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# ON THE RENORMALIZED COUPLING CONSTANT AND THE SUSCEPTIBILITY IN $\phi^4_4$ FIELD THEORY AND THE ISING MODEL IN FOUR DIMENSIONS

Michael Aizenman[1,2]
Departments of Mathematics and Physics
Rutgers University
New Brunswick, New Jersey 08903

Ross Graham[1]
Defence Research Establishment Atlantic
P. O. Box 1012, Dartmouth, N.S., B2Y 3Z7

Abstract: We discuss the Euclidean $\phi^4_4$ field theory, and the critical behavior in ferromagnetic systems in four dimensions. It is rigorously shown that there are at most logarithmic corrections to the mean field law in the behavior of the magnetic susceptibility $\chi = \Sigma S_2(0,x)$. Furthermore, if any such corrections are present in a continuum limit which is used to construct a $\phi^4_4$ field theory the limiting theory would be non-interacting. Our analysis extends to ferromagnetic systems of variables which belong to the Simon-Griffiths class.

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## 1. INTRODUCTION

Recent advances [1, 2] in the rigorous study of the Euclidean $\phi^4_d$ field theory, and the critical phenomena in Ising systems, brought the tools of rigorous analysis up to the threshold of the critical dimension $d = 4$. In this work we present further refined results which are relevant for the study of the logarithmic deviations from the mean-field behavior in four dimensions.

We address here problems which are relevant both for quantum field theory and for statistical mechanics. The two subjects are intricately related to each other, although their perspectives differ significantly. In both fields one is interested in effects which are observed on a scale which is enormously large compared with the one at which the elementary constituents of the systems directly interact with each other. The subtle propagation of the effects across this large gap of scales is a typical feature of systems of infinitely many degrees of freedom, which are at - or very near, a critical point. Characteristicly, such systems have an "upper critical dimension", above which the critical behavior simplifies considerably. It is a consequence of this transition, which for the systems discussed here occurs at $d = 4$ dimensions, that the difficulties in statistical mechanics and the quantum field theory (in the Euclidean regime) are inversly related to each other. In the low dimensions, the critical behavior is non trivial, and does not follow the laws which are simply predicted by a mean-field approximation. On the other hand, field theory can be meaningfully studied by means of a (renormalized) perturbation theory. Above the upper critical dimension the critical exponents are given exactly by the mean field calculation, and the scaling, or continuum, limits are described by Gaussian fields. Thus for $d > d_c$ the main problems of statistical mechanics have simple solutions, but at the same time the task of formulating an interacting local scalar field theory falls beyong the present reach of the available techniques.

This picture, which by now is very familiar to physicists, has recently been further confirmed and supplemented by rigorous results. These include proofs that above the dimension $d = 4$ the mean field approximation yields exact results for the critical behavior in ferromagnetic systems, e.g. the critical exponents $\alpha$[3] and $\gamma$[1]* for Ising systems and $\phi^4$ lattice fields. Similarily, all the continuum (or scaling) limits of such systems are Gaussian, that describe a field theory of non-interacting particles. Conversely, for the low dimension $d = 2$ it has been shown [1] that the critical exponents of general ferromagnetic systems exhibit "hyperscaling", which is a closely related condition to the non-vanishing of the "physical interaction" in the corresponding field theory. For $d < 3$, where the $\phi^4_d$ field theory is super-renormalizable, interacting $\phi^4_d$ fields (which yield Wightman field theories in the corresponding Minkowski space-times) have been constructed; see [4], references therein, and [5].

The results of this paper are directed towards the analysis of the above issues at the critical dimension $d = 4$. In Section 2 we briefly recall the representation of $\phi^4_d$ field theory as a scaling limit of a system of ferromagneticly coupled lattice variables. The physical coupling in the theory can be measured by the renormalized coupling constant $g = |\overline{u}_4|/(\chi^2\xi^d)$, which is dimensionless. Our main results deal with the limiting value of $g$, and the

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*Further such statements (e.g. $\eta = 0$) are presumably true - but have not yet been proven.

critical behavior of the "magnetic susceptibility", $\chi = \sum_x S_2(0,x)$, in general systems of ferromagneticly coupled variables which belong to the Simon-Griffiths class. This class consists of the variables which can be described by means of weighted sums of ferromagnetic Ising spins (see Section 7), and includes both the Ising spins and the $\phi^4$ lattice fields.

The results presented here do not yet fully resolve the main issues, however they show that in $d = 4$ dimensions:

i) In the critical regime, $\chi$ exhibits at most logarithmic deviations from the mean-field law.

ii) In any continuum limit in which $\chi$ does exhibit singular corrections, which are expected to be present in $d < 4$ and $d = 4$ dimensions (as long as $\lambda_0 \xrightarrow{\text{F.T.}} 0$), the renormalized coupling vanishes - and the limit is a (quasi - ) free field.

The paper is organized as follows. The basic set-up is outlined in section 2. The main results are stated and discussed in section 3; and in section 4 it is shown that i) and ii) are consequences of a new inequality - whose derivation is given in sections 6 and 7. Our derivation utilizes the random current formalism of ref. [1], which is briefly recalled in section 5. In section 5 we also mention a geometric interpretation of the effects discussed here.

It might be pointed out that the technique which is introduced in section 6 offers a further simplification of the method of ref. [1]. Consequently, although the inequalities presented here refine previous results, their derivation (for the $\phi^4$ variables) may actually be even simpler.

## 2. THE $\phi^4$ FIELD THEORY AND ISING SYSTEMS

### a. The lattice approximation

The Euclidean $\phi^4_d$ Quantum Field Theory derives from the action:

$$\mathcal{A}(\phi) = \int [\frac{A}{2}|\nabla\phi|^2 + B\phi^2 + \frac{\lambda}{4!}\phi^4]\, d^dx \qquad (2.1)$$

and is described by its Schwinger functions which, formally, are:

$$S_n^{(\text{continuum})}(x_1,\dots, x_n) = \int \prod_{x\in R^d} d\phi(x)\, e^{-\mathcal{A}(\phi)}\, \phi(x_1)\dots\phi(x_n)/\text{Normalization} \qquad (2.2)$$

Thus (2.2) represents an average over "all field configurations", $\phi(x)$, with a weight which is determined by the action. It is well known that such inte grals are rather problematic. An easy case is when $\lambda = 0$, since then the integral in (2.2) reduces to a Gaussian measure, which is well defined for any $A, B > 0$. Gaussian fields are characterized by the Wick indentities,

$$S_{2n}(x_1,\dots,x_{2n}) = \sum_{\substack{\text{pairings of}\\(1,2,\dots,2n)}} S_2(x_{i_1},x_{j_1})\cdots S_2(x_{i_n},x_{j_n}) \qquad (2.3)$$

which express all the Schwinger functions as simple combinations of just the two point function. The physical content of equation (2.3) is that the scattering is trivial and the particles described by the corresponding field theory do not interact.

The texbook prescription for the construction of an interacting field theory usually starts with the perturbation expansion of (2.2) in $\lambda$. The perturbation theory is renormalized in a process which involves the introduction of cut-offs and counter-terms in the actions, which are adjusted simultaneously with the cut-off removal. The well known renormalizability of $\phi_4^4$ means that at the perturbative level such a program can be carried out, with counter-terms which are only of the type which appears in the $\phi^4$ action in equation(2.2). Therefore, this approach can also be presented inthe following way.

To make sense of equation (2.2), one may introduce a lattice approximation to the continuum, approximating it by the cubic lattice $aZ^d$, with the lattice spacing (i.e. short-distance cut-off) a - which is eventually taken to zero. Thus we represent the field $\phi(x)$ by a system of lattice variables $\{\phi_x\}$. To maintain unity with the statistic-mechanical applications of our analysis, we shall typically use the subscript $x$ to denote the position as measured in units of the lattice spacing. Hence, in $\phi_x$ $x$ is always in $Z^d$. The relation with the continuum is given in equation (2.7), below.

In the lattice approximation, the action is given by the (formal) "Riemann-sum" version of equation (2.1), which is obtained by the replacement of $\int dx$ with $\sum_{x \in Z^d} a^d$, and the substitution of the difference operator (times $a^{-1}$) for the gradient in the kinetic term. The resulting distribution of the lattice fields $\{\phi_x\}$ can be expressed as the probability measure

$$\Big(\prod_{x \in Z^d} d\phi_x\ \rho(\phi_x)\Big) \exp\Big[\frac{\beta}{2} \sum_{x,y} J_{x,y}\ \phi_x \phi_y\Big] / \text{Normalization} \tag{2.4}$$

with $\rho(\phi) = \exp[-(\hat{B}\phi^2 + \frac{\hat{\lambda}}{4!}\ \phi^4)]$ and

$$J_{x,y} = \delta_{|x-y|,1} \tag{2.5}$$

(or, rather, the infinite-volume limit of (2.4)). The quantities $(\beta, \hat{B}, \hat{\lambda})$ are simply related to the coupling constants $(A, B, \lambda)$, in the continuum notation of equation (2.1). In particular:

$$\beta = Aa^{d-2}, \quad \frac{\hat{\lambda}}{\beta^2} = \lambda_0 a^{4-d}, \quad \text{with} \quad \lambda_0 = \frac{\lambda}{A^2}. \tag{2.6}$$

In the picture described previously, $(A,B,\lambda)$ correspond to the action with the counter-terms which are appropriate for the given value of the ultraviolet cut-off $a^{-1}$.

The continuum Schwinger functions of (2.2) can now be recovered as the scaling limits of the expectation values:

$$S_n^{(continuum)}(x_1,\ldots,x_n) = \lim_{a\to 0} \langle\phi[x_1/a]\cdots\phi[x_n/a]\rangle \tag{2.7}$$

where $[y]$ is the nearest site in $Z^d$ to $y$.

The main question is whether there is a way of adjusting the parameters $(\hat{\beta},\hat{B},\lambda)$ (or, equivalently, $(A,B,\lambda)$ ) as $a \to 0$, so that the limit (2.7) exists and describes a field theory of interacting particles, i.e. one which is not Gaussian.

