BLOW-UP ESTIMATES FOR NONLINEAR HYPERBOLIC HEAT EQUATION

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BLOW-UP ESTIMATES FOR NONLINEAR HYPERBOLIC HEAT EQUATION*

HAMID BELLOUT† AND AVNER FRIEDMAN‡

Abstract. We consider the Cauchy problem for

\[ \epsilon u_{tt} + u_t - u_{xx} = F(u) ; \]

\( u \) represents the temperature when the standard Fourier law \( q = u_x \) (q flux) is relaxed and \( F(u) \) is a nonlinear source of energy. We establish that the solution exists for \( 0 < t < \phi_\epsilon(x) \), and it blows up as \( t \to \phi_\epsilon(x) \). Further, \( \phi_\epsilon(x) \to T_0 \) as \( \epsilon \to 0 \) where \( T_0 \) is the blow-up time for \( u_t - u_{xx} = F(u) \).

Key words. blow up of solutions, blow up time, hyperbolic equations

AMS(MOS) subject classifications. 35L05, 35L67, 35L70

§1. Introduction.

There has been recently increasing interest in the blow-up of solutions of nonlinear heat equations, such as

\[ u_t - u_{xx} = F(u) , \]  

(1.1)

and nonlinear wave equations, such as

\[ u_{tt} - u_{xx} = F(u) ; \]  

(1.2)

typically \( F(u) \sim Au^p \) \( (p > 1) \) or \( F(u) \sim e^u \) as \( u \to \infty \); see [10] [11] [13] [16] [17] and the references given there regarding (1.1), and [4] [5] [14] regarding (1.2).

Equation (1.1) models the heat equation when the flux \( q \) is given by the Fourier law \( q = -u_x \) and the conservation of energy equation is

\[ u_t + q_x = G \quad (G \quad \text{a source of energy}) . \]  

(1.3)

Fourier's law implies infinite velocity of heat propagation, and there have been a number of modified laws which rule out this feature. One common version is [1] [2] [3] [6] [7] [18] [19]

\[ q(x,t + \epsilon) = -u_x(x,t) \quad (\epsilon > 0) \]

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*This work is partially supported by National Science Foundation Grant DMS-8612880
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or its approximation

\[ q(x, t) + \epsilon q_t(x, t) = -u_x(x, t). \]  

(1.4)

The conservation of energy equation (1.3) then needs also to be modified (see [8]), but for \( \epsilon \) small it is approximately the same as before. From (1.3), (1.4) we deduce

\[ \epsilon u_{tt} + u_t - u_{xx} = F \]  

(1.5)

where \( F = G + \epsilon G_t \) \((G_t = \partial G(u(x, t)) / (\partial t))\); for \( \epsilon \) small, \( F \approx G \).

In this paper we study the Cauchy problem for (1.5) with \( F = F(u) \). If

\[ u(x, 0) = f(x), \]
\[ u_t(x, 0) = g(x) \]

(1.6)

then, since it is natural to assume that initially the temperature satisfies the heat equation, we are led to the assumption that

\[ g = f_{xx} + F(f). \]

(1.7)

In §2 we represent (1.5), (1.6) in two equivalent but useful forms. In §3 we establish the existence of unique solution \( u_\epsilon \) of (1.5), (1.6) for \( 0 < t < \phi_\epsilon(x) \), which blows up to \( +\infty \) as \( t \to \phi_\epsilon(x) \). the method of Caffarelli and Friedman [4] [5] shows that if \( f(u) \sim u^p \) as \( u \to \infty \) then \( \phi_\epsilon(x) \) is continuously differentiable. Additional estimates on \( u_\epsilon \) (independent of \( \epsilon \)) are established in §4.

Next, in §5 we prove that

\[ \liminf_{\epsilon \to 0} \phi_\epsilon(x) \geq T_0 \]

(1.8)

where \( T_0 \) is the blow-up time for (1.1) with \( u(x, 0) = f(x) \). Finally, in §6 we prove that

\[ \limsup_{\epsilon \to 0} \phi_\epsilon(x) \leq T_0 \]

(1.9)

Some generalizations and extensions are given in §7.

Results of the type (1.8), (1.9) have been established for other types of equations in [12] [14] [15].

ASSUMPTIONS. Throughout this paper we assume that \( F \in C^2 \),

\[ F(x) \geq 0, \quad F''(s) \geq 0 \quad \text{if} \quad s \geq 0, \]
\[ F'(s) > 0 \quad \text{if} \quad s > 0, \]

(1.10)

\[ \int_1^\infty \frac{ds}{F(s)} < \infty; \]
\begin{align*}
(1.11) & \quad f \geq 0, \ f + \varepsilon g \geq 0 \quad \text{(for all small } \varepsilon > 0\text{),} \\
(1.12) & \quad f, g \quad \text{belong to } C^3(\mathbb{R}), \\
(1.13) & \quad \begin{cases}
    f(x) + |g(x)| \leq \frac{C}{(1 + |x|)^\alpha}, \\
    \sum_{i=1}^{3} \| f^{(i)}(x) \| + \| g^{(i)}(x) \| \leq C \quad \text{for more constants } C > 0, \alpha > 0.
\end{cases}
\end{align*}

and (1.7) holds. In §6 we shall also need the assumption

\begin{equation}
(1.14) \quad g \geq 0
\end{equation}

The results of this paper extend also to the case where (1.7) is not valid and to space dimension \( \leq 3 \) (under some additional assumptions on \( F \)); see §7.

