HELLY-TEST FOR THE MINIMAL WIDTH OF CONVEX BODIES

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IMA Preprint Series # 357

October 1987

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

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HELLY–TESTS FOR THE MINIMAL WIDTH OF CONVEX BODIES*

PETER GRITZMANN† AND MAREK LASSAK‡

Abstract. Let $\mathcal{K}^d$ denote the family of all convex bodies in Euclidean d-space. The paper deals with the following Helly-type problem for the minimal width w. Let $\mathcal{A} \subset \mathcal{K}^d$ and let n be an integer greater than d. Determine the greatest number $\mu = \Lambda_n(\mathcal{A})$ such that $w(A) \leq \omega$ whenever $A \in \mathcal{A}$ has the property that every n-point subset of A has minimal width at most $\mu\omega$. We give bounds for $\Lambda_n(\mathcal{K}^d)$, for $\Lambda_n(\mathcal{M}^d)$ and for $\Lambda_n(\mathcal{W}^d)$, where $\mathcal{M}^d$, $\mathcal{W}^d$ denote the families of centrally symmetric convex bodies and of bodies of constant width, respectively.

1. INTRODUCTION AND RESULTS

Let $\mathbf{E}^d$ denote the d-dimensional Euclidean space. The scalar product and the Euclidean norm will be denoted by $\langle . \rangle$, $\| \, . \, \|$, respectively, and $\mathbf{B}^d$, $\mathbf{S}^{d-1}$ will indicate the unit ball and the unit sphere in $\mathbf{E}^d$. Let $\mathcal{K}^d$ denote the family of all *convex bodies* of $\mathbf{E}^d$, this is the family of convex compact sets which have interior points and let $\mathcal{M}^d$ denote the family of *centrally symmetric convex bodies* of $\mathbf{E}^d$. The symbols $\mathbf{N}, \mathrm{I\!R}$ denote the sets of positive integers and real numbers, respectively. By $\lfloor \tau \rfloor$ we denote the integer part of real numbers τ.

Now, let $A \subset \mathbf{E}^d$. The convex hull of A will be denoted by $\mathrm{conv}(A)$. The *support functional* s_A of A is the extended real-valued functional of $\mathbf{E}^d$ defined by

$$s_A(z) = \sup\{\langle x, z\rangle | x \in A\}.$$

Geometrically, for $C = \mathrm{conv}(A) \in \mathcal{K}^d$ and $z \neq 0$ the equation $\langle x, z\rangle = s_A(z)$ gives the oriented supporting hyperplane of C of direction z such that the associated positive halfspace does not intersect the interior of C. Accordingly, $b_A : \mathbf{S}^{d-1} \to \mathrm{I\!R} \cup \{\infty\}$ defined by

$$b_A(z) = s_A(z) + s_A(-z)$$

for $A \neq \emptyset$ and $b_\emptyset(z) = 0$ is called the *width* of A. If $K \in \mathcal{K}^d$ and b_K is constant, K is called a *convex body of constant width*. The family of convex bodies of E^d of constant width is denoted by $\mathcal{W}^d$ (a survey of results on $\mathcal{W}^d$ is in [3]). The functionals

$$w(A) = \min\{b_A(z) | z \in \mathbf{S}^{d-1}\}, \quad \mathrm{diam}(A) = \max\{b_A(z) | z \in \mathbf{S}^{d-1}\}$$

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*This paper was written while both authors were visiting the University of Washington, Seattle.

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Research supported in part by the Alexander-von-Humboldt foundation.

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are *called minimal* width of A and *diameter* of A, respectively. Since it is natural and convenient we restrict our considerations to the family

$$\mathcal{S}^d = \{S | S \subset \mathbf{E}^d \wedge 0 < w(S) < \infty\}.$$

The problem of giving bounds for the minimal width of convex bodies in terms of other functionals which are naturally associated with convex bodies has been investigated for a long time (see e.g. [1]). There are also computational aspects of questions like this in computer science (see e.g. [13]).

In the present paper we are dealing with a Helly-type problem concerning the minimal width of convex sets. Ever since Helly, in 1913, stated his famous result ([10]), theorems which guarantee that a family of convex sets has some property whenever for a given positive integer n each n members of the family have a "closely related" property are called *Helly-type theorems*. For a survey of Helly-type theorems see [4]. The problem we are dealing with is of the abstract form: a set has a property if each of its n-point subsets has a "closely related" property.

