COMPACT ATTRACTORS AND SINGULAR PERTURBATIONS

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ABSTRACT
Motivated by fundamental concepts of stability of compact invariant sets, a general class of semigroups which possess a compact attractor are defined. Recent results concerning the flow on the attractor in specific examples are discussed. Particular attention is given to singularly perturbed problems and transition layer solutions.

1. Existence of compact attractors.

Suppose that \( T(t) : X \to X, t \geq 0 \) is a \( C^0 \) semigroup on a Banach space \( X \). If \( B \) is a subset of \( X \), then the \( w \)-limit set of \( B \) is defined to be

\[ \omega(B) = \cap_{s \geq 0} \text{Cl} \cup_{t \geq s} T(t)B \]

It is important to note that \( \omega(B) \) is not equal to the union of \( \omega(x) \) for all \( x \) in \( B \). A subset \( A \) of \( X \) is said to be invariant if \( T(t)A \) is equal to \( A \) for all nonnegative \( t \). Note that this implies that complete orbits are defined for each \( x \) in \( X \); that is, one can define the operation of \( T(t) \) on \( x \) for all \( t \) in \( R \) and \( T(t)x \) belongs to \( A \) for \( t \) in \( R \). A subset \( A \) of \( X \) is said to a \textit{compact attractor} if \( A \) is compact, invariant and, for any bounded set \( B \) in \( X \), the \( w \)-limit set of \( B \) exists and belongs to \( A \).

If a compact attractor exists, then it is clear that the essential part of the flow defined by \( T(t) \) is determined by studying the flow restricted to the attractor. On the other hand, one cannot completely ignore the transient times, of which there are two distinct types. The first transient time is represented by the time that it takes to reach a specified neighborhood of the attractor. This part of the transient behavior occurs on a reasonable time scale—there is no reason to expect the flow to move at either a particularly fast or a particularly slow pace. When we restrict the flow to the attractor, these transient times are being ignored. When the flow on the attractor is being investigated, other types of transient times may be involved. There may be moderately fast time scales, or very slow time scales. This will be especially true if there are parameters in the problem which represent singular perturbations. More will be said about this later. The point that we want to emphasize at this moment is that there are two time scales.
Let us now turn to the following question: What properties must the semigroup $T(t)$ possess in order for there to exist a compact attractor? It is clear that we must be sure that positive orbits of bounded sets are bounded. We also must know that infinity is unstable in some sense. These two properties still are insufficient to ensure that there exists a compact attractor. We need some property on the semigroup which will imply that every bounded set has a compact w-limit set provided that its positive orbit is bounded. Let us turn to a more precise discussion of these concepts.

The semigroup $T(t)$ is said to be asymptotically smooth if, for any positively invariant closed bounded set $B$ in $X$, there is a compact set $J$ in $B$ such that $\text{dist}(T(t)B,J)$ approaches zero as $t$ approaches infinity. One can show that this property implies that $w(B)$ exists, is compact and belongs to $J$. This concept was introduced by Hale, LaSalle and Slemrod [1972] in an effort to modify the theory of dissipative processes for functional differential equations of retarded type that had been developed by Biliotti and LaSalle [1971] so that it would apply to equations of neutral type. The concept can be motivated from fundamental stability concepts (see Hale[1987]) and is a consequence of the following lemma.

Lemma 1. If $J$ is a compact invariant set which is stable, then $J$ is asymptotically stable if and only if there is a positively invariant neighborhood $B$ of $J$ such that $w(K)$ belongs to $J$ for any compact subset $K$ of $B$.

From this result, it is clear that asymptotically smooth maps relate asymptotic stability to uniform asymptotic stability. In fact, one can prove the following result.

Lemma 2. If $T(t)$ is asymptotically smooth and $J$ is a compact invariant set, then $J$ is asymptotically stable if and only if $J$ is uniformly asymptotically stable.