\section*{§2. Equivalent formulation for the Cauchy problem.}

We shall denote by \( u_\varepsilon \) the solution (if existing) of

\begin{equation}
(P^1_\varepsilon) \quad \begin{cases}
    \mathcal{L}_\varepsilon u \equiv \varepsilon u_{tt} + u_t - u_{xx} = F(u), \\
    u(x, 0) = f(x), \\
    u_t(x, 0) = g(x).
\end{cases}
\end{equation}

Setting

\begin{equation}
(2.1) \quad v(x, t) = u(x, t)e^{t/(2\varepsilon)}
\end{equation}

we get the equivalent system

\begin{equation}
(P^2_\varepsilon) \quad \begin{cases}
    \varepsilon v_{tt} - v_{xx} = F(ve^{-t/(2\varepsilon)})e^{t/(2\varepsilon)} + \frac{1}{4\varepsilon} v, \\
    v(x, 0) = f(x), \\
    v_t(x, 0) = \frac{1}{2\sqrt{\varepsilon}} f(x) + g(x).
\end{cases}
\end{equation}

Setting

\begin{equation}
(2.2) \quad w(x, \tau) = v(x, t) \quad \text{where } \tau = \frac{t}{\sqrt{\varepsilon}},
\end{equation}

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we get

\[
\begin{align*}
(P^3_\varepsilon) & \\
& \left\{ \\
& w_{rr} - w_{xx} = F(we^{-\tau/(2\sqrt{\varepsilon})}e^{\tau/(2\sqrt{\varepsilon})}) + \frac{1}{4\varepsilon} w, \\
& w(x, 0) = f(x), \\
& w_r(x, 0) = \frac{1}{2\sqrt{\varepsilon}} f(x) + \sqrt{\varepsilon} g(x).
\end{align*}
\]

The concept of a solution of \((P^3_\varepsilon)\) is always understood in the classical sense.

Using the representation formula

\[
z(x, t) = \frac{1}{2} [z(x + t, 0) + z(x - t, 0)] + \frac{1}{2} \int_{z-t}^{z+t} z_t(\xi, 0) \, d\xi
\]

(2.3)

\[
+ \frac{1}{2\pi} \int_0^t ds \int_{-1}^1 (z_{tt} - z_{xx})(x + (t - s)\eta, s) \, d\eta
\]

for \(z = w\) we then obtain for \(u\) the representation:

\[
u(x_0, t_0) = \frac{e^{-t_0/(2\varepsilon)}}{2\sqrt{\varepsilon}} \iiint_{K^\varepsilon_-(x_0, t_0)} e^{t/(2\varepsilon)} \mathcal{L}_\varepsilon(u) \, dx \, dt
\]

(2.4)

\[
+ \frac{e^{-t_0/(2\varepsilon)}}{8\varepsilon^{3/2}} \iiint_{K^\varepsilon_-(x_0, t_0)} e^{t/(2\varepsilon)} u \, dx \, dt
\]

\[
+ \frac{1}{2} e^{-t_0/(2\varepsilon)} \left[ f(x_0 + \frac{t_0}{\sqrt{\varepsilon}}) + f(x_0 - \frac{t_0}{\sqrt{\varepsilon}}) \right] \]

\[
+ \frac{1}{2} e^{-t_0/(2\varepsilon)} \int_{x_0 - \frac{t_0}{\sqrt{\varepsilon}}}^{x_0 + \frac{t_0}{\sqrt{\varepsilon}}} \left[ \frac{1}{2\sqrt{\varepsilon}} f(x) + \sqrt{\varepsilon} g(x) \right] \, dx,
\]

where \(x_0 \in \mathbb{R}, \, t_0 > 0\) and

\[
K^\varepsilon_-(x_0, t_0) = \{ (x, t) \in \mathbb{R} \times (0, \infty), \ | x - x_0 | < \frac{t_0 - t}{\sqrt{\varepsilon}} \}.
\]

**THEOREM 2.1.** Let (1.7) and (1.10) − (1.13) hold. Then there exist functions \(\phi_\varepsilon(x), u_\varepsilon(x, t)\) satisfying

\[
(2.5) \quad 0 < c \leq \phi_\varepsilon(x) \leq \infty \quad \text{for some} \quad c > 0 \quad (c \ \text{independent of} \ \varepsilon);
\]
\[ (2.6) \begin{cases} \text{if } \phi_\varepsilon(x) \neq \infty \text{ then } \phi_\varepsilon(x) < \infty \text{ for all } x \in \mathbb{R} \\ \text{and } \frac{\phi_\varepsilon(x) - \phi_\varepsilon(x')}{|x - x'|} \leq \sqrt{\varepsilon} \quad \forall x, x' \end{cases} \]

\( u_\varepsilon \) is a solution of \((P^1_\varepsilon)\) in the region

\[ \Omega_\varepsilon = \{(x, t) \in \mathbb{R} \times [0, \infty); \ t < \phi_\varepsilon(x)\} \]

and

\[ (2.7) \quad u_\varepsilon \geq 0 , \]

\[ (2.8) \quad u_\varepsilon(x, t) \to \infty \quad \text{if } t \to \phi_\varepsilon(x) ; \]

The pair \((\phi_\varepsilon, u_\varepsilon)\) is uniquely determined.