PROBLEM. *For a given family $\mathcal{A} \subset \mathcal{S}^d$ and a positive integer n determine the greatest number $\mu = \Lambda_n(\mathcal{A})$ such that for each $A \in \mathcal{A}$ the following holds : if every n-point subset of A has the minimal width at most $\mu\omega$, then $w(A) \leq \omega$.*

We set for $S \in \mathcal{S}^d$ and $n \in \mathbf{N}$

$$\lambda_n(S) = \sup\left\{\frac{w(P)}{w(S)} \middle| P \subset S \wedge |P| = n\right\}.$$

Furthermore, for non-empty subfamilies $\mathcal{A}$ of $\mathcal{S}^d$ we define

$$\Lambda_n(\mathcal{A}) = \inf\{\lambda_n(A) | A \in \mathcal{A}\}.$$

Let us point out that we have the following monotonicity relations for $\mathcal{A}, \mathcal{B} \subset \mathcal{S}^d$ and $n, m \in \mathbf{N}$.

$$(*) \qquad n \leq m \implies \Lambda_n(\mathcal{A}) \leq \Lambda_m(\mathcal{A}),$$

$$(**) \qquad \mathcal{A} \subset \mathcal{B} \implies \Lambda_n(\mathcal{A}) \geq \Lambda_n(\mathcal{B}).$$

In connection with the last implication, observe that if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{S}^d$ and if for every $\varepsilon > 0$ and every $B \in \mathcal{B}$ there exists a subset A_ε of B in $\mathcal{A}$ such that $w(A_\varepsilon)/w(B) \geq 1 - \varepsilon$, then $\Lambda_n(\mathcal{A}) = \Lambda_n(\mathcal{B})$. As we shall see in the next section (see Lemma 2) this implies that for the families $\widetilde{\mathcal{K}}^d$ of all closed convex sets from $\mathcal{S}^d$, and $\widetilde{\mathcal{M}}^d$ of all centrally symmetric convex closed sets from $\mathcal{S}^d$, we have

$$\Lambda_n(\widetilde{\mathcal{K}}^d) = \Lambda_n(\mathcal{K}^d) \text{ and } \Lambda_n(\widetilde{\mathcal{M}}^d) = \Lambda_n(\mathcal{M}^d).$$

So we are dealing mainly with the functionals

$$\Lambda_n(\mathcal{K}^d), \quad \Lambda_n(\mathcal{M}^d) \quad \text{and} \quad \Lambda_n(\mathcal{W}^d).$$

Clearly, for $d = 1$ and $n \geq 2$ all the functionals are 1 and for $n \leq d$ all are 0. So, in the following we always assume

$$d \geq 2 \quad \text{and} \quad n \geq d + 1.$$

THEOREM. *Here are some properties of* $\Lambda_n(\mathcal{A})$*:*

$$\Lambda_n(\mathcal{K}^d) \geq \Lambda_n(\mathcal{S}^d) \geq \begin{cases} \frac{1}{d+1} & \text{for } d \text{ odd,} \\ \frac{1}{d} & \text{for } d \text{ even.} \end{cases} \tag{1}$$

(2) $\Lambda_n(\mathcal{M}^d)$ *is attained for balls and, in particular,*

$$\Lambda_{d+1}(\mathcal{M}^d) = \begin{cases} \frac{1}{\sqrt{d}} & \text{for } d \text{ odd,} \\ \frac{d+1}{d\sqrt{d+2}} & \text{for } d \text{ even.} \end{cases}$$

(3) $\Lambda_3(\mathcal{W}^2) = 0.739\ldots$ *is attained for the Reuleaux-pentagon and* $\Lambda_3(\mathcal{K}^2) > 0.583$*.*

(4) Let $\Delta(k)$ *denote the Hausdorff-distance between* $\mathbf{B}^d$ *and a best approximating*[1] *inscribed polytope with* k *vertices. Then for* $n \geq 2(d+1)$ *we have*

$$\Lambda_n(\mathcal{K}^d) \geq 1 - \Delta(\lfloor \frac{n}{2} \rfloor),$$

and there exists a constant κ *which only depends on* d *but not on* n *such that*

$$\Lambda_n(\mathcal{K}^d) \geq 1 - \kappa n^{\frac{-2}{d-1}}.$$

Let us close this section with a few comments about our Theorem.

