AN EFFICIENT LINEAR SCHEME TO APPROXIMATE
PARABOLIC FREE BOUNDARY PROBLEMS:
ERROR ESTIMATES AND IMPLEMENTATION

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An Efficient Linear Scheme to Approximate Parabolic Free Boundary Problems: Error Estimates and Implementation

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**Abstract** - This paper deals with a fully discrete scheme to approximate multidimensional singular parabolic problems; two-phase Stefan problems and porous medium equations are included. The algorithm consists of approximating at each time step a linear elliptic P.D.E. by piecewise linear finite algebraic elements and making then an element-by-element correction on account of the nonlinearity. Several energy error estimates are derived for the physical unknowns; a sharp rate of convergence of $O(h^{\frac{1}{2}})$ is our main result. The crucial point in implementing the scheme is the efficient resolution of linear systems involved. This topic is discussed and the results of several numerical experiments are shown.

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1 - INTRODUCTION

The aim of this paper is to analyze from both a theoretical and computational viewpoint the performance of a linear scheme to approximate the following multidimensional parabolic problem:

\[
\frac{\partial u}{\partial t} - \Delta \beta(u) = f(\beta(u)) \quad \text{in } Q := \Omega \times (0,T]
\]

\[
\beta(u) = 0 \quad \text{on } \partial \Omega \times (0,T]
\]

\[
u(0) = u_0 .
\]

Here \( \beta \) stands for a nondecreasing Lipschitz continuous function defined on \( \mathbb{R} \) and \( \Omega \) for a polyhedral and convex domain in \( \mathbb{R}^d \). The geometrical constraints upon \( \Omega \) as well as the type of boundary condition considered were chosen only for the sake of simplicity. It is well known that formulation (1.1) is so general as to include two-phase Stefan problems and porous medium equations.

The usual technique to approximate (1.1) (with or without regularization of \( \beta \)) amounts to discretizing a nonlinear elliptic P.D.E. at each time step. Both theoretical and numerical results are now well known for these schemes \([10,23,16,17,18,8,20,25]\), which behave pretty well in approximating not only solutions but also interfaces \([19,20]\). However, when dealing with nonlinear problems one usually tries to linearize them so as to take advantage of efficient linear solvers \([7,15]\). The success of such a trick relies on the smoothness of the solutions \( u \) and \( \theta := \beta(u) \). Consequently, it is not a priori obvious that standard techniques for mildly nonlinear parabolic equations apply in this context because (1.1) is just a low regularity problem. Moreover, high order accuracy schemes in both space and time may be
useless here due again to the lack of regularity. So the question is how to linearize (1.1) properly.

Our present purpose is to discuss the stability and approximating properties of a fully discrete linear scheme associated with the following discrete-time scheme (which is a **nonlinear Chernoff formula**):

$$
U^0 := u_0
$$

$$
0^n - \frac{1}{\mu} \Delta \theta^n = \beta(U^{n-1}) + \frac{1}{\mu} f(\beta(U^{n-1})),
$$

$$
U^n := U^{n-1} + \mu[\theta^n - \beta(U^{n-1})], \quad 1 \leq n \leq N := \frac{T}{\tau}.
$$

Here $\tau > 0$ is the time step and $\mu > 0$ is the **relaxation parameter** which satisfies the stability constraint $\mu \leq L^{-1}_\beta$ ($L_\beta$ Lipschitz constant of $\beta$). This discrete-time algorithm was studied in an abstract and general setting by BREZIS & PAZY [4] and first used in numerical analysis by BERGER, BREZIS & ROGERS [2] (see also VERDI [24] and MAGENES & VERDI [14]), who showed its convergence. Sharp energy error estimates for both singular and mildly nonlinear parabolic problems were recently proved by MAGENES, NOCHETTO & VERDI [13]. The error analysis has been extended by MAGENES [12] to other algorithms suggested by nonlinear semigroup theory. Nonlinear Chernoff formula (1.2) can be regarded as the discrete-time phase relaxation scheme introduced by VISINTIN [27] and VERDI & VISINTIN [26], as well. This point of view was essential in [12] because it exhibits the variational structure of (1.2). Moreover, it allowed the use of variational techniques first applied by NOCHETTO [16,17,18] and NOCHETTO & VERDI [20] for analyzing singular parabolic problems and dealing with minimal regularity properties (say, $u_0 \in L^2(\Omega)$), respectively. Let us say in addition that (1.2) is in the same spirit of the Laplace-modified forward Galerkin method of DOUGLAS &