**§3. Proof of Theorem 2.1.**

Let

\[ F_n(u) = \begin{cases} F(\min(u, n)) & \text{if } u > 0 \quad (n = 1, 2, \ldots) \\ F(0) & \text{if } u \leq 0 \end{cases} \]

and denote by \( u_n \) the solution of \((P^1_\varepsilon)\) corresponding to \( F_n \). The corresponding \( w = w_n \) satisfy:

\[ w^n_{xx} - w^n_{xx} = F_n(w^n e^{-r/(2\sqrt{\varepsilon})}) e^{r/(2\sqrt{\varepsilon})} + \frac{1}{4\varepsilon} w^n , \]

\[ w^n(x, 0) = f(x) , \]

\[ w^n_{r}(x, 0) = \frac{1}{2\sqrt{\varepsilon}} f(x) + \sqrt{\varepsilon} g(x) . \]

Using (1.11) in the representation (2.3) for \( w^n \), we can deduce by a continuity argument that \( w^n(x, t) \geq 0 \) for \( x \in \mathbb{R} \) and all \( t > 0 \).

Next we apply the arguments in [4: pp. 87–88] to \( w^n \) in order to deduce that \( w \equiv \lim w^n \) exists if \( 0 \leq \tau < \tilde{\phi}_\varepsilon(x) \) and \( w \equiv \infty \) if \( \tau > \tilde{\phi}_\varepsilon(x) \), and, if \( \tilde{\phi}_\varepsilon \neq +\infty, \tilde{\phi}_\varepsilon \) is Lipschitz continuous with coefficient 1. The last fact is based on the inequalities

\[ w_{r} \geq \pm w_{x} - c_0 \quad (c_0 \text{ constant}) \]

whose proof is as in [4; p. 86]. Next, instead of (1.22) in [4; p. 88] we have

\[ \frac{w^n_{n,r}}{F(w^n + C_0)} \leq C \quad \text{for some } \ C_0 > 0, C > 0 \]

and this implies (using the last condition in (1.10) that

\[ w(x, \tau) \to \infty \quad \text{it } \tau \to \tilde{\phi}_\varepsilon(x) . \]

In view of (2.1), (2.2) we conclude that the corresponding \( u_n \) converge to \( u_\varepsilon \) which is finite if \( t < \phi_\varepsilon(x) \) and \( +\infty \) if \( t > \phi_\varepsilon(x) \), where \( \phi_\varepsilon(x) = \sqrt{\varepsilon} \tilde{\phi}_\varepsilon(x) \), and \( u_\varepsilon(x, t) \to \infty \) if \( t \to \phi_\varepsilon(x) \).

We have thus established (2.6)–(2.8). To prove the first inequality in (2.5) it suffices to establish:
LEMMA 3.1. There exist positive constants $M, T$ independent of $n, \epsilon$ such that

\begin{equation}
\sup_{\mathbb{R}^n \times (0,T)} u^n(x,t) \leq M.
\end{equation}

Proof. We compare $u^n$ with the solution $\gamma(t) = \gamma_n(t)$ of

\begin{equation}
\begin{cases}
\epsilon \gamma'' + \gamma' = F_n(\gamma), \\
\gamma(0) = a, \\
\gamma'(0) = F_n(a)
\end{cases}
\end{equation}

where $a$ is a sufficiently large positive constant. The functions $z^n = u^n - \gamma_n$ satisfy:

\[ L_\epsilon(z^n) = F_n(u^n) - F_n(\gamma_n) = c(x,t)z^n, \quad c \geq 0, \]
\[ z^n(x,0) < 0, \quad z^n_t(x,0) \leq 0. \]

Representing $z^n$ by the integral formula (2.3) we can establish by continuity in $t$ that $z^n(x,t) < 0$ for all $x, t$. Thus in order to complete the proof of Lemma 3.1 it remains to prove:

\begin{equation}
\gamma_n(T) \leq 2a \quad \text{for some } T \text{ independent of } n, \epsilon
\end{equation}

To prove (3.4) we rewrite the differential equation for $\gamma = \gamma_n$ in the form

\[ \epsilon(e^{t/\epsilon} \gamma')' = F_n(\gamma)e^{t/\epsilon} \geq 0. \]

Since $\gamma'(0) > 0$, we deduce that $\gamma'(t) > 0$ as long as $\gamma(t)$ is positive. Differentiating the equation in (3.3) once in $t$, we also have

\[ \epsilon(e^{t/\epsilon} \gamma''')' = F_n'(\gamma)\gamma'e^{t/\epsilon} \geq 0 \]

and, since $\gamma''(0) = 0$, we deduce that $\gamma''(t) > 0$ as long as $\gamma(t) > 0$. It follows that $\gamma''(t) > 0$ for all $t$. Therefore

\[ \gamma' = F_n(\gamma) - \epsilon \gamma'' \leq F(\gamma). \]

Defining $T_n$ by $\gamma(T_n) = 2a$ we conclude that

\[ \bar{T} = \int_a^{2a} \frac{ds}{F_n(s)} = \int_0^{T_n} \frac{\gamma'(t)}{F(\gamma(t))} \leq T_n. \]

Since $\bar{T} \geq T > 0$, the assertion (3.4) follows.