It should be pointed out here that it is often the custom to keep the parameter $A$ constant, $A = 1$ or $\beta = a^{d-2}$, by means of a "field strength renormalization". In that terminology, the right hand side of (2.7) has an additional coefficient, which is the $n^{th}$ power of an adjustable parameter. We do not have such terms, but instead we treat $\beta$ as freely adjustable. The two terminologies are obviously equivalent.

The formulation of the problem given here offers a natural framework for a non perturbative analysis of the $\phi^4_d$ field theory. It permits to refer to all the possible limits in the parameter space, for which the lattice approximation stays within the class of ferromagnetic probability measures. Of course, even if none of these limits produces a satisfactory field theory, one should not feel that the bounds of human ingenuity have been exhausted. Consideration of assymptotic freedom in $\phi^4_d$, for $d \geq 4$, have led to the suggestion that one should try to make sense of a theory with $\lambda < 0$ [6] - in which case the probabilistic interpretation suffers. A seemingly less radical proposal has been made in ref. [7], where it is pointed out that the renormalized perturbation theory may yield, under certain manipulations, negative values for $\beta$. Our results do not apply to this case, in which the measure (2.4) is no longer ferromagnetic. Such a lattice approximation may however be quite unstable, exhibiting a sensitive dependence on the lattice structure and the way the kinetic term is interpreted (e.g. the range of the interaction $J_{x,y}$ in (2.4)).

The results presented in this work are relevant to the $\phi^4_4$ field theory within the framework outlined above. As we shall see next, the properties of this construction are closely related with the critical behavior in statistic-mechanical models, which are of independent interest.

A striking aspect of (2.7) is that it presents any two sites of the continuum as infinitely separated, on the scale of the lattice. This is a manifestation of the fact that in a local field theory there is no action at a distance, and all the effects propagate only via local interactions. However, it is well known in statistical mechanics that typically correlations in such systems either die out rather fast, or rapidly approach a constant value. Neither case would lead to an interesting continuum limit. The exception to this rule is when the values of the parameters $(\beta,\hat{B},\hat{\lambda})$ approach a critical manifold in the parameter space. In such a case, the continuum field theory acquires a structure and corresponds to the <u>scaling limit</u> of a critical, or nearly critical, lattice system. The physical interaction in the theory is manifested by deviations from the free field law (2.3). A convenient, and widely used measure of the interaction is the renormalized coupling constant $g$, defined by (2.13), below.

## b. Ferromagnetic lattice systems

We shall now focus our attention on lattice systems of variables $\phi_x$, $x \in Z^d$, which interact ferromagneticly, with the Hamiltonian.

$$H = -\frac{1}{2} \sum_{x,y \in Z^d} J_{x,y} \phi_x \phi_y, \qquad J_{x,y} \geq 0 \tag{2.8}$$

The corresponding Gibbs states, at the inverse temperature $\beta$, have precisely the form of (2.4), with $\rho(\phi)d\phi$ describing the non-interacting single site ("a-priori") measure.

The Ising model is of course an example of such a system. In this special case, described by $\rho(\phi) = \delta(\phi^2-1)$, we denote $\phi$ by $\sigma$.

In fact, the $\phi^4$ variables and Ising spins are interwined. Not only is the Ising model the strong coupling limit of a $\phi^4$ system, via

$$\delta(\phi^2 - 1) = \lim_{\lambda \to \infty} e^{-\lambda(\phi^2-1)^2} / \int d\psi \, e^{-\lambda(\psi^2-1)^2},$$

but there is also a converse relation (based on the Simon-Griffiths [8] representation). The $\phi^4$ single-site measure $\rho(\phi)\, d\phi$ (along with others, like $\rho(\phi) = \sum_{k=-n}^{n} \delta(\phi-k)$ or $\rho(\phi) = \chi[|\phi|<1]$ ) belongs to the class of measures which are generated by weighted sums of ferromagnetically coupled Ising spins (see section 7). Lattice fields based on single site measures in the above class may be regarded as describing blocks of Ising spins, in finer Ising models. Consequently, such systems are amenable to analysis which originates in Ising spin systems.

In order to deal simultaneously with both situations we denote

$$S_n(x_1,\ldots,x_n) = \begin{cases} & \langle \phi_{x_1}\cdots\phi_{x_n}\rangle \\ \text{or} & \\ & \langle \sigma_{x_1}\cdots\sigma_{x_n}\rangle \end{cases}$$

for $x_1,\ldots,x_n \in Z^d$.

It is well known that for any finite range ferromagnetic interaction, $J \neq 0$, and $\rho$ of either the $\phi^4$ or Ising type, there is $\beta_c < \infty$(for $d \geq 2$) such that for $\beta < \beta_c$

$$S_2(x,y) \leq \text{const } (\beta)\ e^{-|x-y|/\xi} \qquad (2.9)$$

with $\xi > 0$, and [9]

$$\xi(\beta) \underset{\beta \nearrow \beta_c}{\to} \infty \qquad (2.10)$$

where $\xi(\beta)$ is the correlation length, defined by taking the supremum over $\xi$'s in (2.9). Furthermore, the magnetic susceptibility, $\chi$, diverges at that critical point [10]:

$$\chi = \sum_x S_2(0,x) \to \infty \quad \text{as} \quad \beta \nearrow \beta_c . \qquad (2.11)$$

Thus, there is a critical manifold in the parameter space. The scaling (or continuum) limits discussed above are of interest precisely when the parameters - $(\beta,\hat{\lambda},\hat{B})$ for $\phi^4$ systems, and $\beta$ for the Ising model - approach critical values.

The main results of this paper concern the critical behavior of the quantities $\chi$ and

$$\overline{u_4} = \sum_{x_2,x_3,x_4} u_4(0,\ x_2,\ x_3,\ x_4)\ ,$$

where

$$u_4(x_1,x_2,x_3,x_4) = S_4(x_1,\ldots,x_4)-[S_2(x_1,x_2)S_2(x_3,x_4)+ + S_2(x_1,x_3)S_2(x_2,x_4)+S_2(x_1,x_4)S_2(x_2,x_3)]$$

For <u>any</u> Gaussian system

$$u_4 \equiv 0 \tag{2.12}$$

(see (2.3)) whereas for Ising-generated systems, as the $\phi^4$ field:

$$u_4 \leqslant 0 \quad \text{(Lebowitz inequality [11]).} \tag{2.13}$$

A measure of the physical interaction in the field theory which is described by the scaling limit (2.7) is provided by the following "renormalized coupling constant", which is a normalized value of $u_4$ at "zero momentum".

$$g = |\overline{u}_4|/(\chi^2\xi^d) \tag{2.14}$$

The above expression is dimensionless, and is, therefore, equal to a similar ratio of the continuum versions of $u_4$, $\chi$ and $\xi$.

It is known that if $g$ vanishes <u>at</u> the continuum limit then the theory is Gaussian - i.e. a generalized free field (Newman [12], and [1] - eq. (3.12)). (A hint of this effect is contained in the fact that in a $\phi^4$ field theory $u_4$ has a definite sign). In the next section we shall describe our results on the critical behavior of $\chi$ and $g$.

## 3. MAIN RESULTS

### i) A bound on the susceptibility

Let us first recall that the expected behavior of the susceptibility in the Ising model ,in the limit $t = (\beta_c - \beta)/\beta_c \searrow 0$, is:

$$\chi \approx \begin{cases} t^{-\gamma} & \gamma > 1 \quad d < 4 \\ t^{-\gamma}|\ln t|^{\#} & \gamma = 1 \quad d = 4 \\ t^{-\gamma} & \gamma = 1 \quad d > 4 \text{ - proven [1]} \end{cases} \tag{3.1}$$

(with the value $\# = 1/3$ predicted on the basis of renormalization group arguments [13], and the general bound $1 \leqslant \gamma \leqslant 2$ proven in [14]).

The fact that $\gamma \geqslant 1$ was proven by Glimm and Jaffe [14] - utilizing the following simple consequence of the Lebowitz inequality (2.12):

$$\left|\frac{\partial \chi^{-1}}{\partial \beta}\right| \leqslant \sum_x J_{0,x} \quad , \qquad \text{for} \quad \beta < \beta_c \tag{3.2}$$

The rigorous proof that in $d > 4$ dimensions $\gamma$ takes its mean-field value, i.e. $\gamma = 1$, was obtained by supplementing (3.2) with a uniform positive lower bound [1]. In the following proposition we improve this lower bound - and extend the analysis to $d = 4$ dimensions.

Our results extend to the Simon-Griffiths class of variables which can be generated by Ising spins. Its definition is given in section 7. We use here the symbol of partial-differentiation to emphasize that the derivatives $\partial/\partial\beta$ are taken at zero magnetic field - i.e. the coefficient in the usual symmetry-breaking breaking term $h\sum\sigma_x$, which is omitted in the Hamiltonian (2.6), is $h = 0$.

Proposition 3.1: For Ising, $\phi^4$ and other variables in the Simon-Griffiths class, with the nearest - neighbor interaction (2.4) on $Z^d$:

$$\frac{1}{2d} t^{-1} \leq \beta_c \chi(t) \leq \begin{cases} c\, t^{-1} (|\ln t|+1) & d = 4 \\ [\frac{1 + (2d)^2}{2d} c_d]\, t^{-1} & d > 4 \end{cases} \tag{3.3}$$

where $t = (\beta_c - \beta)/\beta_c$, $c$ is a finite factor, and

$$c_d = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dp\, [4 \sum_{i=1}^{d} \sin^2 (p_i/2)]^{-2} < \infty, \quad \text{for} \quad d > 4.$$

For more general translation invariant ferromagnetic interactions $J_{x,y} \geq 0$, we have, in the single phase regime $(\beta < \beta_c)$:

$$\left|\frac{\partial \chi^{-1}}{\partial \beta}\right| \geq \frac{|J|}{1+(\beta|J|)^2 \sum_x S_2(0,x)^2} \quad [1-\frac{(\beta|J|)^2 \sum S_2 (0,x)^2}{\beta|J|\chi}] \tag{3.4}$$

where $|J| = \sum_x J_{0,x}$. Furthermore, for Ising models the factors $(\beta|J|)^2$ in (3.4) and $(2d)^2$ in (3.3) can also be replaced by $\beta|J|$ and $2d/\beta$, correspondingly.