For $d = 2$ and $n = 3$ in (1) we have $\Lambda_3(\mathcal{S}^2) \geq \frac{1}{2}$. This is a generalization of a property presented in [14].

It was conjectured in [15] that $\Lambda_3(\mathcal{K}^2) = 0.75$ meaning that the disk is extremal for $\Lambda_3(\mathcal{K}^2)$. However, as a consequence of (3) it turns out that this conjecture is false. But the Reuleaux-pentagon is not extremal, either. Rather, it seems that $\Lambda_3(\mathcal{K}^2)$ might be attained for the regular pentagon.

[1] with respect to the Hausdorff-distance

CONJECTURE. $\Lambda_3(\mathcal{K}^2)$ *is attained only for the regular pentagon, thus*

$$\Lambda_3(\mathcal{K}^2) = 6/(3+\sqrt{3}\ \tan 72°) = 0.720\dots.$$

A proof of this conjecture would also show $\Lambda_3(\mathcal{K}^2) > \Lambda_3(\mathcal{S}^2)$.

Let us point out that the second inequality in (4) results from the first by applying asymtotic estimates for the approximation of convex bodies by inscribed polytopes.[2] Explicit bounds for $d = 2$ are given in [11]. For a survey on approximations of convex bodies see [9].

2. SOME PRELIMINARY RESULTS

In the sequel we give some auxilliary results, first three lemmas concerning the width of general bodies and then two lemmas concerning the case of simplices. In particular, we begin by completing the justification of restricting our considerations to the bounded case.

LEMMA 1. *Let $G_1, G_2, \dots$ be an increasing sequence of subsets of E^d and let $G = \bigcup_{i=1}^{\infty} G_i$. Then*

$$w(G) = \sup\{w(G_i) \mid i = 1, 2, \dots\}.$$

PROOF. Denote the above supremum by ω. Clearly, $\omega \le w(G)$. So, suppose in the following $\omega < w(G)$.

Of course, for each $i \in \mathbf{N}$ there is a $z_i \in S^{d-1}$ such that $w(G_i) = b_{G_i}(z_i)$. Assume that the sequence $(z_i)_{i\in\mathbf{N}}$ converges to some $z \in S^{d-1}$ (if not, we can select a convergent subsequence). By our supposition there is a two-point set $D \subset G$ for which $b_D(z) > \omega$. There exists a number j such that $D \subset G_i$ for each $i \ge j$. Since $(z_i)_{i\in\mathbf{N}}$ converges to z, we have $b_D(z_k) > \omega$ for some $k \ge j$ and thus $b_{G_k}(z_k) > \omega$. But this contradicts $b_{G_k}(z_k) = w(G_k) \le \omega$. □

LEMMA 2. *For every $A \in \mathcal{S}^d$ and every $\varepsilon > 0$ there is a bounded $A_\varepsilon \subset A$ such that*

$$w(A_\varepsilon) \ge (1-\varepsilon) w(A).$$

PROOF. Let G_i be the intersection of A with the ball of center 0 and radius i for $i = 1, 2, \dots$. Of course $A = \bigcup_{i=1}^{\infty} G_i$. By Lemma 1, we can take as A_ε a set G_i with i sufficiently large. □

[2] However, in terms of the Hausdorff-distance the applied estimate is asymptotically best possible (see [2, 12] even for $\mathbf{B}^d$.

LEMMA 3. *Let $K \in \mathcal{K}^d$ and $z \in \mathbf{S}^{d-1}$. There exists a segment in K of direction z and length at least $w(K)$.*

PROOF. (See [1].) Let $A = \operatorname{conv}\{a_1, a_2\}$ be a longest segment in K of direction z. So K and $\tilde{K} = a_2 - a_1 + K$ do not have any interior points in common. Thus there exists a hyperplane H which separates K and $\tilde{K}$. But this means that K lies between the two hyperplanes H and $a_1 - a_2 + H$. Since these two hyperplanes have distance less than or equal to the length of A this length has to be at least $w(K)$. □

Let for $\alpha \in \mathbb{R}$ and $z \in \mathbf{S}^{d-1}$

$$H(z;\alpha) = \{x \in \mathbf{E}^d | \langle x, z\rangle = \alpha\} \quad \text{and}$$

$$H^+(z;\alpha) = \{x \in \mathbf{E}^d | \langle x, z\rangle \geq \alpha\}, \quad H^-(z;\alpha) = \{x \in \mathbf{E}^d | \langle x, z\rangle \leq \alpha\}.$$