We have completed the proof of existence of a solution $(u_\epsilon, \phi_\epsilon)$. Uniqueness is proved as in [4] or [5]. (Notice that for uniqueness we need not use the fact that $c$ in (2.5) is independent of $\epsilon$.)

As in [4] we can establish that $u_\epsilon$ is in $C^{2,1}$ in $\Omega_\epsilon$.

The following fact will be used in §6.
THEOREM 3.2. If in addition to the assumptions (1.7), (1.10) –(1.13) we also assume that (1.14) holds then

\begin{equation}
\frac{\partial u_\epsilon}{\partial t} \geq 0 \quad \text{in} \quad \Omega_\epsilon.
\end{equation}

Proof. Set

\begin{equation}
z(x, \tau) = u^n_t(x, t) e^{t/(2\epsilon)}, \quad \tau = \frac{t}{\sqrt{\epsilon}}.
\end{equation}

Then

\begin{equation}
z_{\tau \tau} - z_{xx} = F'_n(u^n)z + \frac{1}{4\epsilon} z,
\end{equation}

\begin{align*}
z(x, 0) &= g(x), \\
z_\tau(x, 0) &= \frac{1}{2\sqrt{\epsilon}} g(x).
\end{align*}

Consider first the case where \( g(x) \geq \delta > 0 \). Then representing \( z \) by an integral formula (2.3) and proceeding by continuity on \( \tau \) we can establish that \( z(x, \tau) > 0 \) for all \( x, \tau \).

Applying this to (3.7) with \( g \geq 0 \) replaced by \( g + \delta \), and letting \( \delta \to 0 \), it follows that \( z \geq 0 \) where \( z \) is given by (3.6), and (3.5) is then proved by taking \( n \to \infty \).

§4. Additional estimates on \( u_\epsilon \).

LEMMA 4.1. Assume that for some positive constants \( M, T \) the solution of \( (P'_t) \) (established in Theorem 1.1) satisfies:

\begin{equation}
u_\epsilon(x, t) \leq M \quad \text{in} \quad \mathbb{R} \times [0, T], \quad \text{for all small} \ \epsilon.
\end{equation}

Then there exists a positive constant \( C_1 \) independent of \( \epsilon \) such that

\begin{equation}
u_\epsilon(x, t) + |u_\epsilon, t(x, t)| \leq \frac{C_1}{(1 + |x|)^\alpha} \quad \text{in} \quad \mathbb{R} \times [0, T],
\end{equation}

\begin{equation}
|u_\epsilon, x| + |u_\epsilon, t| + |u_\epsilon, xx| + |u_\epsilon, tx| + |u_\epsilon, xxx| + |u_\epsilon, txx| \leq C_1 \quad \text{in} \quad \mathbb{R} \times [0, T].
\end{equation}

Proof. Consider the function

\[ W_\alpha(x, t) = \frac{e^{At}}{(1 + x^2)^{\alpha/2}}, \quad A > 0. \]
For any $A$ (no matter how large), if $\epsilon$ is small enough then

\[ \mathcal{L}_\epsilon W_\alpha \geq \frac{A}{2} W_\alpha. \]

On the other hand

\[ \mathcal{L}_\epsilon u_\epsilon = \tilde{F}(u_\epsilon) u_\epsilon \quad \left( \tilde{F}(v) = \frac{F(v)}{v} \right) \]

and therefore the function $z = W - u_\epsilon$ satisfies

\[ \mathcal{L}_\epsilon z \geq \frac{A}{2} W_\alpha - \tilde{F}(u_\epsilon) u_\epsilon \geq \tilde{F}(u_\epsilon) z \quad \text{if} \quad t < T \]

provided we choose $A$ such that $A > 2\tilde{F}(M)$. Noting that

\[ z(x,0) > 0, \quad z_t(x,0) > 0 \]

If $A$ is large, we can represent $z$ by the integral representation (2.3) and then deduce by continuity on $t$, that $z \geq 0$ if $t < T$. Thus

\[ u_\epsilon \leq W_\alpha = \frac{e^{\alpha T}}{(1 + x^2)^{\alpha/2}}. \]

Similarly, from the equation

\[ \mathcal{L}_\epsilon(u_\epsilon, t) = F'(u_\epsilon) u_\epsilon, t \]

and the fact that $|F'(u_\epsilon)| \leq F'(M)$ we can proceed as before to estimate $u_\epsilon, t$ from above by the same function $W$ (with a different constant $A$). The function $-u_\epsilon, t$ are estimated similarly. Thus (4.2) is proved.

The function $u_\epsilon, x$ is estimated similarly using the comparison function $W_\alpha$ with $\alpha = 0$.