Remarks: 1) It is known that

$$\beta_c\, |J|\, \langle\phi^2\rangle_o \geq 1 \tag{3.5}$$

(i.e. a mean - field approximation produces a lower bound on $\beta_c$), where $\langle - \rangle_o$ is the $\beta = 0$ state. Therefore the above substitution for Ising models leads to a slight improvement of the bounds.

2) It should be noted that while the $\phi^4$ field theory offers the freedom of rescaling $\phi$ by a constant, the quantities $\beta\chi$, $\frac{\partial \chi^{-1}}{\partial \beta}$ and $\beta^2 \sum S_2(0,x)^2$ are invariant under such a "field strength" renormalization.

3) The "dressed bubble sum" which appears in the denominator in (3.4) is a pivotal term for various insights on the criticalily of $d = 4$ dimensions. It is finite in $d > 4$, and is expected to diverge for $d \leq 4$ (although only logarithmically at $d = 4$). It is convenient to express it by the Plancherel identity:

$$\Sigma S_2(0,x)^2 = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dp \; |G(p)|^2 \quad , \tag{3.6}$$

in terms of the Fourier-transform

$$G(p) = \sum e^{ipx} S_2(0,x).$$

For the nearest neighbor model one has the "Gaussian bound" of Fröhlich, Simon and Spencer [15]:

$$G(p) \leq \frac{1}{2\beta \sum_{i=1}^{d} 2 \sin^2(p_i/2)} + c(\beta)\delta(p) \tag{3.7}$$

with $c(\beta) = 0$ for $\beta < \beta_c$. This implies that for $\beta < \beta_c$:

$$\beta^2 \sum S_2(0,x)^2 \leq \begin{cases} c_d & d > 4 \\ \bar{c}[1+\ln_+(\beta\chi)] & d = 4 \end{cases} \tag{3.8}$$

with some $\bar{c} < \infty$, where for the case $d = 4$ we combined (3.6) with the inequality

$$G(p) \leq \frac{2}{G(p)^{-1}+\chi^{-1}} .$$

Thus, while proposition 3.1 simplifies the proof of ref. [1] that $\left|\frac{\partial \chi}{\partial \beta}\right|^{-1}$ is bounded away from zero for $d > 4$, it is also consistent with the expected behavior (for Ising models)

$$\left|\frac{\partial\chi^{-1}}{\partial\beta}\right|_{\beta\nearrow\beta_c} \to 0, \quad \text{in } d = 4 \text{ dimensions.} \tag{3.9}$$

ii) A bound on the renormalized coupling constant

The next result deals with the renormalized coupling constant $g$ - defined by (2.13). It has already been shown [1] that, in the nearest - neighbor case:

$$g \leqslant \frac{2(2d)^2}{\xi^{d-4}}\left[1+0\left(\frac{1}{\xi^2}\right)\right] \tag{3.10}$$

(To be more precise - the prefactor 2 is our improvement over the factor 3 in the bound proven in ref. [1], and rederived in ref. [2]. Otherwise, the results of ref. [1] are even better than (3.10), which can be strengthened by the factor $(1-\exp[-c\hat{\lambda}/\beta^2])$ for $\phi^4$ fields, and by dropping the term $(2d)^2$ for Ising models.) The bound (3.10) demonstrates that in $d > 4$ dimensions

$$\lim_{\substack{\xi\to\infty \\ (\beta\nearrow\beta_c)}} g = 0 \tag{3.11}$$

Since $g$ is known to be universally bounded (Glimm and Jaffe [16]), (3.10) is clearly a very inefficient bound for $d < 4$ (although the above-mentioned factor drastically improves the situation for a $\phi^4$ field at a fixed $\lambda_0^{\text{F.T.}} = \lambda/A^2$), yet it is <u>marginal</u> in $d = 4$ dimensions. The following result is an improvement of (3.10) whose main relevance to $g$ is in that critical dimension.

<u>Proposition 3.2:</u> In the single-phase regime of a ferromagnetic system of variables in the Simon-Griffiths class ($\phi^4$ lattice fields and Ising spins included), with the nearest neighbor interaction (2.4), the renormalized coupling constant satisfies:

$$g \leq \frac{2[2d]}{\xi^{d-4}} \left( \left|\frac{\partial \chi^{-1}}{\partial \beta}\right| + 0\left(\frac{1}{\xi^2}\right) \right) \tag{3.12}$$

More generally, for any translation invariant interaction:

$$|\overline{u_4}| \leq 2\beta[\beta|J|]\ \chi^4 \left( \left|\frac{\partial \chi^{-1}}{\partial \beta}\right| + \frac{1}{\beta\chi} \right) . \tag{3.13}$$

For Ising models, the factors $[2d]$ in (3.12) and $[\beta|J|]$ in (3.13) can be ommitted, and the other factors $\beta$ in (3.13) can be replaced by $\tanh \beta$.

<u>Remarks:</u> 1) The only additional information which is needed to deduce (3.12) from (3.13) is the bound

$$\beta\chi \leq \xi^2\left(1+0\left(\frac{1}{\xi^2}\right)\right) \tag{3.14}$$

which is proven for the former case using consequences of reflection positivity (Sokal [17]).

The new bound (3.13) differs from (3.10) by the factor $\frac{1}{2d}\left|\frac{\partial \chi^{-1}}{\partial \beta}\right|$ - which, by (3.2), can only lead to an improvement. In fact, the expected behavior (3.9) suggests that the improvement can be very significant. Thus, (3.13) contains the following striking manifestation of criticality. If indeed $d = 4$ is similar to $d < 4$ - in the sense of existence of singular corrections to the mean-field behavior of $\chi(t)$ (expressed by (3.9)) then, when judged by $g$ (which is a measure of the physical interaction in the corresponding field theory) - $d = 4$ resembles $d > 4$, i.e. (3.11) holds there as well.

In section 5 we offer a heuristic picture of this situation.

## 4. REDUCTION OF THE MAIN RESULTS TO A NEW INEQUALITY

The main results of this paper, described in the preceeding section, are consequences of the following proposition, for which we join the variables $\beta$ and $J$, in (2.3), into

$$K_{x,y} = \beta J_{x,y}$$

Proposition 4.1: In a finite ferromagnetic system of variables which belong to the Simon-Griffiths class:

$$\begin{aligned} |u_4(x_1,\ldots,x_4)| \leq & \sum_{u,v,w} S_2(x_4,w)\, K_{w,v}\, S_2(x_2,v)\, K_{v,u} \frac{\partial}{\partial K_{v,u}} S_2(x_1,x_3) \\ & + \sum_{w} S_2(x_4,w)\, K_{w,x_1}\, S_2(x_3,x_1)\, S_2(x_2,x_1) \qquad (4.1) \\ & + \sum_{w} S_2(x_4,w)\, K_{w,x_3}\, S_2(x_2,x_3)\, S_2(x_1,x_3) \end{aligned}$$

For Ising systems, the following slightly improved inequality is also valid

$$\begin{aligned} |u_4(x_1,\ldots,x_4)| \leq & \sum_{u,v} S_2(x_4,v)\, S_2(x_2,v) \tanh(K_{v,u}) \frac{\partial}{\partial K_{v,u}} S_2(x_1,x_3) \\ & + S_2(x_4,x_1)\, S_2(x_3,x_1)\, S_2(x_2,x_1) \qquad (4.1') \\ & + S_2(x_4,x_3)\, S_2(x_2,x_3)\, S_2(x_1,x_3) \end{aligned}$$

The explanation and derivation of the inequality (4.1) are postponed to sections 5 and 6, where we use the geometric techniques developed in ref. [1]. Here we shall show how it implies the main results discussed in section 3. Let us, however, first remark that the last two terms in (4.1) play no role in our applications - being negligible in the cases of most interest. Yet these terms do belong there, since in the saturating limit $\beta \to \infty$ we clearly have (for Ising models): $\langle\sigma\sigma\rangle \to 1$ and $|u_4| \to 2$, while $\frac{\partial}{\partial K_{v,u}} \langle\sigma_{x_1}\sigma_{x_3}\rangle \to 0$.

Proof of Proposition 3.1: For simplicity let us denote

$$\langle xy\ldots\rangle = S_n(x,y\ldots)$$

Formally (differentiating (2.3)):

$$\frac{\partial\chi}{\partial K_{y,z}} = \sum_x \langle 0xyz\rangle - \langle 0x\rangle\langle yz\rangle$$

and

$$\frac{\partial\chi}{\partial\beta} = \frac{1}{2} \sum_{y,z} J_{y,z} \frac{\partial\chi}{\partial K_{y,z}} \quad .$$

In the infinite volume limit these relations require some justification, and so does the continuity of (4.1). This however is an easy task for $\beta < \beta_c$ - since all the correlation functions obey exponential upper bounds which are uniform in the infinite-volume limit. We shall omit here this part of the argument - which is standard (see, e.g. [17]).

Thus:

$$\frac{\partial\chi}{\partial\beta} = \frac{1}{2} \sum_{x,y,z} J_{y,z}[\langle 0xyz\rangle - \langle 0x\rangle\langle yz\rangle] =$$

$$= \frac{1}{2} \sum_{x,y,z} J_{y,z}[\langle 0y\rangle\langle xz\rangle + \langle 0z\rangle\langle xy\rangle - |u_4(0,x,y,z)|]$$

(where $-|u_4| = u_4$, by the Lebowitz inequality (2.13))

$$\geq \chi^2|J| - \frac{1}{2} \sum_{\substack{x,y,z \\ u,v,w}} J_{y,z} \langle zw\rangle K_{w,v} \langle yv\rangle \beta J_{v,u} \frac{\partial}{\partial K_{v,u}} \langle 0x\rangle$$

$$- \frac{1}{2} \sum_{\substack{x,y,z \\ w}} J_{y,z}[\langle zw\rangle K_{w,0} \langle y0\rangle\langle x0\rangle + \langle zw\rangle K_{w,x} \langle yx\rangle\langle 0x\rangle]$$

(by (4.1), with $(x_1,x_2,x_3,x_4) = (0,y,x,z)$)

$$= \chi^2|J| - \sum_{\substack{y,z \\ w}} K_{y,z} \langle zw\rangle K_{w,0} \langle 0y\rangle \left(\frac{\partial\chi}{\partial\beta} + \frac{\chi}{\beta}\right) \tag{4.2}$$

(using translation invariance).