$H(z;\alpha)$ is the hyperplane orthogonal to z and with distance α from the origin, while $H^+(z;\alpha), H^-(z;\alpha)$ are the associated halfspaces. Furthermore, let for $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$

$$S(z;\alpha_1,\alpha_2) = H^+(z;\alpha_1) \cap H^-(z;\alpha_2).$$

So, $S(z;\alpha_1,\alpha_2)$ denotes a *strip* of minimal width $\alpha_2 - \alpha_1$ orthogonal to z. The vector z is called *normal* of the strip $S(z;\alpha_1,\alpha_2)$. Let $A \in \mathcal{S}^d$. Then a strip $S = S(z;\alpha_1,\alpha_2)$ is called *minimal* (with respect to A) if $A \subset S$ and $w(A) = w(S)$.

Let $T \subset E^d$ be a simplex. By the *height* of T from a vertex $v \in T$ we mean the distance from v to the hyperplane carrying the remaining vertices of T. Denote by $h(T)$ the *minimal height* of T.

LEMMA 4. *For every d-simplex $T \subset E^d$ we have*

$$\frac{w(T)}{h(T)} \geq \begin{cases} \frac{2}{d+1} & \text{for } d \text{ odd,} \\ \frac{2}{d} & \text{for } d \text{ even.} \end{cases}$$

PROOF. Enclose T in a strip $S(z;\alpha_1,\alpha_2)$, where $\alpha_2 - \alpha_1 = w(T)$. Denote the set of vertices of T by V. We may assume without loss of generality that

$$k = \left|V \cap H(z;\alpha_1)\right| \leq \begin{cases} \frac{d+1}{2} & \text{for } d \text{ odd,} \\ \frac{d}{2} & \text{for } d \text{ even.} \end{cases}$$

Let $v_1, \ldots, v_k = V \cap H(z;\alpha_1)$. Since $S(z;\alpha_1,\alpha_2)$ is a minimal strip for T, there are points $x_1 \in T \cap H(z;\alpha_1)$ and $x_2 \in T \cap H(z;\alpha_2)$ such that $\| x_2 - x_1 \| = w(T)$. Let $\mu_1, \ldots, \mu_k \in [0,1]$ with $\sum_{i=1}^k \mu_i = 1$ such that $x_1 = \sum_{i=1}^k \mu_i v_i$. Assume that $\mu_1 \geq 1/k$ and denote by H the hyperplane containing $V \setminus \{v_1\}$. Since $x_2 \in H$, the distance $\delta(x_1, H)$

of x_1 and H does not exceed $w(T)$. If $\mu_1 = 1$, we have $h(T) = w(T)$. So consider the case $\mu_1 < 1$. Set

$$y = \frac{1}{1-\mu_1}x_1 - \frac{\mu_1}{1-\mu_1}v_1.$$

Obviously, we have $\| v_1 - y \| / \| x_1 - y \| \leq 1/\mu_1 \leq k$. Since $y \in T \cap H$ it follows that $\delta(v_1, H) \leq kw(T)$. Thus $w(T)/h(T) \geq 1/k$ which proves the assertion. $\square$

Let T denote a d-simplex in $\mathbf{E}^d$ and let $V = V(T) = \{v_0, \ldots, v_d\}$ be the set of its vertices. For each $p \in \{1, \ldots, d\}$ set $q = d + 1 - p$ and define

$$\mathcal{V}_p = \{(P, Q) | P, Q \subset V \wedge P \cap Q = \emptyset \wedge |P| = p \wedge |Q| = q\}.$$

Now, let

$$\zeta_p(T) = \sum_{(P,Q)\in\mathcal{V}_p} \| \gamma(P) - \gamma(Q) \|^2,$$

where

$$\gamma(P) = \frac{1}{p}\sum_{v_i\in P} v_i$$

denotes the center of the face conv(P) (and respectively for Q).