Next we differentiate $\mathcal{L}_\epsilon u_\epsilon = F$ once in $t$ and once in $x$ and obtain

\[ \mathcal{L}_\epsilon u_\epsilon, t x = F'(u_\epsilon) u_\epsilon, t x + F''(u_\epsilon) u_\epsilon, t u_\epsilon, x. \]

Noting that

\[ |F''(u_\epsilon) u_\epsilon, t u_\epsilon, x| \leq C, \]

we can proceed as before to compare $u_\epsilon, t x$ with $W_0 \equiv e^{At}$ provided $A$ is sufficiently large. We thus obtain the estimate

\[ |u_\epsilon, t x| \leq C_1, \quad C_1 \quad \text{constant}. \]

Similarly we establish the estimate

\[ |u_\epsilon, xx| \leq C_1. \]

Differentiating $\mathcal{L}(u_\epsilon) = F$ three times and using the estimates derived so far, we can again compare $u_\epsilon, xxx$ and $u_\epsilon, txx$ with $W_0$ and thus complete the proof of (4.3).
Remark 4.1. Lemma 4.1 implies that any sequence $\epsilon \to 0$ has a subsequence such that

$$u_\epsilon \to u, \quad u_{\epsilon, x} \to u_x, \quad u_{\epsilon, xx} \to u_{xx}$$

uniformly in compact subsets of $\mathbb{R} \times [0, T]$.

(4.4)

However, we cannot establish the boundedness of $u_{\epsilon, tt}$ by the method of Lemma 4.1 (since $u_{\epsilon, tt}(x, 0)$ is unbounded as $\epsilon \to 0$), and thus we cannot assert that

(4.5)  \quad u_{\epsilon, t} \to u_t \quad \text{uniformly in compact subsets of } \mathbb{R} \times [0, T].

Lemma 4.2. Under the assumption of Lemma 4.1

(4.6)  \quad u_\epsilon(x, t) \to u(x, t), \quad u_{\epsilon, t}(x, t) \to u_t(x, t)

as $\epsilon \to 0$, uniformly in compact subsets of $\mathbb{R} \times (0, T]$, where $u$ is a solution of (1.2).

Proof. Multiplying $\mathcal{L}_\epsilon(u_\epsilon) = F(u_\epsilon)$ by $e^{t/\epsilon}$ and integrating in $t$ we find that

$$u_{\epsilon, t}(x, t_0) = \frac{1}{\epsilon} e^{-t/\epsilon} g(x)$$

$$+ \frac{1}{\epsilon} e^{-t/\epsilon} \int_0^t [u_{\epsilon, xx}(x, s) + F(u_\epsilon(x, s))] e^{s/\epsilon} \, ds.$$

Set

$$H_\epsilon = u_{\epsilon, xx} + F(u_\epsilon)$$

and write

(4.7)  \quad u_{\epsilon, t} = \frac{1}{\epsilon} e^{-t/\epsilon} g(x) + \frac{1}{\epsilon} \int_0^t [H_\epsilon(x, s) - H(x, t)] e^{s/\epsilon} \, ds

$$+ H_\epsilon(x, t)[1 - e^{t/\epsilon}] \equiv J_1^\epsilon + J_2^\epsilon + J_3^\epsilon.$$

Then

$$|J_1^\epsilon| \leq \frac{C}{\epsilon} e^{-t/\epsilon} \to 0$$

uniformly in $t \in [\delta, T]$.

Next, by Lemma 4.1, $|H_\epsilon, t| \leq C_0$ where $C_0$ is independent of $\epsilon$. Hence

$$|J_2^\epsilon| \leq \frac{C_0}{\epsilon} \int_0^t (t - s) e^{(s-t)/\epsilon} \, ds

= C_0 \epsilon [-\frac{t}{\epsilon} e^{-t/\epsilon} - e^{-t/\epsilon} + 1] \leq C_0 \epsilon.$$
Finally, by Remark 4.1, any sequence $\epsilon \to 0$ has a subsequence such that (4.4) holds; therefore

$$J^\epsilon_3 \to u_{xx} + F(u).$$

Thus, by (4.7),

$$u_{\epsilon,t} \to u_{xx} + F(u)$$

uniformly in compact subsets of $\mathbb{R} \times (0, T]$; the right hand side must coincide with $u_t$ (since $u_\epsilon \to u$ uniformly) and thus $u$ is a solution of (1.2).

Since $u$ is also continuous up to $t = 0$ and $u(x, 0) = f(x)$, and since

$$|u| \leq M \quad \text{by (4.1)},$$

$u$ is uniquely determined [9; Chap 2]. It follows that (4.6) (and (4.4)) hold for the full range of the parameter $\epsilon$.

Consider now the parabolic equation

$$u_t - u_{xx} = F(u),$$

$$u(x, 0) = f(x),$$

and set

$$N(t) \equiv \sup_{0 < \epsilon < 1} \sup_{x \in \mathbb{R}} u(x, t).$$

Then there exists a $T_0$ such that

$$N(t) < \infty \quad \text{for all} \quad t < T_0.$$

We assume that $T_0 < \infty$; $T_0$ is then called the blow-up time for (4.8).