By the Schwarz inequality:

$$\sum_{\substack{y,z \\ w}} K_{y,z} \langle zw\rangle K_{w,0} \langle 0y\rangle \leq$$

$$\leq \left(\sum_{\substack{y,z \\ w}} K_{y,z} K_{w,0} \langle zw\rangle^2\right)^{1/2} \left(\sum_{\substack{y,z \\ w}} K_{y,z} K_{w,0} \langle 0y\rangle^2\right)^{1/2} \tag{4.3}$$

$$= (\beta|J|)^2 \sum\langle 0x\rangle^2$$

Substituting (4.3) in (4.2), and isolating the term $\frac{\partial\chi}{\partial\beta}$ we obtain the claimed relation (3.4):

$$\left|\frac{\partial\chi^{-1}}{\partial\beta}\right| \geq \frac{|J|}{1+[\beta|J|]\ \beta|J| \sum_x S_2(0,x)^2} \left[1 - \frac{[\beta|J|]\ \beta|J| \sum S_2(0,x)^2}{\beta|J|\chi}\right]$$

For Ising systems, where one may use (4.1′), the factors $\beta|J|$ in the square brackets may be replaced by 1.

To proceed towards (3.3) further information is needed about the behavior of $\beta^2 \sum_x S_2(0,x)^2$ as $\beta \nearrow \beta_c$. For the nearest - neighbor model, which is endowed with reflection - positivity we have the "Gaussian bound" (3.7) and its consequence -- the upper bound (3.8). Substituting it in the relation (3.4), which was derived above, we obtain the following differential inequalities, for $0 < \beta < \beta_c$,

$$2d \geq \left|\frac{\partial \chi^{-1}}{\partial \beta}\right| \geq \begin{cases} \dfrac{2d}{1+(2d)^2 C_d} \, [1-0(\chi^{-1})] & d > 4 \\[2ex] \dfrac{\hat{c}}{1+\ln_+(\beta\chi)} & d = 4 \end{cases} \tag{4.4}$$

(the upper bound is by (2.10)). $\hat{c}$ is a finite constant, and $\ln_+ x = \max(\ln x, 0)$.

The initial value for (4.4) is: $\lim_{\beta \nearrow \beta_c} \chi^{-1}(\beta) = 0$ [9,10], supplemented with the condition: $\chi(\beta) > 0$ for $0 < \beta < \beta_c$. The claimed bounds, (3.3), are now obtained by an integration of (4.4) - with the aid, for $d = 4$, of the lemma which follows. ∎

<u>Lemma 4.1</u>: Let $f(t) \geq 0$ be a differentiable function over $[0,t_0]$, such that

i) $f(0) = 0$

ii) $\dfrac{df}{dt} \geq \dfrac{1}{1+\ln_+(f^{-1})}$

Then, for some universal constant $c > 0$:

$$f(t) \geq \min\left\{ \frac{ct}{\ln_+ t^{-1}} \, , \, e^{-2} \right\} \tag{4.5}$$

Proof: Let $h(t) = f(2+\ln f^{-1})$. Then $h(t)$ is differentiable and

i) $h(0) = 0$

ii) $\frac{d\,h(t)}{dt} = (1+\ln f^{-1}) \frac{df}{dt} > 1$, as long as $f < e$.

Thus

$$t < f(2+\ln_+ f^{-1}),$$

for $0 < t < t_0$. In the smaller regime where $f < e^{-2}$

$$t < 4f \ln f^{-1},$$

and therefore

$$\ln t < 4c \ln f \quad (< 0)$$

with an explicitely calculable constant $c > 0$. Combining the last inequalities we obtain:

$$f > \frac{ct}{4c \ln f^{-1}} > \frac{ct}{\ln t^{-1}},$$

which proves the claim. ∎

Proof of Proposition 3.2: The bound (3.13), on $|\overline{u^4}|$, is a direct consequence of the new inequality (4.1). One needs just to sum over $(x_2, x_3, x_4)$, and apply the translation invariance. (See also the opening remark in the proof of proposition 3.1.)

For the nearest - neighbor model we also have the relation (3.14), whose substitution in the bound on $g = |\overline{u^4}|/(\chi^2 \xi^d)$ which results from (3.13), leads directly to (3.12). ∎

## 5. A RANDOM - CURRENT REPRESENTATION

Our derivation of the above mentioned results, and their heuristic explanation, are obtained by means of the random-current representation which was developed in ref. [1]. We start by introducing it for finite Ising systems with the general two-body ferromagnetic interaction:

$$H = -\frac{1}{2} \Sigma J_{x,y} \sigma_x \sigma_y, \quad J_{x,y} \geq 0 \tag{5.1}$$

We refer to pairs of sites with $J_{x,y} \geq 0$ as bonds, and denote $b = \{x,y\}$.

One way of thinking about a current configuration is as a collection of closed loops - each formed by lattice bonds - and a number of open-ended lines. We denote by $n(b)$ the total number of times a bond is covered, which we regard as the flux through $b$. The contribution of open lines (which do not connect) is manifested by the presence of points at which $\sum_{b \ni x} n(b)$ (≈"div $\underline{n}$") is odd. Thus, we regard

$$\partial \underline{n} = \{x \mid (-1)^{\sum_{b \ni x} n(b)} = -1\} \tag{5.2}$$

as the set of sources of a flux configuration $\underline{n}$ (which is an integer valued function of bonds).

The ensembles of flux configurations which we consider are actually simpler than the above picture suggests, since the multiplicity of the decomposition into loops, etc., will not play a role. The probability of a flux configuration will always be proportional to the Poisson weight:

$$w(\underline{n}) = \prod_b \frac{(\beta J_b)^{n(b)}}{n(b)!} \, . \tag{5.3}$$

By imposing proper source constraints one obtains an ensemble of random currents (given by flux numbers) which in a very exact sense can be viewed as the mediators of correlations in the corresponding Ising model.

The weights (5.3) first appear in the standard representation (generated by the expansion $\exp(\beta J_b \sigma_x \sigma_y) = \sum_{n=0}^{\infty} (\sigma_x \sigma_y)^{n(b)} (\beta J_b)^{n(b)} / n(b)!$):

$$Z \equiv \prod_x \left(\frac{1}{2} \sum_{\sigma_x = \pm 1}\right) e^{-\beta H(\sigma)} = \sum_{\partial \underline{n} = \phi} w(\underline{n}) \tag{5.4}$$

and

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle = \sum_{\partial \underline{n} = \{x_1, \ldots, x_n\}} w(\underline{n})/Z \tag{5.5}$$

where $\underline{n} = \{n(b)\}$, $n(b) = 0,1,2,\ldots$, and $\phi$ is the <u>empty set</u>.

The analysis in this representation is greatly facilitated by an identity of Griffiths, Hurst and Sherman [18] (which is also widely used in ref. [1]). It states that for every function $f$ of the combined fluxes:

$$\sum_{\substack{\partial \underline{n}_1 = A \\ \partial \underline{n}_2 = \{x,y\}}} w(\underline{n}_1) w(\underline{n}_2) f(\underline{n}_1 + \underline{n}_2) = \sum_{\substack{\partial \underline{n}_1 = A \Delta \{x,y\} \\ \partial \underline{n}_2 = \phi}} w(\underline{n}_1) w(\underline{n}_2)\ \chi[C_{\underline{n}_1 + \underline{n}_2}(x) \ni y] f(\underline{n}_1 + \underline{n}_2) \tag{5.6}$$

where $C_{\underline{m}}(x)$ is the cluster of sites which are connected to $x$ by a path along bonds with $m_b \neq 0$, $\chi[E]$ is the characteristic function which is 1 if the condition $E$ is satisfied, and $\Delta$ is the set difference operation: $A \Delta B = (A \cup B) \setminus (A \cap B)$.

The identity (5.6) leads to various relations which may be simply expressed by considering pairs of independent systems of random currents, with weights proportional to $w(\underline{n})$ and source constraints $\partial \underline{n}_i = A_i$ (ref. [1,19]). We denote by Prob. $(E | A_1, A_2)$ the normalized probability that in such an ensemble a condition E is satisfied. For $A = \{y,z\}$ (in which case $A\Delta\{x,y\} = \{x,z\}$), (5.6) may be rewritten as follows:

$$\langle\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_z\rangle = \langle\sigma_x\sigma_z\rangle \text{ Prob. } (C_{\underline{n}_1+\underline{n}_2}(x)\ni y \mid \{x,z\},\phi), \tag{5.7}$$

and one also has:

$$\langle\sigma_x\sigma_y\rangle^2 = \text{Prob. } (C_{\underline{n}_1+\underline{n}_2}(x)\ni y \mid \phi,\phi). \tag{5.8}$$

Equation (5.8) shows that the phenomenon of long-range-order coincides with percolation, in the duplicated system of random currents, and (5.7) provides us with a very useful expression for the analog of a "hitting probability".

Relations (5.7) and (5.8) can be unified by a convention that for pairs of sources: $\{x,x\} = \phi$. We adopt this convention for the rest of this paper.