The algorithm (1.2) gives rise actually to an effective numerical scheme after discretizing in space; namely variable $\theta^n$ is approximated by continuous piecewise linear finite elements and variable $U^n$ by piecewise constants. The primary aim of this paper is to show that the resulting scheme is stable and preserves the approximation properties of (1.2). The tools in deriving the rates of convergence are essentially those in [13,26]. The crucial step in implementing the fully discrete scheme is the efficient resolution of the linear elliptic P.D.E. in (1.2). We used an incomplete Cholesky factorization for preconditioning the resulting matrix coupled with a conjugate gradient method [1,15]. We chose this iterative technique because we have in mind an automatic decomposition of a general domain $\Omega$ rather than a particular geometry. Therefore, we cannot take advantage of geometrical properties of $\Omega$ which allow direct methods to be competitive. The numerical experiments confirm the ability of our scheme to approximate solutions. The observed orders of convergence agree with the theoretical ones. The approximation of the interfaces is not as good as for the usual nonlinear scheme [20]; this is due to artificial diffusion added by the linear P.D.E. in (1.2).

The outline of the paper is as follows. Section 2 is devoted to state assumptions and continuous and fully discrete problems. The stability of the discrete scheme in energy and maximum norms is proved in section 3. Several energy error estimates for both physical unknowns $u$ and $\theta=\theta(u)$ are derived in section 4. Finally, the implementation of the method as well as the results of several numerical experiments are discussed in section 5.
2 - **FORMULATION OF THE PROBLEM**

In this section we shall establish the hypotheses upon the data and state the continuous and the fully discrete problems.

2.1 - **Basic assumptions, notation and finite element properties**

Assume the following hypotheses.

\[(H_\Omega)\] \(\Omega \subset \mathbb{R}^d (d \geq 1)\) is a polyhedral and convex domain.
Set \(Q := \Omega \times [0,T[,\) where \(0 < T < +\infty\) is fixed.

\[(H_\beta)\] \(\beta : \mathbb{R} \rightarrow \mathbb{R}\), \(\beta(0) = 0\), is a nondecreasing and Lipschitz continuous function, more precisely

\[(H_\beta)_1\] 0\(\beta'(s)\beta_L < +\infty\) for a.e. \(s \in \mathbb{R}\);
moreover \(\beta\) grows at least linearly at infinity, namely

\[(H_\beta)_2\] \(\exists C_1,C_2 > 0:\) \(\forall s \in \mathbb{R},\) \(|s| \leq C_1 + C_2 |\beta(s)|\).

\[(H_F)\] \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a uniformly Lipschitz continuous function, namely

\[|f(s_1) - f(s_2)| \leq L_f |s_1 - s_2|, \quad \forall s_1,s_2 \in \mathbb{R} .\]

The geometrical constrains upon \(\Omega\) as well as the fact that \(\beta\) and \(f\) are independent of the space and time variables \((x,t)\) were assumed only for the sake of simplicity. Consequently a more general treatment is still possible (see Remark 11). The basic regularity required upon the initial datum is

\[(H_{u_0})\] \(u_0 \in L^2(\Omega)\).
This assumption will be straightened later on in order to
prove some of our results. Let \( \{S_h\}_h \) be a family of decompositions \( S_h = \{S_k\}_{k=1}^K \) of \( \Omega \) into closed \( d \)-simplices so that \( \Omega = \bigcup_{k=1}^K S_k \); as usual, \( h \) stands for the mesh-size. We assume that

\( (H_{S_h}) \) the family \( \{S_h\}_h \) is regular \([5, \text{p.132}]\).