**Lemma 4.3.** If the assumptions of Lemma 4.1 hold with $T < T_0$, then

$$u_\epsilon|_{t < 0} \leq A_T \quad \text{in} \quad \mathbb{R} \times [0, T]$$

for all $\epsilon$ sufficiently small, where

$$A_T = \sup_{\mathbb{R} \times [0, T]} (u_+ | u_t |) + 1$$

(which is a positive constant independent of $M$).

**Proof.** By Lemma 4.2, if $\rho$ is sufficiently large then

$$u_\epsilon(x, t) + |u_\epsilon,t(x, t)| \leq A_T \quad \text{if} \quad |x| > \rho, \ 0 \leq t \leq T.$$

On the other hand, if $|x| \leq \rho, \ 0 \leq t \leq T$ then, by Lemma 4.2, (4.9) holds provided $\epsilon$ is sufficiently small.
§5. \( \lim \inf \phi_\varepsilon \geq T_0. \)

**Theorem 5.1.** Under the assumptions of Theorem 2.1, for any \( T_1 < T_0, \phi_\varepsilon(x) > T_1 \) for all \( x \in \mathbb{R} \) provided \( \varepsilon \) is small enough, and

\[
(5.1) \quad u_\varepsilon \to u \quad \text{uniformly in} \quad \mathbb{R} \times [0, T_1]
\]

as \( \varepsilon \to 0. \)

**Proof.** From the proof of Theorem 2.1 we have (see (3.2)) that the conditions of Lemma 4.1 hold for some small \( T. \) Lemma 4.3 thus implies that \( M \) in (4.1) can be replaced by the constant

\[
A = \sup_{\mathbb{R} \times [0, T_1]} (u^+ + |u_t|) + 1,
\]

provided \( \varepsilon \) is small enough.

Let \( v(t) \) be the solution of

\[
\begin{align*}
\varepsilon v_{tt} + v_t &= F(v), \\
v(0) &= A, \\
v_t(0) &= F(A).
\end{align*}
\]

By (3.3), (3.4),

\[
(5.3) \quad v(t) < 2A \quad \text{if} \quad 0 < t \leq \sigma,
\]

where \( \sigma \) is a positive constant independent of \( \varepsilon. \)

We wish to compare \( u_\varepsilon \) with \( v(t + T - \delta) \) (for any \( \delta > 0 \)) provided \( \varepsilon \) is sufficiently small (so that (4.9) is valid) in order to deduce that \( u_\varepsilon(x, t) \) exist in \( \mathbb{R} \times [0, \tilde{T}] \) for \( \tilde{T} = T - \delta, \) and

\[
u_\varepsilon(x, t) \leq M \quad \text{in} \quad \mathbb{R} \times [0, \tilde{T}].\]

To do this we work with the solutions \( u^n \) of the truncated problems and proceed precisely as in the proof of Lemma 3.1, with \( t = 0 \) replaced by \( t = T - \delta. \)

Since \( \delta \) is arbitrary we deduce that the conditions of Lemma 4.1 hold with \( T \) replaced by \( T + \sigma. \) We can proceed in this way step-by-step until we reach the value \( t = T_1. \)

**Corollary 5.2.** Under the assumptions of Theorem 2.1

\[
(5.4) \quad \lim \inf_{\varepsilon \to 0} \left[ \inf_x \phi_\varepsilon(x) \right] \geq T_0.
\]

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§6. \( \lim \sup \phi_{\epsilon} \leq T_0 \).

**Theorem 6.1.** Let the assumptions of Theorem 2.1 hold and assume also that (1.14) holds. Then, for any \( x \in \mathbb{R} \),

\[
(6.1) \quad \lim_{\epsilon \to 0} \sup \phi_{\epsilon}(x) \leq T_0 .
\]

**Proof.** Suppose the assertion is not true. Then there exist \( x_0 \in \mathbb{R} \) and \( \delta > 0 \) such that for a sequence \( \epsilon \to 0 \),

\[
\phi_{\epsilon}(x_0) > T_0 + 2\delta .
\]

By (2.6) we then get, for any \( \rho > 0 \),

\[
(6.2) \quad \phi_{\epsilon}(x) > T_0 + \delta \quad \forall \ x \in (-\rho, \rho) ,
\]

provided \( \epsilon \) is small enough.

From the definition of \( T_0 \) it follows that there is a sequence \( (x_n, \eta_n) \) with \( \eta_n \downarrow 0 \) such that

\[
u(x_n, T_0 - \eta_n) \to \infty \quad \text{if} \quad n \to \infty
\]

Choose any large positive constant \( M \) and let \( n_0 \) be such that

\[
u(x_{n_0}, T_0 - \eta_{n_0}) > M .
\]

By Theorem 3.2, \( u_{\epsilon}(x, t) \) is monotone increasing in \( t \) and therefore

\[
(6.3) \quad u_{\epsilon}(x_{n_0}, t) > M \quad \text{if} \quad T_0 - \eta_{n_0} \leq t \leq \phi_{\epsilon}(x_{n_0}) .
\]