A striking manifestation of the usefulness of the random-current representation is obtained when one deals with the quantities which are the main concern of this work - $u_4$ and $\partial\chi/\partial\beta$. These are originally given by differences of products of spin correlation functions. It turns out that the delicate cancellations may be performed exactly, leading to the following expressions of definite sign [1]:

$$-u_4(x_1,\ldots,x_4) = 2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle \text{ Prob. } (C_{\underline{n}_1+\underline{n}_2}(x_1)\ni x_3 \mid \{x_1,x_2\},\{x_3,x_4\})$$

$$\equiv 2 \sum_{\substack{\partial\underline{n}_1=\{x_1,x_2\} \\ \partial\underline{n}_2=\{x_3,x_4\}}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_1)\ni x_3]/Z^2 \tag{5.9}$$

and

$$\frac{1}{\beta}\frac{\partial\langle\sigma_x\sigma_y\rangle}{\partial J_{u,v}} = \left[\sum_{\substack{\partial\underline{n}_1=\{x,u\}\\ \partial\underline{n}_2=\{v,y\}}} + \sum_{\substack{\partial\underline{n}_1=\{x,v\}\\ \partial\underline{n}_2=\{u,y\}}}\right] w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(x)\ni y]/Z^2 \qquad (5.10)$$

$$\equiv \langle\sigma_x\sigma_u\rangle\langle\sigma_v\sigma_y\rangle \text{ Prob. } (C_{\underline{n}_1+\underline{n}_2}(x)\cap C_{\underline{n}_1+\underline{n}_2}(y) = \phi \,|\, \{x,u\},\{v,y\}) +$$

$$+ \text{ a } (u,v) \text{ permutation.}$$

A summation of (5. 10) leads to the following expression for $\chi = \sum_x \langle\sigma_o\sigma_x\rangle$

$$\left|\frac{\partial\chi^{-1}}{\partial\beta}\right| = |J|\ \overline{\text{Prob.}}\ (C_{\underline{n}_1+\underline{n}_2}(0)\cap C_{\underline{n}_1+\underline{n}_2}(x) = \phi \,|\, \{0,u\},\{v,x\}) \qquad (5.11)$$

where the bar over $\overline{\text{Prob.}}$ represents an average over the sources - with the weights $\langle\sigma_o\sigma_u\rangle J_{u,v}\langle\sigma_v\sigma_x\rangle/|J|\chi^2$.

The expressions (5.9) and (5.11) present us with a geometric picture for the critical behavior of $g$ and $|\partial\chi^{-1}/\partial\beta|$. The former, as can be seen from (5.9) (e.g. by the argument used in section 15 of ref. [1]), measures the probability of <u>intersection</u> of two currents which join widely <u>separate</u> sources; while $|\partial\chi^{-1}/\partial\beta|$ is the probability that such currents <u>do not intersect</u> when two of their sources are <u>close</u>.

For comparison, and as a source of inspiration, let us mention the corresponding known results for the intersection properties of random paths, $\omega_x$, $\omega_y$ - generated as the trajectories of independent random walks which start at $x$ and $y$. The analog of $g$ is the probability that two paths which start at $x = 0$ and $y = R$, respectively, intersect inside a region of the order $|R|$ (which mimics the correlation length), say $\Lambda_R = \{x\in Z^d \,|\, |x| \le |R|\}$. In the limit $|R| \to \infty$, this obeys:

$$\lim_{|R|\to\infty} \text{Prob.}\ (\omega_0 \cap \omega_R' \cap \Lambda_R \neq \phi) \quad \begin{cases} = 0 & d \geq 4 \\ \neq 0 & d < 4 \end{cases} \tag{5.12}$$

At the same time, for random walks which start at fixed sites u, v:

$$\lim_{|R|\to\infty} \text{Prob.}\ (\omega_u \cap \omega_v' \cap \Lambda_R = \phi) \quad \begin{cases} \neq 0 & d > 4 \\ = 0 & d \leq 4 \end{cases} \tag{5.13}$$

the latter being the analog of $|\partial\chi^{-1}/\partial\beta|$ - by (5.11). As we see, the behavior in $d = 4$ dimensions is similar to either $d < 4$, or $d > 4$, - depending on whether the sources are tied together or not. Notice that this corresponds precisely to the expected qualitative behavior of $g$ and $|\partial\chi^{-1}/\partial\beta|$ which we discussed in section 3 ((3.9) and (3.11)).

## 6. PROOF OF THE INEQUALITY FOR ISING SYSTEMS

We start by introducing a very useful construction, associating to each pair of currents with the sources $\partial \underline{n}_1 = \{x,y\}$, $\partial \underline{n}_2 = \phi$ a path $x(s)$ $s = 0,\ldots,t$, with $x(0) = x$ and $x(t) = y$. In essence, a flux configuration with a pair of (distant) sources differs from a sourceless configuration by having a long current line - which we now extract. (Although this decomposition is not unique, we shall make it well defined.)

The path corresponds to a walk which starts at $x$ and moves along bonds with an <u>odd</u> value of $m(b) = n_1(b)+n_2(b)$, without traversing any bond twice. It is chosen as follows:

1) For each site, we order the set of bonds which connect to $x$, and for which $m(b)$ is odd. We choose a standard order (e.g., using the lexicographic order of the lattice). One could also do it stochastically.

2) The walk is generated by starting at $x$, and at each step following the earliest bond, in the above order, among the bonds for which $m(b)$ is odd, and which have not yet been traversed.

3) The walk stops when it reaches y (for the first time). We denote by $\omega(s)$, $s = 0,1,\ldots,t$, the sequence of sites which are visited $(\omega(0) = x$, $\omega(t) = y)$, and by $b(s) = \{\omega(s-1), \omega(s)\}$ the bonds which are traversed.

The feasibility of the above construction follows from the fact (which has been used in the analysis of Euler's Königsberg bridge puzzle) that if $\partial \underline{m} = \{x,y\}$ then any non-repeating walk which starts at $x$, and moves along bonds with odd values of $m_b$, has (in a finite system) to eventually reach $y$.

Using the above construction, we now associate to each pair $\underline{n}_1$, $\underline{n}_2$, with $\partial\underline{n}_1 = \{x,y\}$ and $\partial\underline{n}_2 = \phi$, the sequence $(\underline{n}_1^{(s)}, \underline{n}_2^{(s)})$, $s = 1,\ldots,t$, which is obtained by interchanging $n_1$, $n_2$ along the path defined above. Explicitly:

$$\underline{n}_i^{(s)}(b) = \begin{cases} \underline{n}_j(b) & \text{if} \quad t \geq s \geq 1, \quad \text{and} \quad b \in \{b(1),\ldots,b(s)\} \\ \underline{n}_i(b) & \text{otherwise} \end{cases} \tag{6.1}$$

where $(i,j) = (1,2)$ or $(2,1)$.

Since for any $b$ $\{b(1),\ldots,b(t)\}$ $(\underline{n}_1+\underline{n}_2)_b$ is <u>odd</u>, it is clear that the above construction at each step shifts the sources of the two configurations. Thus

$$\partial\underline{n}_1^{(s)} = \begin{cases} \{x_s, y\} & 1 \leq s < t \\ \phi & s = t \end{cases} \tag{6.2}$$

and

$$\partial\underline{n}_2^{(s)} = \begin{cases} \{x, x_s\} & 1 \leq s < t \\ \{x,y\} & s = t \end{cases} \tag{6.3}$$

By (6.1), for each $1 \leq s \leq t$:

$$w(\underline{n}_1^{(s)})\, w(\underline{n}_2^{(s)}) = w(\underline{n}_1)w(\underline{n}_2) \tag{6.4}$$

Furthermore, the mapping $(\underline{n}_1,\underline{n}_2) \to (\underline{n}_2^{(t)},\underline{n}_1^{(t)})$ reproduces $(\underline{n}_1,\underline{n}_2)$ when reapplied to $(\underline{n}_2^{(t)},\underline{n}_1^{(t)})$ (since the path $b(s)$ depends only on $\underline{n}_1+\underline{n}_2$). It is therefore an isomorphism of the set of current configurations with the sources $\partial\underline{n}_1 = \{x,y\}$, $\partial\underline{n}_2 = \phi$. In particular, we have the following identity.

Lemma 6.1: In the above notation:

$$\sum_{\substack{\partial \underline{n}_1=\{x,y\} \\ \partial \underline{n}_2=\phi}} w(\underline{n}_1)\, w(\underline{n}_2)\, f(\underline{n}_1,\underline{n}_2) = \sum_{\substack{\partial \underline{n}_1=\{x,y\} \\ \partial \underline{n}_2=\phi}} w(\underline{n}_1)\, w(\underline{n}_2)\, f(\underline{n}_2^{(t)},\underline{n}_1^{(t)}) \qquad (6.4)$$

for every function f.

The main benefit of the above construction is that it permits one to apply a stopping-time technique. The following result is derived by such an argument.