Let us give some examples of asymptotically smooth maps. If there exists a constant $s$ such that $T(t)$ is compact for $t$ greater than $s$, then $T(t)$ is asymptotically smooth. This situation occurs, for example, when $T(t)$ is the semigroup generated by ordinary differential equations, retarded functional differential equations with a finite delay, and systems of reaction diffusion equations.

Another example of an asymptotically smooth semigroup is the case when $T(t)$ is an $\alpha$-contraction, where $\alpha$ is the Kuratowski measure of noncompactness. We do not define this concept precisely, but only remark that $T(t)$ is an $\alpha$-contraction if $T(t) = S(t) + U(t)$, where $U(t)$ is compact for all $t$ and $S(t)$ is a linear semigroup with norm approaching zero as $t$
approaches infinity. This type of semigroup occurs in many retarded functional differential equations with infinite delays, neutral functional differential equations, wave equations and beam equations with linear or nonlinear damping (for references, see, for example, Hale[1987],[1985]).

In order to state the next result, we need a definition. We say that the semigroup $T(t)$ is point dissipative if there is a bounded set $B$ in $X$ such that, for any $x$ in $X$, there is an $s = s(x,B)$ such that $T(t)x$ belongs to $B$ for $t$ larger than $s$. The basic result on the existence of a compact attractor is contained in the following theorem.

Theorem 3. If

(i) $T(t)$ is asymptotically smooth,
(ii) $T(t)$ is point dissipative, and
(iii) positive orbits of bounded sets are bounded,

then there is a compact attractor. Furthermore, the attractor is connected and there is an equilibrium point of $T(t)$.

For the historical origins of Theorem 3, consult the above references.

2. Examples.

As we have remarked above, a fundamental problem is to discuss the flow that is obtained by restricting the semigroup to the attractor. Up to the present time, there have been only a few cases where researchers have either attempted or have been able to discuss the complete flow on the attractor. On the other hand, each such attempt to discuss the complete flow either has led to significantly new results or at least to a much deeper understanding of the system involved. In addition, the new ideas that occur and the new methods that are developed in such efforts have played an important role in the formulation and solution of problems in completely different situations.

Let us mention a few specific situations in which there have been some success. Consider the scalar parabolic equation in one space dimension

$$ u_t = u_{xx} + f(x,u,u_x) \quad \text{on } (0,1) $$

with separated boundary conditions; that is, the boundary conditions are specified at $x = 0$ and $x = 1$. If $u_0$ is an equilibrium point of this equation which is hyperbolic, we let $W^U(u_0)$ be the unstable manifold of $u_0$ and let $W^S(u_0)$ be the stable manifold of $u_0$. The following theorem was first proved by Henry [1985] and independently was proved in a different way by Angenent [1985].
Theorem 4. For each pair of hyperbolic equilibria \( u_0 \) and \( v_0 \), the manifold \( W^U(u_0) \) is transversal to the stable manifold \( W^S(v_0) \).

This rather surprising result was discovered because various people were trying to understand the global flow on the attractor for the so-called Chafee-Infante problem corresponding to the case with homogeneous Dirichlet boundary conditions and \( f(u) = a(u-u^3) \), where \( a \) is a real parameter. The above theorem shows that the qualitative changes in the dynamics of the flow can change only at those values of the parameter \( a \) at which an equilibrium becomes nonhyperbolic. As a consequence of the transversality, the flow on the attractor for the Chafee-Infante problem was completely analyzed by Henry [1985]. The general situation has been solved recently by Brunovsky and Fiedler [1987] and Fusco and Rocha [1987].

Other directions of research have been inspired by the above result. For example, Fusco [1985] was able to associate a special system of ordinary differential equations with a class of scalar parabolic equations and to show that the dynamics were equivalent. In this way, a general class of ordinary differential equations were obtained for which the stable and unstable manifolds were always transversal. These results were extended further by Fusco and Oliva [1986].

Angenet and Fiedler [1986] have adapted some of the ideas in the proofs of the above results to discuss the \( w \)-limit sets of solutions of a scalar reaction diffusion equation with periodic boundary conditions. In particular, they show that the \( w \)-limit set of any bounded orbit must contain a rotating wave. They also discuss some of the orbit connections. The result on the \( w \)-limit set was obtained independently by Matano [1986] and Massatt [1986].