We choose \( \rho \) in (6.2) such that \( \rho > |x_{n_0}| + 1 \). Then

\[
(6.4) \quad \phi_{\epsilon}(x) > T_0 + \delta \quad \text{if} \quad |x - x_{n_0}| \leq 1 .
\]

Introduce the function

\[
\psi(x) = \frac{\pi}{2} \sin \pi(x - x_{n_0}) .
\]

It satisfies

\[
\psi'' = -\pi^2 \psi, \quad \psi > 0 \quad \text{in} \quad x_n < x < x_{n_0} + 1 ,
\]

\[
\psi(x_{n_0}) = \psi(x_{n_0} + 1) = 0 ,
\]

\[
(6.5) \quad \int_{x_{n_0}}^{x_{n_0} + 1} \psi(x) \, dx = 1 .
\]
Multiplying $\mathcal{L}_\varepsilon(u_\varepsilon) = F(u_\varepsilon)$ by $\psi$ and integrating over $\{x_{n_0} < x < x_{n_0} + 1\}$, we find that the function

$$a(t) = \int_{x_{n_0}}^{x_{n_0}+1} u_\varepsilon(x, T_0 - \eta_{n_0} + t)\psi(x) \, dx$$

satisfies:

$$\varepsilon a'' + a' = -\pi^2 a + F(a) + [u_\varepsilon(x, T_0 - \eta_{n_0} + t)\psi_x(x)]_{x_{n_0}}^{x_{n_0}+1}.$$

Since

$$\psi_x(x_{n_0}) = -c < 0, \quad \psi_x(x_{n_0} + 1) > 0,$$

it follows that

$$(6.6) \quad \varepsilon a'' + a' \geq -\pi^2 a + F(a) + cM$$

where (6.3) was used; also

$$(6.7) \quad a(0) > 0, \quad a'(0) \geq 0, \quad a'(t) \geq 0.$$

**Lemma 6.2.** The solution $a(t)$ of (6.6), (6.7) blows up in time $t \leq \delta$ provided $M$ is sufficiently large and $\varepsilon$ is sufficiently small.

Assuming the lemma we conclude that

$$\phi_\varepsilon(x) < T_0 - \eta_{n_0} + \delta \quad \text{for some} \quad x \in (x_{n_0}, x_{n_0+1}),$$

which is a contradiction to (6.4).

**Proof of Lemma 6.2.** Let $b(t)$ denote the solution of

$$(6.8) \quad \varepsilon b'' + b' = c_0(F(b) + M),$$

$$(6.9) \quad 0 \leq b(0) < a(0), \quad b''(0) = a'(0)$$

where

$$(6.10) \quad c_0 = \min\left(\frac{1}{2}, \frac{c}{2}\right), \quad c \text{ as in (6.6)}.$$

Writing (6.8) in the form

$$\varepsilon(e^{t/\varepsilon}b')' = c_0 e^{t/\varepsilon}(F(b) + M) \geq 0$$

we see that $b' \geq 0$.  

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We claim that if \( M \) is large enough then, for any \( \epsilon > 0 \),
\[
(6.11) \quad b(t) \text{ blows up in time } \leq \delta.
\]
Indeed, suppose \( b(t) \) exists for all \( t \leq \delta \). We claim that there exists a \( t_0 \) such that
\[
(6.12) \quad t_0 \in (0, \frac{\delta}{2}), \quad b''(t_0) \geq 0.
\]
Indeed, otherwise we have
\[
b''(t) < 0 \quad \text{for all } t \in (0, \delta/2)
\]
and therefore, by (6.8),
\[
b' \geq c_0(F(b) + M).
\]
Hence
\[
\int_0^{\delta/2} \frac{b'}{c_0(F(b) + M)} \geq \frac{\delta}{2}.
\]
But the left-hand side is bounded above by
\[
\int_{a(0)}^{\infty} \frac{ds}{c_0(F(s) + M)}
\]
which is < \( \delta/2 \) if \( M \) is sufficiently large; a contradiction.

Having proved (6.12), we differentiate (6.8) in \( t \) and obtain, after multiplying by \( e^{t/\epsilon} \),
\[
\epsilon(e^{t/\epsilon}b'')' = c_0e^{t/\epsilon}F'(b)b' \geq 0
\]
Using (6.12) we deduce that
\[
b''(t) \geq 0 \quad \text{if } t > t_0
\]
Hence
\[
b'' + b' = c_0(F(b) + M) + (1 - \epsilon)b'' \geq c_0(F(b) + M) \quad \text{if } t \geq t_0
\]
Denoted by \( \gamma(t) \) the solution of
\[
(6.13) \quad \gamma'' + \gamma' = c_0(F(\gamma) + M),
\]
\[
\gamma(t_0) = b(t_0), \quad \gamma'(t_0) = b'(t_0)
\]
we deduce that
\[
\epsilon(e^{t/\epsilon}(b - \gamma)')' \geq 0,
\]
\[
(b - \gamma)(t_0) = (b - \gamma)'(t_0) = 0.
\]
It follows that \( b(t) \geq \gamma(t) \). On the other hand it is easily seen that if \( M \) is sufficiently large then \( \gamma(t) \) blows up in time \( t \leq t_0 + \gamma/2 \). Therefore

\[
(6.14) \quad b(t) \text{ blows up in time } t < \delta.
\]

In order to complete the proof of the lemma we compare \( a(t) \) with \( b(t) \). From (6.10) and (6.6), (6.8) we find that

\[
\epsilon(e^{t/\epsilon}(a - b)')' \geq c_0 F'(a - b)e^{t/\epsilon}.
\]

Since also \((a - b)(0) > 0, (a - b)'(0) \geq 0\), we easily deduce that \( a(t) \geq b(t) \) for all \( t \) for which \( b(t) \) exists. It follows that \( a(t) \) blows up in time < \( \delta \).