Lemma 6.2: For any finite Ising system:

$$|u_4(x_1,\ldots,x_4)| \leq 2 \sum_{u,v} \langle \sigma_{x_2}\sigma_v\rangle \tanh(\beta J_{u,v}) \sum_{\substack{\partial \underline{n}_1=\{x_1,u\} \\ \partial \underline{n}_2=\{x_3,x_4\}}} w(\underline{n}_1)\, w(\underline{n}_2)$$

$$\times\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \ni v]\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \not\ni x_1]/Z^2$$

$$+\ 2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_1}\sigma_{x_4}\rangle \qquad (6.5)$$

Proof: We start from equation (5.9), which we modify by adding a dummy variable $\underline{n}_2$.

$$\frac{1}{2}\,|u_4(x_1,\ldots,x_4)| = \sum_{\substack{\partial \underline{n}_1=\{x_1,x_2\} \\ \partial \underline{n}_2=\phi \\ \partial \underline{n}_3=\{x_3,x_4\}}} w(\underline{n}_1)w(\underline{n}_2)w(\underline{n}_3)\ \chi[C_{\underline{n}_1+\underline{n}_3}(x_3) \ni x_1]/Z^3 \qquad (6.6)$$

We shall now break this sum into two parts, using the above method of associating with each pair $(\underline{n}_1,\underline{n}_2)$ a sequence $(\underline{n}_1^{(s)}, \underline{n}_2^{(s)})$, $s = 1,\ldots,t = t(\underline{n}_1,\underline{n}_2)$. First, however, we replace $C_{\underline{n}_1+\underline{n}_3}$ in (6.6) by $C_{\underline{n}_2^{(t)}+\underline{n}_3}$ (using lemma 6.1). Thus

$\frac{1}{2}|u_4(x_1,\ldots,x_4)| =$

$$= \sum_{\text{(as above)}} w(\underline{n}_1)w(\underline{n}_2)w(\underline{n}_3)\ \chi[C_{\underline{n}_2+\underline{n}_3}(x_3)\ni x_1]\ \chi[C_{\underline{n}_2^{(t)}+\underline{n}_3}(x_3)\ni x_1]/Z^3$$

$$+ \sum_{\text{(as above)}} w(\underline{n}_1)w(\underline{n}_2)w(\underline{n}_3)\ \chi[C_{\underline{n}_2+\underline{n}_3}(x_3)\not\ni x_1]\ \chi[C_{\underline{n}_2^{(t)}+\underline{n}_3}(x_3)\ni x_1]/Z^3$$

$$\equiv \quad I + II \tag{6.7}$$

Since $\chi[\ \ ] \leq 1$, the first sum in (6.7) is bounded by the following expression

$$I \leq \sum_{\substack{\partial\underline{n}_1=\{x_1,x_2\}\\ \partial\underline{n}_2=\phi\\ \partial\underline{n}_3=\{x_3,x_4\}}} w(\underline{n}_1)w(\underline{n}_2)w(\underline{n}_3)\ \chi[C_{\underline{n}_2+\underline{n}_3}(x_3)\ni x_1]/Z^3 =$$

$$= \langle\sigma_{x_1}\sigma_{x_2}\rangle \sum_{\substack{\partial\underline{n}_2=\phi\\ \partial\underline{n}_3=\{x_3,x_4\}}} w(\underline{n}_2)w(\underline{n}_3)\ \chi[C_{\underline{n}_2+\underline{n}_3}(x_3)\ni x_1]/Z^2 =$$

$$= \langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_1}\sigma_{x_3}\rangle\langle\sigma_{x_1}\sigma_{x_4}\rangle \tag{6.8}$$

(by (5.5) and (5.6)).

The second term in (6.7) is more significant. To bound it, we observe that for each contributing pair $(\underline{n}_1,\underline{n}_2)$ there is at least one $s\in\{1,\ldots,t\}$ for which

$$C_{\underline{n}_2^{(s-1)}+\underline{n}_3}(x_3)\not\ni x_1 \quad \text{and} \quad C_{\underline{n}_2^{(s)}+\underline{n}_3}(x_3)\ni x_1 \tag{6.9}$$

(with $n_2^{(0)} = n_2$).

Let $T = T(\underline{n}_1+\underline{n}_2)$ be the smallest $s$ for which (6.9) is satisfied, and let $u = \omega(T-1)$, $v = \omega(T)$ and $\hat{\underline{n}}_i = \underline{n}_i^{(T-1)}$, $i = 1,2$. Obviously

$$\partial\hat{\underline{n}}_1 = \{u,x_2\}, \quad \partial\hat{\underline{n}}_2 = \{x_1,u\} \tag{6.10}$$

unless $u = x_1$ or $x_2$ – when the sources cancel. The condition (6.9) implies that

$$\text{i)} \quad C_{\hat{\underline{n}}_2+\underline{n}_3}(x_3) \not\ni x_1$$

$$\text{ii)} \quad \hat{n}_1(b) \neq 0, \quad \text{for} \quad b = \{u,v\} \tag{6.11}$$

and

$$\text{iii)} \quad C_{\hat{\underline{n}}_2+\underline{n}_3}(x_3) \ni v$$

Furthermore, the correspondence $(\underline{n}_1,\underline{n}_2,\underline{n}_3) \to (\hat{\underline{n}}_1,\hat{\underline{n}}_2,\underline{n}_3)$ is invertible (to invert it one needs only to know $\underline{n}_3$ and $\underline{n}_1+\underline{n}_2 = \hat{\underline{n}}_1+\hat{\underline{n}}_2$), and the analog of (6.3) is satisfied. Thus we obtain the following upper bound on the second term in (6.7):

$$\begin{aligned} II \leq \sum_{u,v} \sum_{\substack{\partial\underline{n}_1=\{u,x_2\} \\ \partial\underline{n}_2=\{x_1,u\} \\ \partial\underline{n}_3=\{x_3,x_4\}}} & w(\underline{n}_1)w(\underline{n}_2)w(\underline{n}_3) \ \chi[C_{\underline{n}_2+\underline{n}_3}(x_3) \ni v] \\ & \times \chi[C_{\underline{n}_2+\underline{n}_3}(x_3) \not\ni x_1] \ \ \chi[n_1(\{u,v\}) \neq 0]/Z^3 \end{aligned} \tag{6.12}$$

The sum over $\underline{n}_1$, which is decoupled from $(\underline{n}_2,\underline{n}_3)$, is estimated in lemma 6.3. Separating this factor, and shifting down the remaining indices we obtain

$$\begin{aligned} II \leq \sum_{u,v} \langle\sigma_{x_2}\sigma_v\rangle \tanh(\beta J_{u,v}) \sum_{\substack{\partial\underline{n}_1=\{x_1,u\} \\ \partial\underline{n}_2=\{x_3,x_4\}}} & w(\underline{n}_1)w(\underline{n}_2) \\ & \times \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \ni v] \ \ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \not\ni x_1]/Z^2 \end{aligned} \tag{6.13}$$

The substitution of (6.8) and (6.13) in (6.7) leads to the claimed bound (6.5). ∎

In the transition from (6.12) to (6.13) we used:

Lemma 6.3:

$$\sum_{\partial \underline{n}=\{u,x\}} w(\underline{n})\ \chi[n(\{u,v\}) \neq 0]/Z \ \leq\ \langle\sigma_v\sigma_x\rangle \tanh(\beta J_{u,v}) \tag{6.14}$$

whose proof we leave as an exercise for the reader. (One may derive (6.14) by a direct argument, or by an application of the generalization of (5.6) which appears in ref. [19]).

We are now ready for the proof of the new inequality (4.1).

Proof of Proposition 4.1 - for Ising systems.

Let us start from the bound (6.5) of lemma 6.2. To perform the sum

$$S = \sum_{\substack{\partial \underline{n}_1=\{x_1,u\} \\ \partial \underline{n}_2=\{x_3,x_4\}}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \ni v]\ \ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \ni x_1]$$

we split, for each contributing pair $(\underline{n}_1,\underline{n}_2)$, the bonds into two sets. Denoting by $\overline{A}$ the set of bonds which contain at least one site in a set $A$ we write $\underline{n}_i = \underline{n}'_i+\underline{n}''_i$ (i=1,2), with $\underline{n}''_i \equiv 0$ on $\overline{C}_{\underline{n}_1+\underline{n}_2}(x_1)$ and $\underline{n}'_i = 0$ on the complement. In this notation, for each $\{x_1,\ldots,x_4,u,v\}$,

$$S = \sum_{\substack{A\ni x_1,u \\ \not\ni x_3,x_4,v}} \ \sum_{\substack{\partial \underline{n}'_1=\{x_1,u\} \\ \partial \underline{n}'_2=\phi}} w(\underline{n}'_1)w(\underline{n}'_2)\ \chi[C_{\underline{n}'_1+\underline{n}'_2}(x_1) = A]$$

$$\times \sum_{\substack{\partial \underline{n}''_1=\phi \\ \partial \underline{n}''_2=\{x_3,x_4\}}} w(\underline{n}''_1)w(\underline{n}''_2)\ \chi[\underline{n}''_1+\underline{n}''_2 \equiv 0 \text{ on } \overline{A}]\ \chi[C_{\underline{n}''_1+\underline{n}''_2}(x) \ni v] \tag{6.15}$$

The very last sum is, by (5.6), exactly $\langle\sigma_{x_3}\sigma_v\rangle'_{A^c}\ \langle\sigma_{x_4}\sigma_v\rangle'_{A^c}\ Z^2_{A^c}$; where $\langle - \rangle'_{A^c}$ represents the expectation value in a system obtained by setting $J_b = 0$ for all $b \in \bar{A}$. Using the Griffiths inequality

$$\langle\sigma_{x_4}\sigma_v\rangle'_{A^c} \leq \langle\sigma_{x_4}\sigma_v\rangle, \tag{6.16}$$

we obtain:

$$\sum_{\substack{\partial \underline{n}_1'' = \phi \\ \partial \underline{n}_2'' = \{x_3, x_4\}}} w(\underline{n}_1'')w(\underline{n}_2'')\ \chi[\underline{n}_1''+\underline{n}_2'' \equiv 0 \ \text{ on } \ \bar{A}]\ \ \chi[C_{\underline{n}_1''+\underline{n}_2''}(x_3) \ni v] \leq$$

$$\leq \langle\sigma_{x_4}\sigma_v\rangle \sum_{\substack{\partial \underline{n}_1'' = \phi \\ \partial \underline{n}_2'' = \{x_3, V\}}} w(\underline{n}_1'')w(\underline{n}_2'')\ \ \chi[\underline{n}_1''+\underline{n}_2'' \equiv 0 \ \text{ on } \ \bar{A}] \tag{6.17}$$

and therefore

$$S \leq \langle\sigma_{x_4}\sigma_v\rangle \sum_{\substack{\partial \underline{n}_1 = \{x_1, u\} \\ \partial \underline{n}_2 = \{x_3, v\}}} w(\underline{n}_1)w(\underline{n}_2)\ \ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \not\ni x_1] \tag{6.18}$$