As another illustration, let us consider the delay equation

\[ x'(t) = -bx(t) - g(x(t-1)) \]

where \( b \) is a constant and the function \( g \) has negative feedback; that is,

\[ xg(x) > 0 \text{ for } x \neq 0, \quad g'(0) > 0. \]

Assume also that \( x = 0 \) is hyperbolic and \( g(x) \geq -k \) for all \( x \). Mallet-Paret [1986] has proved the following fact.

Theorem 5. For a delay equation satisfying the above hypotheses, there is a compact attractor \( A \) and the flow on \( A \) has a Morse decomposition.
In the proof of the above theorem, it is shown that the number of zeros in a delay interval \((t-1,t]\) of a function corresponding to a solution of the equation on the attractor is a nonincreasing function of \(t\). This unexpected result has been a stimulus for many new directions of research and is playing an important role in understanding examples; for example, the spectral theory for linear equations with periodic coefficients (see Mallet-Paret and Sell [1986]).

We should mention also the important work of Babin and Vishik [1983], [1985] on the wave equation with linear damping. They discussed this equation as a dynamical system and obtained some results on the existence of an attractor and showed that it was the union of the unstable manifolds of the equilibria if they are hyperbolic. These papers stimulated further research which has led to a more complete understanding of that problem as well as the development of a general theory of dissipative gradient systems. Applications of these latter results have been made to other problems (see, for example, Haraux [1985], Hale [1985], [1987], Lopes and Ceron [1984]).

Over the last ten years, there has been some very important work of several Japanese mathematicians concerned with the structure of the flow on the attractor for a pair of reaction diffusion equations (see the article of Nishiura, Mimura and Fujii [1986] for references and the people involved). They have analyzed in detail the bifurcation diagram for equilibria with the parameters being the diffusion coefficients. The analysis was performed both theoretically and computationally. New ideas in singular perturbation theory were developed and have led to the study of other types of singular problems.


Due to the complexity of the problems involved, we often concentrate on invariant sets of the dynamical system which are stable. More specifically, we attempt to analyze in detail the stability and instability of equilibrium points and sometimes periodic orbits. We then turn to numerical procedures to gain insight into the behavior of the orbits. Numerical procedures will only detect stable invariant sets if the initial data is chosen in a random way. On the other hand, if there are more than one stable invariant set, then the basins of attractions of these stable invariant sets play a very important role in the understanding of the dynamics. The boundaries of these basins of attraction are determined by the unstable manifolds of some unstable invariant sets. Of course, these unstable manifolds are difficult to discuss theoretically and difficult to obtain computationally. However, knowing the behavior of these unstable manifolds can lead to a better understanding of the global dynamics as well as the manner in which the global dynamics changes as parameters are varied. This is important especially in the understanding of the manner in which complicated oscillatory phenomena such as chaos can
occur through successive bifurcations. For given maps or for periodically forced ordinary differential equations, many researchers have discussed the global dynamics with the help of numerically computed unstable manifolds. Hale and Sternberg [1987] have performed such computations to understand the onset of chaos in delay equations. More specifically, they used a static continuation method to locate the unstable periodic orbit of least period and then showed that a transverse homoclinic orbit was generated for this orbit. The same method should apply to gradient systems with the computations being easier because the only unstable invariant sets that need to be considered are equilibria.

4. Singly perturbed problems and quasistability.

The above discussion has emphasized the importance of investigating the behavior of the unstable manifolds of invariant sets. When there are some singularly perturbed terms in the evolutionary equation which defines the dynamical system, these unstable manifolds may be the key to the understanding of a phenomenon that is observed in applications and numerical computations; namely, the observation that the system moves rather quickly to some point in the phase space, remains in an apparent state of equilibrium for a very long time, moves at a moderate rate to another point, remaining again in an apparent state of equilibrium and continues this process until finally reaching a state of equilibrium. For the purpose of reference, we refer to this phenomenon as quasistability.