Since quasi-uniformity is not required, local refinements are allowed. A further property we need only in Lemma 2 is the acuteness of \( \{S_h\}_h \); this guarantees the discrete maximum principle to hold \([6]\). Let us now define the discrete spaces we shall work with:

\[
V^1_h := \{x \in C^0(\Omega) : x|_{S_k} \text{ is linear } \forall k = 1, \ldots, K, \ x = 0 \text{ on } \partial \Omega\},
\]

\[
V^0_h := \{\psi : \psi|_{S_k} \text{ is constant } \forall k = 1, \ldots, K\}.
\]

We denote by \( \langle ., . \rangle \) both the inner product in \( L^2(\Omega) \) and the duality pairing between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \). The corresponding discrete inner product is defined by

\[
(2.1) \quad \langle x, \phi \rangle_h := \sum_{k=1}^K \int_{S_k} \Pi_h(x\phi) \, dx,
\]

for any piecewise uniformly continuous functions \( x \) and \( \phi \), where \( \Pi_h \) stands for the local linear interpolant operator. Notice that the previous integral can be evaluated easily by means of the vertex quadrature rule which is exact for piecewise linear functions \([5, \text{p.182}]\). It is well known that \( \langle ., . \rangle_h \) is an inner product in \( V^1_h \) which satisfies \([22, \text{p.260}]\):

\[
(2.2) \quad \| x \|_{L^2(\Omega)}^2 \leq \langle x, x \rangle_h \leq C \| x \|_{L^2(\Omega)}^2, \quad \forall x \in V^1_h,
\]

where \( C : 1 \) is a constant independent of \( h \). The following well
known error bound takes into account the effect of numerical integration:

\begin{equation}
\|\phi - \phi_h\|_H^2 \leq Ch^2 \|\phi\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}, \quad \forall \phi \in V_h.
\end{equation}

We now introduce the discrete $H^1$-projection operator $P_h$; more precisely, for any $z \in H^1_0(\Omega)$, let $P_h z \in V_h$ be defined by

\begin{equation}
\langle P_h z, \phi \rangle = \langle z, \phi \rangle, \quad \forall \phi \in V_h.
\end{equation}

Since \{\text{S}_h\} and the Green operator associated with the inner product in $H^1_0(\Omega)$ are regular, the following approximation property hold for any $z \in H^1_0(\Omega)$ [5, p.138]:

\begin{equation}
\|z - P_h z\|_{H^r(\Omega)} \leq Ch^{r+1} \|z\|_{H^{r+1}(\Omega)}, \quad r=0,1.
\end{equation}

We also introduce the $L^2$-projection operator $P_h^0$ onto $V_h$, which, for any $z \in L^2(\Omega)$, is defined by

\begin{equation}
\langle P_h^0 z, \psi \rangle = \langle z, \psi \rangle, \quad \forall \psi \in V_h,
\end{equation}

and satisfies

\begin{equation}
\|z - P_h^0 z\|_{H^{-s}(\Omega)} \leq Ch^{r+s} \|z\|_{H^{r}(\Omega)}, \quad 0 \leq s, r \leq 1.
\end{equation}

Finally we conclude with some notation concerning the time discretization. Let $\tau := T/N$ be the time step (N positive integer) and set $t^n = n\tau, \quad t^n := [t^{n-1}, t^n]$ for $1 \leq n \leq N$. We also set

\begin{equation*}
z^n := z(., t^n), \quad z^n := \frac{1}{\tau} \int_{t^{n-1}}^{t^n} z(., t) \, dt \quad (z^0 := z^0)
\end{equation*}

for any continuous (resp. integrable) function in time defined in $Q$, and
\[ \delta z^n = \frac{z^n - z^{n-1}}{\tau}, \quad 1 \leq n \leq N \]

for any given family \( \{z^n\}_{n=0}^N \).

2.2 - The continuous problem. Regularity

Let us now state the variational formulation of problem (1.1) we shall work with.