\section*{§7 Generalizations.}

7.1. The results of this paper extend to the case where (1.7) is replaced by

\[
g = f_{xx} + F(f) + \epsilon h
\]

provided \( h \) satisfies

\[
(7.1) \quad |h(x)| \leq \frac{C}{(1 + |x|)^\alpha} \quad \text{for some } \alpha > 0,
\]

\[
(7.2) \quad \sum_{i=1}^{3} |h^{(i)}(x)| \leq C,
\]

and

\[
(7.3) \quad g \geq 0.
\]

These conditions ensure that (1.13) holds and that

\[
\sum_{i=1}^{3} |D_x^i u^{e}_{tt}(x, 0)| \leq M < \infty,
\]

which is the only condition that \( u^{e}_{tt}(x, 0) \) needed to satisfy in the previous analysis.

7.2. The results of this paper extend to the case where \( x \) is \( N \)-dimensional with \( N = 2 \) or \( N = 3 \), provided \( F, f \) and \( g \) satisfy the following additional conditions:

\[
(7.4) \quad sF'(s) - F(s) \leq 0 \quad \forall s \geq 0,
\]
\[ (7.5) \quad \frac{t}{2\varepsilon} + 1) f(x) + \varepsilon g(x) \geq \frac{t}{\sqrt{\varepsilon}} \left| \nabla f(x) \right| \quad \forall \, t \geq 0, \, x \in \mathbb{R}^N, \]
\[ \frac{1}{2\sqrt{\varepsilon}} f(x) + \sqrt{\varepsilon} g(x) - \lambda \left| \nabla f(x) \right| + \frac{t}{\varepsilon} \left[ g(x) + \frac{1}{4\varepsilon} f(x) \right] \]
\[ (7.6) \quad - \lambda \left( \frac{1}{\sqrt{\varepsilon}} \left| \nabla f(x) \right| + \sqrt{\varepsilon} \left| \nabla g(x) \right| \right) \]
\[ \geq \frac{t}{\sqrt{\varepsilon}} \left| \nabla g(x) \right| + \lambda \frac{t}{\sqrt{\varepsilon}} \left| \nabla^2 f(x) \right| \quad \forall \, t \geq 0, \, x \in \mathbb{R}^N \]
where \( \lambda > 1 \) (\( \lambda \) constant)

(here \( g = \Delta f + F(f) \)). These conditions are satisfied if
\[ F(s) = e^{\theta s}, \quad 0 < \theta \leq 1, \]
\[ f(x) = A + f_1(x), \quad |D^\alpha f_1(x)| \leq \frac{C}{(1 + |x|)^\beta} \]
for \( 0 \leq |\alpha| \leq 5 \) and some \( \beta > 0, \) and \( A \) is
a sufficiently large positive constant.

Under these conditions the existence and uniqueness of a solution \( u_\varepsilon \) can be established using the formulation \( (P^3_\varepsilon) \) and the approximating sequence considered in [5]. Condition (7.5) ensures that \( u_\varepsilon \geq 0. \) The existence of \( \phi_\varepsilon(x) \) and (2.5), (2.6) result from the inequality
\[ w_\varepsilon, t \geq \lambda \left| \nabla w_\varepsilon \right| \]
which is proved using the extension of the representation formula (2.3) to \( N \) dimensions [5] and the conditions (7.4), (7.5). The results of §4,5 extend with minor changes to dimension \( N. \) Finally, in §6 we need a stronger assumption than (1.14), namely,
\[ (7.8) \quad g(x) \geq \delta_0 > 0; \]
using this we can prove as in [5] that \( \exists \delta_1 > 0 \) such that
\[ u_\varepsilon, t \geq \delta_1 \left| \nabla u_\varepsilon \right| \]
This guarantees that, for any \( t > T_0 + \frac{\sigma}{2} \) (\( \sigma \) arbitrarily small),
\[ u_\varepsilon(x, t) > M \quad \text{for } x \text{ in a fixed ball } B \text{ of radius } \delta_1 \sigma/4. \]

Introducing the function
\[ a(t) = \int_B u_\varepsilon(x, T_0 + \frac{\sigma}{4} + t) \psi(x) \, dx \]
where \( \psi \) is the principal eigenfunction of \( -\Delta \) in \( B, \) \( \psi > 0 \) in \( B, \int_B \psi = 1, \) we again derive (6.6), (6.7), and conclude that \( a(t) \) blows up in time \( \leq T_0 + \sigma. \)
REFERENCES