Let us now give a philosophical discussion of how quasistability might occur. These ideas evolved from conversations with J.Carr, G. Fusco and R. Pego. Suppose D is a smooth bounded domain in $\mathbb{R}^n$, $L$ is a linear differential operator of some order and $f$ is a linear or nonlinear differential operator of lower order. Consider the partial differential equation

$$u_t = eL u + f(u) \text{ in } D$$

together with some boundary conditions on the boundary of D. We suppose that $e$ is a very small positive real number and that the differential equation together with the boundary conditions defines a gradient semigroup on a Banach space $X$.

If we suppose that there is also a compact attractor $A$ and that each equilibrium point is hyperbolic, then $A$ is the union of the unstable manifolds of the equilibria. For problems of this type, there often are many equilibrium solutions which develop transition layers as $e$ approaches zero. Most of these are unstable and, in fact, the unstable equilibria are precisely the ones that determine the dimension of the attractor $A$. Furthermore, if $u_e$ is an equilibrium solution, then $\dim W^u(u_e)$
is often closely related to the number of transition layers that the solution develops as \( e \) approaches zero. This dimension is either equal to the number of transition layers or a constant plus this number. Furthermore, it often happens that the solutions with fewer transition layers develop transition layers more quickly. Thus, their unstable manifolds have low dimension, which in turn implies that their stable manifolds are large.

Now suppose that \( u(t) \) is an arbitrary solution of the equation. Since \( A \) is the attractor, \( u(t) \) must approach \( A \) as \( t \) approaches infinity. It is possible that \( u(t) \) approaches an unstable equilibrium point. However, the set of initial values for which this occurs lies on a manifold with a codimension which is nonzero; that is, it is very unlikely that one begins with such initial values. The most likely situation is that the solution approaches one of the stable equilibria. On the other hand, it is to be expected that many solutions will come very close to unstable manifolds of equilibria and then proceed to a stable point. The path that is followed could be the one that goes successively from the neighborhood of one unstable manifold to the neighborhood of another which has a smaller dimension, etc.

If the above scenario is correct, then it is clear that it is necessary to understand the behavior of the flow on the unstable manifolds of equilibrium points. Since the solutions with fewer transition layers have unstable manifolds with smaller dimension, it follows that it is likely that many solutions for \( e \) very small will eventually approach a stable equilibrium along the unstable manifold of an equilibrium point which has already developed very steep transition layers. In many cases, one can show that a solution with \( n \) transition layers has only \( n \) positive eigenvalues and these approach zero as \( e \) approaches zero. This shows that the flow on the unstable manifold is very slow near the unstable equilibrium. If this slow behavior persists when we are far from the equilibrium point, then the flow is very slow far away from equilibrium and we begin to get an explanation of quasistability.

Let us attempt to be more specific in the case where the equilibrium point \( u_e \) has \( \dim W^U(u_e) = 1 \) and the positive eigenvalue \( s(e) \) approaches zero as \( e \) approaches zero. Let \( u(t) \) be a solution with initial data on the unstable manifold and very close to \( u_e \). Let \( r(u(t)) \) be the instantaneous rate of expansion of \( u(t) \) along the unstable manifold (this could be obtained from the linear variational equation about \( u(t) \)). When \( t \) is very small, we must have \( r(u(t)) \) very nearly equal to \( s(e) \). Suppose that \( u(t) \) approaches a stable equilibrium point whose dominant eigenvalue is \( -s_0 \). Then \( r(u(t)) \) should be close to \( s_0 \) for very large \( t \). Thus, the solution could move very slowly for a long time and then move more quickly when it arrives near to the stable equilibrium.
A precise meaning for the concept of quasistability in this simple case perhaps is the following: Parametrize a component of $W^u(u_e)$ by a distance parameter $d$ with $d = 0$ corresponding to $u_e$ and order points on the unstable manifold by their distance from $u_e$. Now choose any two points $v$ and $w$ on the unstable manifold with $v$ smaller than $w$. We say quasistability occurs if, for any positive $T$, there is an $e_0$ such that, if $e$ belongs to $(0, e_0)$ and $u(0)$ belongs to the order interval $(u_e, v)$, then $u(t)$ belongs to the order interval $(u_0, w)$ for all $t$ in $[0, T]$. This implies that the solution can be made to move as slowly as desired by choosing $e$ small enough.