Problem (P): find \( \{u, \theta\} \) such that

\[ (2.6) \quad u \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \quad \theta \in L^2(0,T;H^1_0(\Omega)), \]

\[ (2.7) \quad \theta(x,t) = \beta(u(x,t)) \quad \text{for a.e.} \quad (x,t) \in Q, \]

\[ (2.8) \quad u(.,0) = u_0 \]

and for a.e. \( t \in ]0,T[ \) and for all \( \phi \in H^1_0(\Omega) \) the following equation holds

\[ (2.9) \quad \langle \frac{\partial u}{\partial t}, \phi \rangle + \langle \nabla \theta, \nabla \phi \rangle = \langle f(\theta), \phi \rangle. \]

Existence and uniqueness are well known for (P) (see, e.g., [9, 11] and the references given therein). Now, we recall the further regularity results we shall use in the sequel. Letting assumptions \( (H_n), (H^1), (H^1_f) \) and \( (H^1_0) \) hold, we have that:

\[ \text{(R1)} \quad \text{if} \quad u_0 \in L^\infty(\Omega), \text{ then } u, \theta \in L^\infty(Q); \]

\[ \text{(R2)} \quad \text{if} \quad \Delta \beta(u_0) \in L^1(\Omega), \text{ then } \frac{\partial u}{\partial t} \in L^\infty(0,T;M(\Omega)), \quad \text{where } M(\Omega) \]

denotes the set of regular Baire measures [10, 23; 9, 11].
Remark 1 - We stress that the assumption upon the initial datum in (R2) may be weakened somewhat by taking $\Delta \beta(u_0) = \xi + \mu$, where $\xi \in L^1(\Omega)$ and $\mu \in M(\Omega)$ provided that the set $S_0 := \{x \in \Omega: \beta(u_0(x)) = 0\} \cap \Omega$ is sufficiently smooth and $\text{supp} \mu \subset S_0$. The proof proceeds as in [11] after a convenient regularization.

2.3 - The fully discrete nonlinear Chernoff formula

Finally let us state the fully discrete algorithm precisely. Let $0 < \mu \leq L^{-1}_\beta$ be a fixed number (the so-called relaxation parameter).

Problem $(P_{h, \tau})$: for any $1 \leq n \leq N$, find $\{U^n, \phi^n\}$ such that $U^n \in V^0_h$, $\phi^n \in V^1_h$ and, setting

\begin{equation}
U^0 := P^0_h u_0,
\end{equation}

we have

\begin{equation}
<\phi^n, x>_h + \frac{1}{\mu} <\nabla \phi^n, \nabla x>_h = <\beta(U^{n-1}) + \frac{1}{\mu} f(\beta(U^{n-1})), x>_h,
\end{equation}

for all $x \in V^1_h$, and

\begin{equation}
U^n = U^{n-1} + \mu [P^0_h \phi^n - \beta(U^{n-1})].
\end{equation}

Since the matrix of linear system (2.11) is symmetric and positive definite the solution of $(P_{h, \tau})$ exists and is unique (see also section 5). Moreover, since both $U^n$ and $P^0_h \phi^n$ are piecewise constants, equation (2.12) may be regarded as an element-by-element algebraic correction which takes the intrinsic nonlinearity into account.

Remark 2 - For computational convenience we would like to take the piecewise constant interpolant of $u_0$ as initial
datum $U^0$ rather than $P^0_h U_0$. This is simply impossible in general due to the lack of regularity of $u_0$, but it is still possible in some cases of relevant interest for both Stefan problems and porous medium equations. Namely, assume that the phases are initially separated by smooth surfaces, say Hölder continuous, and also that $u_0$ is regular in each phase, say $u_0 \in C^{0,\frac{1}{2}}$. We then define $U^0$ to be either the value of $u_0$ at the barycenter $y_k$ of the simplex $S_k$, whenever $y_k$ belongs to the interior of the phases, or any value in the range of $u_0$ restricted to $S_k$ provided $y_k$ lies on the initial interfaces. It is easily seen that

$$\|u_0 - U^0\|_{L^2(\Omega)} = O(h^{\frac{1}{2}});$$

this error estimate suffices for later purposes.