The substitution of (6.18) in the inequality (6.5) leads to the following result

$$|u_4(x_1,\ldots,x_4)| \leq 2 \sum_{u,v} \langle\sigma_{x_2}\sigma_v\rangle\ \langle\sigma_{x_4}\sigma_v\rangle\ \tanh(\beta J_{u,v})$$

$$\times \sum_{\substack{\partial \underline{n}_1 = \{x_1, u\} \\ \partial \underline{n}_2 = \{x_3, v\}}} w(\underline{n}_1)w(\underline{n}_2)\ \ \chi[C_{\underline{n}_1+\underline{n}_2}(x_1) \cap C_{\underline{n}_1+\underline{n}_2}(x_3) = \phi]/Z^2$$

$$+ 2\ \langle\sigma_{x_1}\sigma_{x_2}\rangle\ \langle\sigma_{x_1}\sigma_{x_3}\rangle\ \langle\sigma_{x_1}\sigma_{x_4}\rangle \tag{6.19}$$

While $u_4(\ldots)$ is symmetric under permutations, the right hand side of (6.19) clearly is not. Averaging it with respect to the permutation $(x_1,x_3)$ one arrives at an expression which by (5.10) is exactly equal to the claimed bound in (4.1′). ∎

## 7. SYSTEMS GENERATED BY ISING SPINS

In this section we prove the inequality (4.1) for $\phi^4$ variables, and in fact - for the more general class of variables which may be represented as sums of Ising spins:

$$\phi_x^{(N)} = \sum_{\alpha=1}^{N} Q_\alpha \sigma_x^{(\alpha)} \tag{7.1}$$

(or limits of such sums) where $\{\sigma_x^{(\alpha)}\}$ are coupled ferromagnetically with the Hamiltonian $H = H_1+H_2$ $(I,J \geq 0)$

$$H_1 = - \sum_{x,\alpha,\beta} I_{x;\alpha,\beta} \sigma_x^{(\alpha)} \sigma_x^{(\beta)}$$

$$H_2 = -\sum_{x,y} J_{x,y} \phi_x\phi_y \equiv -\sum Q_\alpha Q_\beta \hat{J}_{x,\alpha;y,\beta} \sigma_x^{(\alpha)} \sigma_y^{(\beta)} \tag{7.2}$$

Thus, $H_2$ forms a ferromagnetic coupling of the variables $\{\phi_x\}$, whose "a-priori" distribution $\rho(\phi)d\phi$ is generated by $H_1$. Distributions which can be obtained in this manner form the Simon-Griffiths class.

This class includes the $\phi^4$ measure of (2.3). Further details of this representation of $\phi^4$ variables as limits of "block spins", i.e. the explicit form of $Q(N)$ and $I(N)$, are not relevant for our discussion. (They can be found in refs.[8,1]). The main facts are that in the above limit $J_{x,y}$ is constant, and

$$\langle \phi_{x_1}^{(N)} \cdots \phi_{x_n}^{(N)} \rangle \to S_n(x_1,\ldots,x_n) \tag{7.3}$$

Thus, we now consider systems of spins associated with sites which are parametrized as $\underline{x} = (x,\alpha)$. Sites $x$ of the original lattice may be thought of as blocks - which we denote by $B_x = \{(x,\alpha)\}_{\alpha=1,\dots,N}$, of "microscopic sites". The quantities $\frac{\partial\chi^{-1}}{\partial\beta}$ and $u_4(x_1,\dots,x_4)/[S_2(x_1,x_2)S_2(x_3,x_4)]$, with the $\phi$ variables, inherit the simple geometric interpretation discussed above. They still measure intersection probabilities of random currents - which now move via bonds which join the "microscopic sites".

The role previously played by the identity (5.7) would now be taken by the following bound on block-hitting probability.

Proposition 7.1: In the above system of Ising spins

$$\text{Prob.}\ (C_{\underline{n}_1+\underline{n}_2}(\underline{x}) \cap B_z \neq \phi \,|\, \{\underline{x},\underline{y}\},\phi) \leq$$

$$\leq \sum_{w(\neq z)} \langle\sigma_{\underline{x}}\phi_w\rangle\, \beta J_{w,z}\, \langle\phi_z\sigma_{\underline{y}}\rangle / \langle\sigma_{\underline{x}}\sigma_{\underline{y}}\rangle$$

for any $z \neq x$. More explicitly:

$$\sum_{\substack{\partial\underline{n}_1=\{\underline{x},\underline{y}\}\\ \partial\underline{n}_2=\phi}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(\underline{x}) \cap B_z \neq \phi] \leq \tag{7.4}$$

$$\leq \sum_{\substack{\underline{w}=(w,\gamma),w\neq z\\ \underline{z}=(z,\delta)}} \beta \hat{J}_{\underline{w},\underline{z}} \sum_{\substack{\partial\underline{n}_1=\{\underline{x},\underline{w}\}\\ \partial\underline{n}_2=\{\underline{z},\underline{y}\}}} w(\underline{n}_1)w(\underline{n}_2)$$

(where in the sum over $\underline{z} = (z,\delta)$ only $\delta$ is varied).

Proof: In the proof we return to the simpler setup of the previous section - where sites are labeled just as $x,y$, etc. $B_z$ is regarded now as merely a collection of sites - $B$.

By (5.3), $\beta J_{w,z} w(\underline{n}) = w(\underline{n}')\ n'(\{w,z\})$ where $n'(b) = n(b)+\delta_{b,\{w,z\}}$. Clearly if $\partial\underline{n} = \{x,w\}$ then $\partial\underline{n}' = \{x,z\}$. Thus (7.4) may be rewritten as follows

$$\sum_{\substack{\partial \underline{n}_1=\{\underline{x},\underline{y}\} \\ \partial \underline{n}_2=\phi}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(x)\cap B \neq \phi] <$$

$$< \sum_{\substack{w\in B^c \\ z\in B}} \sum_{\substack{\partial \underline{n}_1=\{x,z\} \\ \partial \underline{n}_2=\{z,y\}}} w(\underline{n}_1)w(\underline{n}_2)\ n_1(\{w,z\})$$

Let us now change the variables of summation to $\underline{m} = \underline{n}_1+\underline{n}_2$ and $\underline{k} = \underline{n}_2$ (as in the proof of lemma 3.2 in ref. [1]). Since $w(\underline{n}_1)w(\underline{n}_2) = w(\underline{m})\binom{\underline{m}}{\underline{k}}$, the claimed relation becomes:

$$\sum_{\partial \underline{m}=\{x,y\}} w(\underline{m})\ \chi[C_{\underline{m}}(x)\cap B \neq \phi] \sum_{\substack{0\le \underline{k}\le \underline{m} \\ \partial \underline{k}=\phi}} \binom{\underline{m}}{\underline{k}} < \sum_{\partial \underline{m}=\{x,y\}} w(\underline{m})\ F(\underline{m}) \tag{7.5}$$

where

$$F(m) = \sum_{\substack{w\in B^c \\ z\in B}} \sum_{\substack{0\le \underline{k}\le \underline{m} \\ \partial \underline{k}=\{z,y\}}} \binom{\underline{m}}{\underline{k}} k(\{w,z\})$$

We shall now show that (7.5) is satisfied even at the level of the partial sums which correspond to each fixed $\underline{m}$, assuming only that $\partial \underline{m} = \{x,y\}$.

The proof of the following pair of identities is left as a exercise,

$$\sum_{\substack{k=0 \\ \text{even(odd)}}}^{m} \binom{\underline{m}}{\underline{k}} k = \frac{m}{2} \sum_{\substack{k=0 \\ \text{even(odd)}}}^{m} \binom{\underline{m}}{\underline{k}}, \tag{7.6}$$

for $m \neq 1$, i.e. $m = 0,2,3,\ldots$ . (Hint: use $(1 \pm x)^m$ as generating functions.)

Since the source condition is only a constraint on the parity of $\{k_b\}$, we may use (7.6) in F. Therefore

$$F(m) = \sum_{\substack{w\varepsilon B^c \\ z\varepsilon B}} \sum_{\substack{0\leq \underline{k}\leq \underline{m} \\ \partial\underline{k}=\{z,y\}}} \binom{\underline{m}}{\underline{k}} \left\{\frac{m(\{w,z\})}{2} \chi[m(\{w,z\}) \neq 1] + k(\{w,z\}) \chi[m = 1]\right\} \tag{7.7}$$

It has been shown in ref. [1], proof of lemma 3.2, that if $z\epsilon C_{\underline{m}}(y)$ then

$$\sum_{\substack{0\leq \underline{k}\leq \underline{m} \\ \partial\underline{k}=\{z,y\}}} \binom{\underline{m}}{\underline{k}} = \sum_{\substack{0\leq \underline{k}\leq \underline{m} \\ \partial\underline{k}=\phi}} \binom{\underline{m}}{\underline{k}} \tag{7.8}$$

Furthermore, the argument of the above proof shows that if $m(\{w,z\}) = 1$ and if $\{w,z\}$ is a link in some closed loop of bonds $b$ with $m(b) \neq 0$, then the value ofthe sum (7.7) is not changed if $k(\{w,z\})$ is replaced there by $\frac{m}{2}$.

For a lower bound on $F$ we shall take into account only those pairs $(w,z)$ in (7.7) for which $\{w,z\}$ is the "first entry" bond for some path from $y$ to $B$ - which proceeds only along bonds with $m(b) \neq 0$. We shall distinguish between two cases.

i) For every pair $(w,z)$ in the above class, with $m(\{w,z\}) = 1$, there is a closed loop of bonds with $m_b \neq 0$, which passes through $\{w,z\}$.

In this case, (7.8) and the above substitution lead to:

$$F(\underline{m}) = \sum_{\substack{w\varepsilon B^c \\ z\varepsilon b}} \frac{m(\{w,z\})}{2} \chi[z \in C_{\underline{m}}(y)] \sum_{\substack{0\leq k\leq m \\ \partial k=\phi}} \binom{\underline{m}}{\underline{k}} \geq$$

$$\geq \chi[C_{\underline{m}}(x)\ B \neq \phi] \sum_{\substack{0\leq \underline{k}\leq \underline{m} \\ \partial\underline{k}=\phi}} \binom{\underline{m}}{\underline{k}} \tag{7.9}$$

where the inequality is due to the fact that the first factor is half the total flux "into" $B$ along bonds which connect to $C_{\underline{m}}(y)$. Our assumption implies that if $B$ is connected to $y$, then that flux is at least 2 (even if $x \in B$).