5. Transition layer solutions.
The above discussion clearly indicates that we need a better understanding of the dynamics of the flow on the unstable manifolds of those equilibria which have steep transition layers. We need a geometric theory of dynamical systems which will apply to situations in which singularly perturbed terms appear. In particular, we need to know that center manifold theory as well as the method of Liapunov-Schmidt can be applied.

Let us illustrate these remarks in a particular example for which we wish to determine the existence and stability of a solution with a transition layer. Consider the equation

$$u_t = e^2 u_{xx} + f(u, x) \quad \text{on } (-1, 1)$$

with the boundary conditions

$$u_x = 0 \quad \text{at } x = -1 \text{ and } +1.$$

The constant $e$ is assumed to be small and positive, and for simplicity, we assume that

$$f(u, x) = u(1 - u)(u - a(x))$$

$$a(x) \ln (0, 1) \text{ for each } x \text{ in } (-1, 1), \quad a(x) = 1/2 \text{ for } x = 0,$$

$$a(0) = 1/2, \quad a'(0) = 0$$

There are methods available for the existence and stability of solutions with transition layers. One approach to existence is to consider two distinct Dirichlet problems on the intervals $(-2, 0)$ and $(0, 2)$ with the initial data at the end points being a parameter which is used to match the
derivatives of the two functions at $x = 0$. This procedure has been used by Mimura, Tabata and Hosono [1980] by an appropriate modification of the method of Fife [1974],[1976] for the Dirichlet problem. The stability of the solution can be determined by the SLEP method of Nishiura and Fujii [1985].

For the cubic function $f$ above, Angenent, Mallet-Paret and Peletier [1987] have discussed stable solutions of the equation. They obtained existence of the solution by using results of Matano [1979] and the comparison principle. The stability of the solution was determined by estimation of the principal eigenvalue.

Hale and Sakamoto [1987] have shown recently that it is possible to obtain the existence of the equilibrium solution by applying the method of Liapunov-Schmidt. This procedure carries with it some of the stability properties of the solution. Some of the basic difficulties encountered in the procedures mentioned above are encountered here also, but our resolution of these problems are somewhat different. In addition, the application of the method of Liapunov-Schmidt imposes some geometric restrictions upon the choice of the initial approximation which seem to be of fundamental importance. Let us now describe the procedure in more detail.

Our first objective is to discuss the existence of a transition layer solution $u(x,e)$ with the property that, for any positive number $b$, it approaches the value $0$ uniformly in $[-1,-b]$ and the value $1$ uniformly in $[b,1]$ as $e$ approaches zero. The techniques below also will yield a solution with $0$ and $1$ interchanged.

It is very easy to see why one should suspect that there is such a transition layer solution. In fact, if we rescale variables by $x = ey$, then

$$u_{yy} + f(u,ey) = 0$$

$$u_y = 0 \text{ at } y = 1/e \text{ and } -1/e.$$  

For $e = 0$, this equation becomes

$$u_{yy} + f(u,0) = 0$$

with the boundary conditions $u_y = 0$ at plus and minus infinity. From the hypotheses on the function $a$, there is a heteroclinic orbit $z(y)$ connecting the saddle points $0$ and $1$. The function $z(x/e)$ is certainly a good candidate for an approximate transition layer solution to the original equation. However, it does not satisfy the boundary conditions. We, therefore, adjust the approximate solution to
\[ u_0(x,e) = c_0(x)z(x/e) + c_1(x) \]

where \( c_0 \) and \( c_1 \) are appropriate cutoff functions.