3 - STABILITY OF THE DISCRETE SCHEME

Let us start by combining the equations (2.11) and (2.12) and rewriting the discrete problem as follows:

$$\langle \partial U^n, \chi \rangle + \langle \nabla \Phi^n, \nabla \chi \rangle = \frac{\mu}{\tau} \langle [P^0_h - I] \phi^n, \chi \rangle + \langle f(\beta(U^{n-1})), \chi \rangle, \quad \forall \chi \in V_h^1.$$  

Since the relaxation parameter $\mu$ is chosen so that $0 < \mu \leq L^1_\beta$, the following property holds:

$$\alpha := I - \mu \beta \text{ satisfies } 0 \leq \alpha'(s) \leq 1, \text{ for a.e. } s \in \mathbb{R}.$$  

The proof of the a priori estimates proceeds along the same lines as that for the semidiscrete approximation analyzed in [13]. However we present here a detailed proof for the sake of completeness.
3.1 - Stability in Energy Norms

Given an absolutely continuous function \( \lambda: \mathbb{R} \rightarrow \mathbb{R} \) so that \( \lambda(0) = 0 \) and \( 0 \leq \lambda' \leq \lambda \leq \infty \), \( \phi_{\lambda} \) stands for the convex function defined by

\[
\phi_{\lambda}(s) = \int_{0}^{s} \lambda(z) dz , \quad \text{for } s \in \mathbb{R}.
\]

Moreover, \( \phi_{\lambda} \) has the properties

\[
\frac{1}{2\lambda} \lambda^2(s) \leq \phi_{\lambda}(s) \leq \frac{\lambda}{2} s^2 , \quad \text{for } s \in \mathbb{R},
\]

which are easy to prove. The following elementary relations will be used in the sequel

\[
2ab \leq \frac{a^2 + b^2}{\eta} \quad \text{for } a, b \in \mathbb{R}, \quad \eta > 0,
\]

\[
2a(a-b) = a^2 - b^2 + [a-b]^2 , \quad \text{for } a, b \in \mathbb{R}.
\]

Lemma 1 - Assume that \( (H_{\eta}), (H_{\beta})_1, (H_{\tau}), (H_{u_0}) \) and \( (H_{S_h}) \) hold. Then there exist a constant \( C > 0 \) independent of the discretization parameters such that

\[
\max_{1 \leq n \leq N} \| \beta(U^n) \|_{L^2(\Omega)} + \sum_{n=1}^{N} \| U^n - U^{n-1} \|_{L^2(\Omega)} + \sum_{n=1}^{N} \| \tau \delta \|_{H^1(\Omega)} \leq C.
\]

The norm in the middle may be viewed as a discrete \( H^1(0,T;L^2(\Omega)) \) norm.

**Proof.** Function \( \phi^n \) is an admissible test function in (3.1) because \( \phi^n \in V^1_h \). So let us take \( \chi = \tau \phi^n \) and sum (3.1) over \( n \) from 1 to \( m \leq N \). We now proceed to estimate each resulting term. First note that
\[ P_h^0, n = \frac{1}{\mu} [U^n - U^{n-1}] + \phi(U^{n-1}) = \]
\[ = \frac{1}{2} \beta(U^n) - \frac{1}{2\mu} \alpha(U^{n-1}) + \frac{1}{2\mu} U^n + \frac{1}{2\mu} [\phi(U^n) - \phi(U^{n-1})]. \]

(3.7)

Then using the definition of the \( L^2 \)-projection \( P_h^0 \), the convexity of \( \phi_\beta \) and \( \phi_\alpha \) and the identity (3.5) we can bound from below the first term in (3.1); namely

\[ 2 \sum_{n=1}^{m} \langle U^n - U^{n-1}, \theta, n \rangle = 2 \sum_{n=1}^{m} \langle U^n - U^{n-1}, P_h^0, n \rangle \geq \]
\[ \geq \int_{\Omega} \sum_{n=1}^{m} \{ [\phi_\beta(U^n) - \phi_\beta(U^{n-1})] + \frac{1}{\mu} [\phi_\alpha(U^{n-1}) - \phi_\alpha(U^n)] \} + \]
\[ + \frac{1}{2\mu} \| U^n \|_{L^2(\Omega)}^2 - \| U^0 \|_{L^2(\Omega)}^2 + \sum_{n=1}^{m} \| U^n - U^{n-1} \|_{L^2(\Omega)}^2. \]