LEMMA 5. *Let T be an arbitrary d-simplex inscribed $\mathbf{B}^d$ and let T_0 denote the regular simplex inscribed $\mathbf{B}^d$. Then $\zeta_p(T) \leq \zeta_p(T_0)$.*

PROOF. Let for $i = 0, \ldots, d$ and $j = 1, \ldots d$ the j-th coordinate of v_i be denoted by ν_{ij}. Since for $(P, Q) \in \mathcal{V}_p$ we have

$$\| \gamma(P) - \gamma(Q) \|^2 = \sum_{j=1}^{d} \left(\frac{1}{p} \sum_{v_i\in P} \nu_{ij} - \frac{1}{q} \sum_{v_i\in Q} \nu_{ij} \right)^2,$$

we get

$$\zeta_p(T) = \frac{1}{p^2q^2}\left[\left(p^2\binom{d}{p} + q^2\binom{d}{q} \right) \sum_{i=0}^{d}\sum_{j=1}^{d} \nu_{ij}^2 + \right.$$
$$\left. + 2\tau \left(p^2\binom{d-1}{p} + q^2\binom{d-1}{q} - 2pq\binom{d-1}{p-1} \right) \right],$$

with

$$\tau = \sum_{j=1}^{d} \sigma_2(\nu_{0j}, \ldots, \nu_{dj}).$$

As usual, $\sigma_2(.)$ denotes the second elementary symmetric function. After some simplification (using $\| v_i \|= 1$) we obtain

$$\zeta_p(T) = \frac{1}{p^2q^2}\left[p(d+1)^2\binom{d}{p} - 2\tau(d+1)\binom{d-1}{p-1}\right].$$

Since $\| v_i \|= 1$, we get $\tau \geq -\frac{d+1}{2}$. Moreover,

$$\| \gamma(P) - \gamma(Q) \|^2 = \frac{(d+1)^2}{pqd} \quad \text{for} \quad (P,Q) \in \mathcal{V}_p(T_0).$$

Hence,

$$\zeta_p(T) \leq \frac{(d+1)^2}{p^2q^2}\left[p\binom{d}{p} + \binom{d-1}{p-1}\right] = \frac{(d+1)^2}{p^2q^2}\binom{d+1}{p} = \zeta_p(T_0),$$

which proves the assertion. □

3. Proof of the Theorem

(1) The left inequality results from (**). To prove the right inequality let $A \in \mathcal{S}^d$. For convenience, put $\rho = 1/(d+1)$ for d odd and $\rho = 1/d$ for d even.

First consider the case when A is compact. Let T denote a simplex of greatest volume among all simplices with vertices in A. Denote by v a vertex of T such that the height of T from v is $h(T)$. Let $H(z;\alpha)$ be the hyperplane carrying all vertices of T except v. The maximality of T implies

$$A \subset S\big(z; \alpha - h(T), \alpha + h(T)\big).$$

Thus $w(A) \leq 2h(T)$. So by Lemma 4 we get $w(T)/w(A) \geq w(T)/2h(T) \geq \rho$.

Now, we are ready to work with an arbitrary $A \in \mathcal{S}^d$. Let $\varepsilon > 0$. By Lemma 2 there is a bounded $A_\varepsilon \subset A$ such that

$$w(A_\varepsilon) \geq (1-\varepsilon)w(A).$$

Of course, for the closure $\mathrm{cl}(A_\varepsilon)$ of A_ε we have $w(\,\mathrm{cl}(A_\varepsilon)) = w(A_\varepsilon)$. Let T_ε be a d-simplex of maximal volume with all vertices in $\mathrm{cl}(A_\varepsilon)$. We can find in A_ε a $(d+1)$-point set P_ε such that $\mathrm{conv}(P_\varepsilon)$ contains a translate of $(1-\varepsilon)T_\varepsilon$. Hence

$$w(P_\varepsilon) \geq (1-\varepsilon)w(T_\varepsilon).$$

From the above inequalities and the compact case considered before we get

$$\frac{w(P_\varepsilon)}{w(A)} \geq \frac{(1-\varepsilon)w(T_\varepsilon)}{w(A)} \geq (1-\varepsilon)^2\frac{w(T_\varepsilon)}{w(\mathrm{cl}\, A_\varepsilon)} \geq (1-\varepsilon)^2\rho.$$

Since $A \in \mathcal{S}^d$ and $\varepsilon > 0$ are arbitrary, we obtain $\Lambda_{d+1}(\mathcal{S}^d) \geq \rho$.