Let us now investigate the possibility of obtaining an exact solution of the original equation which is close to \( u_0(x,e) \). If we let \( u = u_0(x,e) + v \), then \( v \) must satisfy the equation

\[
L^e v + G(e) + F(v,e) = 0
\]

\[
L^e v(x) = e^2 v'' + f(u_0(x,e),x)v
\]

\[
G(e)(x) = e^2 u_0''(x,e) + f(u_0(x,e),x)
\]

\[
F(v,e)(x) = f(u_0(x,e) + v,x) - f(u_0(x,e),x) - f_1(u_0(x,e),x)v
\]

One can show that \( G(e) = O(e) \) as \( e \) approaches zero and that \( F(v,e) = O(|v|^2) \) as \( v \) approaches zero. Therefore, it is natural to consider the equation for \( v \) as a perturbation of the equation \( L^e v = 0 \) for \((v,e)\) in a neighborhood of \((0,0)\).

The first task is to understand the spectrum of the operator \( L^e \) as a linear map from the weighted space \( C^2_{e}[-1,1] \) to the space \( C[-1,1] \). The norm in \( C^2_{e}[-1,1] \) is given by

\[
\sup\{|v(x)| + e|v'(x)| + e^2|v''(x)|\}
\]

By using a Prufer transformation and analyzing the behavior of the corresponding angle, Hale and Sakamoto show that there is exactly one eigenvalue \( s(e) \) which approaches zero as \( e \) approaches zero. Furthermore, \( s'(0) \) is proportional to \( a'(0) \), a fact which allows one to determine the stability of the transition layer solution once it is known to exist. Furthermore, there are positive constants \( r \) and \( e_0 \) such that the remaining eigenvalues are less than \(-r\) for \( e \) in \((0,e_0)\). The use of the Prufer transformation in the study of stability of solutions of parabolic equations has been used previously by Fusco and Hale [1985], Hale and Rocha [1985], Jones [1984], and Rocha [1985], [1986].

After obtaining this information about the linear part of the equation, we are in a position to apply the method of Liapunov-Schmidt. If \( g(e) \) is a normalized eigenfunction corresponding to the eigenvalue \( s(e) \), then we let \( v = cg(e) + w \) with \( w \) orthogonal to \( g \) and apply the method of Liapunov-Schmidt in the usual way to determine \( w = w(c,e) \) and a bifurcation function \( B(c,e) \) with the property that there is transition
layer solution of the original equation near $u_0(x,e)$ if and only if there is a solution $c = c(e)$ of the equation $B(c,e) = 0$ with $c(0) = 0$.

When the above analysis is completed, one discovers that there is no solution of the bifurcation equation of the desired type. This means that there is no transition layer solution close to $u_0(x,e)$ within order $e$ which can be obtained by perturbing in the direction of the eigenfunction $g(e)$. In appropriately scaled variables, the function $g(e)$ approaches the function $z'(y)$. For steep transition layers, this is therefore the natural direction to make the perturbations. What is to be done? The basic reason for the difficulty is that the bifurcation function $B(c,e)$ has a Taylor series which has the approximate form

$$G(e) + ekc$$

where $k$ is a nonzero constant. The function $G(e)$ is $O(e)$ and not $O(e^2)$ and so there there can be no zero $c$ of $B(c,e)$ which is close to zero when $e$ is close to zero. To obtain such a solution, one must improve the rate at which $G(e)$ approaches zero as $e$ approaches zero. This means that we must improve the original approximation for the transition layer solution. Therefore, we attempt to obtain a function $u_1(x,e)$ so that the function

$$U(x, e) = u_0(x, e) + eu_1(x, e)$$

has the following property: When we let $u = U(x, e) + v$ in the original equation, the corresponding function $G(e)$ is $O(e^2)$ as $e$ approaches zero. This is equivalent to improving the outer approximation to the solution and is always possible (see, for example, Fife [1974]). This improvement of the initial approximation does not change the spectral analysis of the linear operator $L^e$. Consequently, one obtains a new bifurcation function $B(c,e)$ and it has a zero $c(e)$ with $c(0) = 0$. This gives a complete solution to the problem that was posed (see Hale and Sakamoto [1987] for details).
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Additional Instructions:
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