Case ii) There is a pair $(w,z)$ in above-mentioned class, with $m(\{w,z\}) = 1$, for which there is no loop with the properties mentioned in i).

In such a situation, any path from $z$ to $y$ along bonds with $m(b) \neq 0$ has to pass via the bond $\{w,z\}$. Hence $k(\{w,z\}) = 1$ for any contributing $\underline{k}$ in (7.7), and thus (using again (6.19)), the contribution from just this pair $(w,z)$ shows that

$$F(\underline{m}) \geq \sum_{\substack{0<\underline{k}<\underline{m} \\ \partial\underline{k}=\phi}} \binom{\underline{m}}{\underline{k}}$$

This completes the proof of (7.5). ∎

To demonstrate the power of proposition 7.1, let us point out that it directly leads to the following bound, which has the optimal coefficient 2 - improving in this sense the previous results of both ref. [1] and ref. [2] (where 2 is in effect replaced by 3).

<u>Corrollary 7.1:</u> In ferromagnetic systems of variables in the Simon-Griffith class ($\phi^4$ included):

$$|u_4(x_1,\ldots,x_4)| \leq$$

$$\leq 2 \sum_{\substack{y\notin\{x_1,\ldots,x_4\} \\ u,v(\neq y)}} S_2(x_1,u)\ \beta J_{u,y}\ S_2(x_3,v)\ \beta J_{v,y}\ \ S_2(y,x_2)\ S_2(y,x_4) +$$

$$+ 2\Big[\sum_{u\neq x_1} S_2(x_1,x_2)\ S_2(x_1,x_3)\beta J_{x_1,u}\ S_2(x_u,x_4) + 3 \text{ similar terms}\Big] \qquad (7.10)$$

for noncoincidental points $\{x_i\}$.

<u>Proof:</u> $|u_4|$ is related by (5.10) to the probability of intersection of two <u>independent</u> random currents. (7.10) is derived by overestimating this probability by the <u>number</u> of blocks which are visited by both currents. ∎

Remarks: 1) The four last terms in the right hand side may be omitted by extending the sumation also over $y = x_i$. The split was made to avoid terms with $S_2(x,x)$ - which in certain situations are specially singular.

2) As in ref. [1], one may improve (7.10) by a factor which estimates the probability that two currents actually intersect inside a block - given that both visit it. For $\phi^4$ variables this leads to an improvement by the factor $\{1-\exp[-c\hat{\lambda}/(\beta|J|)^2]\}$. A generalization and the discussion of this effect would be given elsewhere.

3) The bound (7.10) is of course not as good as the main results of this paper - especially for $d \leqslant 4$. It demonstrates, however, the advantage of proposition 7.1 over the comparable lemma 11.1 of ref. [1].

We now proceed towards the proof of proposition 4.1 - repeating the steps which were taken in the simple case discussed in section 6. The only modification which is needed is a systematic replacement of site conditions by block conditions. In this manner one arrives at the following straightforward extension of lemma 6.2.

Lemma 7.2: For systems introduced above:

$$|\langle\sigma_{\underline{x}_1},\dots,\sigma_{\underline{x}_4}\rangle-(\langle\sigma_{\underline{x}_1}\sigma_{\underline{x}_2}\rangle\langle\sigma_{\underline{x}_3}\sigma_{\underline{x}_4}\rangle+2\text{ permutations})| \leqslant$$

$$\leqslant 2 \sum_{\underline{u},\underline{v}} \langle\sigma_{\underline{x}_2}\sigma_{\underline{y}}\rangle\beta\hat{J}_{\underline{u},\underline{v}} \sum_{\substack{\partial\underline{n}_1=\{\underline{x}_1,\underline{u}\}\\ \partial\underline{n}_2=\{\underline{x}_3,\underline{x}_4\}}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3)\cap B_v \neq \phi]\ \chi[C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3)\ni\underline{x}_1]/Z^2$$

$$+\ 2\langle\sigma_{\underline{x}_1}\sigma_{\underline{x}_2}\rangle \sum_{\substack{\partial\underline{n}_1=\{\underline{x}_3,\underline{x}_4\}\\ \partial\underline{n}_2=\phi}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3)\cap B_{x_1} \neq \phi]/Z^2 \qquad (7.11)$$

Or, more succintly:

$$|u_4(x_1,\dots,x_4)| \leq 2 \sum_{u,v} S_2(x_2,v)\ \beta J_{u,v}\ S_2(x_1,u)\ S_2(x_3,x_4)$$

$$\times\ \overline{\text{Prob.}}\ (C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \cap B_v \neq \phi \text{ and } C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \not\ni \underline{x}_1 \mid \{\underline{x}_1,\underline{u}\},\{\underline{x}_3,\underline{x}_4\})$$

$$+\ 2\ S_2(x_1,x_2)\ S_2(x_3,x_4)\ \overline{\text{Prob.}}\ (C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \cap B_{x_1} \neq \phi \mid \{\underline{x}_3,\underline{x}_4\},\phi) \qquad (7.12)$$

where $u$ and $S_2$ refer to the $\phi$ variables, and the bar over $\overline{\text{Prob.}}$ indicates that $x_i = (x_i,\alpha_i)$ are averaged over $B_{x_i}$ with some probability weights.

The expression (7.12) is derived from (7.11) by summation over $\alpha_1,\dots,\alpha_4$, with the factors $Q_{\alpha_1},\dots,Q_{\alpha_4}$. We have replaced $\tanh(\beta J)$ of (6.5) by $(\beta J)$, which of course is an upper bound.

Proof of Proposition 4.1 – for systems generated by ferromagneticly coupled Ising spins.

A trivial modification of the argument which lead to (6.17) produces the following analogous bound.

$$\sum_{\substack{\partial\underline{n}_1=\{\underline{x}_1,\underline{u}\}\\ \partial\underline{n}_2=\{\underline{x}_3,\underline{x}_4\}}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \cap B_v \neq \phi]\ \chi[C_{\underline{n}_1+\underline{n}_2}(x_3) \not\ni x_1]/Z^2 \leq$$

$$\leq \sum_{\substack{w\\ \underline{v}'\in B_v}} \langle \sigma_{\underline{x}_4}\sigma_{\underline{w}}\rangle \beta \hat{J}_{\underline{w},\underline{v}'} \sum_{\substack{\partial\underline{n}_1=\{\underline{x}_1,\underline{u}\}\\ \partial\underline{n}_2=\{\underline{x}_3,\underline{v}'\}}} w(\underline{n}_1)w(\underline{n}_2)\ \chi[C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \not\ni \underline{x}_1] \qquad (7.13)$$

or, in the notation of (7.12),

$$S_2(x_1,u)\ S_2(x_3,x_4)\ \text{Prob.}(C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \cap B_v \neq \phi \text{ and } C_{\underline{n}_1+\underline{n}_2}(x_3) \not\ni x_1 \mid \{\underline{x}_1,\underline{u}\},\{\underline{x}_3,\underline{x}_4\}) \leq$$

$$\leq \sum_{w} S_2(x_4,w)\beta J_{w,v}\ S_2(x_1,u)\ S_2(x_3,v)\ \text{Prob.}(C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \not\ni \underline{x}_1 \mid \{x_1,u\},\{x_3,v\}) \qquad (7.14)$$

The main difference between the derivation of (7.13) and that of (6.17) is that the site hitting probability, given by (5.6), is replaced by the estimate of the block hitting probability given by lemma 7.2.

The substitution of (7.13) in (7.11) leads, after summation over $\alpha_1$-with the weights $Q_{\alpha_1}$, and the $(x_1,x_3)$ symmetrization, to an expression which may be summarized as follows:

$$|u_4(x_1,\ldots,x_4)| \leq \sum_{\substack{u,v\\w}} S_2(x_2,v)\, S_2(x_4,w)\beta J_{w,v}\, \beta J_{v,u}$$

$$\times \Big[S_2(x_1,v)\, S_2(x_3,v)\, \overline{\text{Prob.}}\, (C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \ni \underline{x}_1 \,|\, \{\underline{x}_1,\underline{u}\},\{\underline{x}_3,\underline{v}\}) +$$

$$+ S_2(x_1,u)\, S_2(x_3,u)\, \overline{\text{Prob.}}\, (C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \ni \underline{x}_1 \,|\, \{\underline{x}_1,\underline{v}\},\{\underline{x}_3,\underline{u}\})\Big]$$

$$+ \Big[S_2(x_1,x_2)\, S_2(x_3,x_4)\, \overline{\text{Prob.}}\, (C_{\underline{n}_1+\underline{n}_2}(\underline{x}_3) \cap B_{x_1} \neq \phi \,|\, \{\underline{x}_3,\underline{x}_4\},\phi) + \text{a } (1,3) \text{ permutation}\Big]$$

Using (5.10), the expression in the first square braket is easily seen to be exactly equal to $\frac{1}{\beta}\,\frac{\partial}{\partial J_{u,v}}\, S_2(x_1,x_3)$. Applying (7.4) to bound the second term, one arrives at (4.1). ∎

<u>Remark:</u> Notice that the replacement of $\overline{\text{Prob.}}$ in (7.15) by 1 leads back to (7.10). However, as explained in the discussion in the early sections, we expect this probability to be <u>vanishingly small</u> in $d \leq 4$ dimensions, except for manifestly Gaussian limits where $g$ vanishes due to the the other factor $\{1-\exp[-c\hat{\lambda}/(\beta|J|)^2]\}$, of ref. [1].

Note: At an AMS summer seminar (1982), where the results of this work were reported, we learned that somewhat similar considerations were being independently applied by Aragao de Carvalho, Caraciolo and Fröhlich [20] to improve the upper bound on $g$ by the methods of ref. [2]. However, the relation of the factor by which ref. [20] improves the previous bound with the critical behavior of $\chi$, has not been established for the one component theory. This apparently, is due to the greater difficulty to produce within that framework, a "random-walk" lower bound on $|\partial\chi^{-1}/\partial\beta|$.

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