So $(*)$ implies $\Lambda_n(\mathcal{S}^d) \geq \rho$. □

(2) Let $K \in \mathcal{M}^d$. We may assume that 0 is the center of K. We will show that $B = \left(\frac{1}{2}w(K)\right) \cdot \mathbf{B}^d$ is contained in K. Suppose, there is a point $b \in \operatorname{int}(B) \setminus K$. Since K is convex there exists a supporting hyperplane $H(z_0; \alpha)$ through b. By symmetry of K we have $K \subset S(z_0; -\alpha, \alpha)$. But $w\big(S(z_0; -\alpha, \alpha)\big) = 2\alpha < w(K)$. This contradiction shows that, indeed, $B \subset K$. Thus we have proved that $\Lambda_n(\mathcal{M}^d)$ is attained for balls.

Let T_0 be a regular simplex inscribed $\mathbf{B}^d$. An easy calculation shows that

$$w(T_0) = \parallel \gamma(P) - \gamma(Q) \parallel$$

for each $(P, Q) \in \mathcal{V}_{\lfloor \frac{d+1}{2} \rfloor}(T_0)$, which, in turns, equals the right hand side of the second assertion of (2). Furthermore, for an arbitrary simplex T inscribed $\mathbf{B}^d$ we get by the above equality and Lemma 5

$$\sum_{(P,Q)\in\mathcal{V}_{\lfloor \frac{d+1}{2} \rfloor}} w^2(T) \leq \zeta_{\lfloor \frac{d+1}{2} \rfloor}(T) \leq \zeta_{\lfloor \frac{d+1}{2} \rfloor}(T_0)) = \sum_{(P,Q)\in\mathcal{V}_{\lfloor \frac{d+1}{2} \rfloor}} w^2(T_0).$$

This implies $w(T) \leq w(T_0)$, which ends the proof of (2). □

(3) As it is shown in [7], every $K \in \mathcal{W}^2$ of width 1 contains an equilateral triangle T of side l, where l is the side of a largest equilateral triangle contained in the Reuleaux-pentagon of width 1. Since $0.85338 < l < 0.85345$, as evaluated by the authors of [7], we have $0.73904 < w(T) < 0.73911$, which immediately implies the first assertion of (3).

On the other hand, by [6], every plane convex body of minimal width 1 contains a convex body of constant width $1/(3 - \sqrt{3})$. This implies the rest of (3). □

(4) Let K be an arbitrary convex body. Let $k = \lfloor n/2 \rfloor$. Further, let Q be a polytope with k vertices $v_1, \ldots, v_k$ inscribed $\frac{1}{2}\mathbf{B}^d$ such that the Hausdorff-distance between $\mathbf{B}^d$ and Q is $\Delta(k)/2$. This means, in particular,

$$1 - \Delta(k) \leq w(Q).$$

Now, let $\tilde{A}_i = \operatorname{conv}\{-v_i, v_i\}$, for $i = 1, \ldots, k$, and set $\tilde{Q} = \operatorname{conv}(\tilde{A}_1 \cup \ldots \cup \tilde{A}_k)$. Obviously,

$$w(Q) \leq w(\tilde{Q}).$$

By Lemma 3, there exists a segment A_i of length greater than or equal to $w(K)$ in K which is parallel to $\tilde{A}_i$ for $i = 1, \ldots, k$. Let $P = \operatorname{conv}(A_1 \cup \ldots \cup A_k)$. Then P is a polytope with at most n vertices. In order to estimate the minimum width of P observe that all widthes of a convex body do not change under central symmetrization, so in particular for $P^* = \frac{1}{2}(P + (-P))$ we have $w(P) = w(P^*)$. Denote by c_i the center of A_i. Set $A'_i = -c_i + A_i$, and $P' =$ conv$(A'_1 \cup \ldots \cup A'_k)$. Of course $P' \subset P^*$. Thus

$$w(P') \leq w(P^*).$$

Observe that $w(K) \cdot \tilde{Q} \subset P'$, which implies

$$w(K)w(\widetilde{Q}) \leq w(P').$$

Combining the obtained inequalities we get

$$w(K)(1 - \Delta(k)) \leq w(K)w(Q) \leq w(K)w(\tilde{Q}) \leq w(P') \leq w(P^*) \leq w(P),$$

which proves the inequality $\Lambda_n(\mathcal{K}^d) \geq 1 - \Delta(\lfloor n/2 \rfloor)$.

The second assertion in (4) follows from the first one and from the fact that, by [5] and [2], every convex body can be approximated by an inscribed polytope with at most n vertices such that the Hausdorff-distance is less that

$$\kappa' n^{\frac{-2}{d-1}},$$

where κ' is a constant that depends only on the dimension and on the circumradius of K. □

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