g_1 - g_2\|_{C([T_1, T_2]; L^p(\omega))} \int_{T_1}^t \|\varphi(\hat{\sigma}(g_1))(s) - \varphi(\hat{\sigma}(g_2))(s)\|_{L^p(\omega)} ds \right) \\
\leq C_p \left( \|g_1 - g_2\|_{C([T_1, T_2]; L^p(\omega))} + M \int_{T_1}^t \|\hat{\sigma}(g_1)(s) - \hat{\sigma}(g_2)(s)\|_{L^p(\omega)} ds \right).
\]

Using Gronwall's lemma, we obtain

\[
\|\hat{\sigma}(g_1) - \hat{\sigma}(g_2)(t)\|_{L^p(\omega)} \leq C_p e^{C_p M (t - T_1)} \|g_1 - g_2\|_{C([T_1, T_2]; L^p(\omega))},
\]

and this proves (i).

(ii) Let \( g \) and \( h \) belong to \( C([T_1, T_2]; [L^q(\omega)]^d) \) and let us set

\[
\tau = \hat{\sigma}(g + h) - \hat{\sigma}(g) - \varphi_\tau(g)(h).
\]

We have

\[
\tau(t) = P \int_{T_1}^t \left[ \varphi(\hat{\sigma}(g + h)(s)) - \varphi(\hat{\sigma}(g)(s)) - \psi_\tau(g)(\varphi_\tau(g)(h)(s)) \right] ds \\
= P \int_{T_1}^t \psi_\tau(g)(\tau(s)) ds,
\]

where

\[
k(t) = - \int_{T_1}^t \left[ \varphi(\hat{\sigma}(g + h)(s)) - \varphi(\hat{\sigma}(g)(s)) - \psi_\tau(g)(\varphi_\tau(g)(h)(s) - \hat{\sigma}(g)(s)) \right] ds.
\]
Then

$$\tau = \xi_{\delta(h)}(k) ,$$

which makes sense because one can easily show that $k \in C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4$. We obtain, using Theorem B1 (i)

$$\| \tau \|_{C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4} \leq C_p \| k \|_{C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4}.$$ 

In order to establish (ii), we therefore have to prove that

$$\| k \|_{C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4} = O(\| h \|_{C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4}).$$

Due to (A.5), we have

$$k(t) = -\int_{T_1}^t \int_0^1 \left[ \Psi(\delta(g)(s) + u(\delta(g + h)(s) - \delta(g)(s))) - \Psi(\delta(g)(s)) \right] \delta(g + h)(s) - \delta(g)(s) \, du \, ds.$$ 

Then

$$\| k(t) \|_{L^0(\omega)} = \int_{T_1}^t \int_0^1 \delta(g)(s) + u(\delta(g + h)(s) - \delta(g)(s))) - \Psi(\delta(g)(s)) \|_{L^0(\omega)} \, du \, ds,$$

and using (i) (which has already been proved),

$$\| k(t) \|_{L^0(\omega)}^4 \leq C \| h \|_{C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4} \int_{T_1}^T \| \epsilon(h)(s) \|_{L^0(\omega)}^4 ds ,$$

where

$$\epsilon(h)(s) = \int_0^1 \Psi(\delta(g)(s) + u(\delta(g + h)(s) - \delta(g)(s))) - \Psi(\delta(g)(s)) \|_{L^0(\omega)} \, du .$$

It remains to show that $\epsilon(h)$ converges to $0$ in $L^1(T_1, T_2)$ if $h$ converges to $0$ in $C([T_1, T_2]; \mathbb{L}_s^0(\omega))^4$). From (A.4) we have

$$\| \epsilon(h)(s) \| \leq M \quad \forall s \in [T_1, T_2].$$
On the other hand, for every \( s \in [T_1, T_2] \) and every \( u \in [0, 1] \), as \( h(s) \to 0 \) in \( [L^q(\omega)]^4_s \),
\( u(\hat{\sigma}(g + h)(s) - \hat{\sigma}(g)(s)) \to 0 \) in \( [L^q(\omega)]^4_s \) (c.f. (i) above) and using Theorem A 1(ii),
\( \epsilon(h)(s) \to 0 \). This shows that \( \epsilon(h) \to 0 \) in \( L^1(T_1, T_2) \), and proves that \( \hat{\sigma} \) is differentiable from \( C([T_1, T_2]; [L^q(\omega)]^4_s) \) into \( C([T_1, T_2]; [L^p(\omega)]^4_s) \) with
\[
\forall g \in C([T_1, T_2]; [L^q(\omega)]^4_s), \quad \hat{\sigma}'(g) = \xi\hat{\sigma}(g).
\]
From Theorem B1(i), we see that \( \hat{\sigma}'(g) \in L(\ C([T_1, T_2]; [L^q(\omega)]^4_s)) \).

In order to complete the proof of (ii), let us show that \( g \to \hat{\sigma}'(g) \) is continuous from \( C([T_1, T_2]; [L^q(\omega)]^4_s) \) to \( L(\ C([T_1, T_2]; [L^q(\omega)]^4_s); C([T_1, T_2]; [L^p(\omega)]^4_s)) \). We have already proved that \( g \to \hat{\sigma}(g) \) is continuous from \( C([T_1, T_2]; [L^q(\omega)]^4_s) \) into itself. Using Theorem B1(ii) and the expression of \( \hat{\sigma}'(g) \) the conclusion follows.

(iii) From (B.4), we have
\[
\Sigma(h) = \hat{\sigma}(g_0 + h) - \hat{\sigma}(g_0) - \hat{\sigma}'(g_0)[h] = \int_0^1 (\hat{\sigma}'(g_0 + uh) - \hat{\sigma}'(g_0))[h] \, du.
\]
Therefore
\[
\| \Sigma(h) \|_{C([T_1, T_2]; [L^p(\omega)]^4_s)} \leq \int_0^1 \| \hat{\sigma}'(g_0 + uh) - \hat{\sigma}'(g_0) \|_{L(\ C([T_1, T_2]; [L^q(\omega)]^4_s); C([T_1, T_2]; [L^p(\omega)]^4_s))} \, du.
\]

From (ii) and Theorem B1(iii), we obtain for \( h \in C([T_1, T_2]; [L^{pa/q-p}(\omega)]^4_s) \cap [L^q(\omega)]^4_s \),
the estimate (B.5).
REFERENCES


Jean-Pierre PUEL et Annie RAOULT
Laboratoire d'Analyse Numérique
Tour 55-65, 5ème étage
Université Pierre et Marie Curie
75252 PARIS CEDEX 05, FRANCE.