Here we have also used that \( \alpha \) is increasing in order to eliminate the contribution of last term in (3.7). The terms involving functions \( \phi_\beta \) and \( \phi_\alpha \) can be further bounded by means of (3.2) and (3.3) as follows:

\[ \int_{\Omega} \sum_{n=1}^{m} [\phi_\beta(U^n) - \phi_\beta(U^{n-1})] = \int_{\Omega} [\phi_\beta(U^m) - \phi_\beta(U^0)] \geq \]
\[ \geq \frac{1}{2L_\beta} \| U^m \|_{L^2(\Omega)}^2 - \frac{1}{2} \| U^0 \|_{L^2(\Omega)}^2. \]

and

\[ \frac{1}{\mu} \int_{\Omega} \sum_{n=1}^{m} [\phi_\alpha(U^{n-1}) - \phi_\alpha(U^n)] = \frac{1}{\mu} \int_{\Omega} [\phi_\alpha(U^0) - \phi_\alpha(U^m)] \geq \]
\[ \geq \frac{1}{2\mu} \| U^0 \|_{L^2(\Omega)}^2 - \frac{1}{2\mu} \| U^m \|_{L^2(\Omega)}^2. \]

Inserting these estimates into the previous expression and using assumption \((H_{u_0})\) leads to
\[
\sum_{n=1}^{m} \langle U^n - U^{n-1}, \varphi^n \rangle_{L^2(\Omega)}^2 + \frac{1}{4\mu} \sum_{n=1}^{m} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2.
\]

The next term on the left hand side of (3.1) provides the \(H_1\)-estimate, because Poincaré inequality holds. The first term on the right is nonpositive; indeed the orthogonality property of the \(L^2\)-projection \(P^0_h\) implies

\[
\mu \sum_{n=1}^{m} \langle [P^0_h - I] \varphi^n, \varphi^n \rangle_h = \mu \sum_{n=1}^{m} \langle [P^0_h - I] \varphi^n, [I - P^0_h] \varphi^n \rangle_h \leq 0.
\]

Let us now analyze the contribution due to the source term. In view of the definition of \(P^0_h\) and assumption (H_1) combined with the first line in (3.7) plus (3.4), we get

\[
| \sum_{n=1}^{m} \langle f(\beta(U^{n-1})), \varphi^n \rangle | = | \sum_{n=1}^{m} \langle f(\beta(U^{n-1})), P^0_h \varphi^n \rangle | \leq C + C \sum_{n=1}^{m} \| \beta(U^{n-1}) \|_{L^2(\Omega)}^2 + \frac{1}{8\mu} \sum_{n=1}^{m} \| U^n - U^{n-1} \|_{L^2(\Omega)}^2.
\]

Since the last term may be absorbed into the left, the assertion (3.6) follows as a consequence of the discrete Gronwall inequality.

**Remark 3** - From (3.6) and the first line in (3.7), it is easily seen that

\[
\max_{1 \leq n \leq N} \| P^0_h \varphi^n \|_{L^2(\Omega)} \leq C.
\]

**Remark 4** - If \(\beta\) grows at least linearly at infinity as stated in (H_1), the first term in (3.6) provides the a priori bound

\[
\max_{1 \leq n \leq N} \| U^n \|_{L^2(\Omega)} \leq C.
\]
3.2 - Stability in Maximum Norm

The present goal is to prove an a priori estimate in $L^\infty(\Omega)$. To this aim we need a further assumption upon the triangulation, say that $S_h$ is of acute type. In other words

$$\text{The projection of the vertices of any } d\text{-simplex } S \in S_h \text{ onto the hyperplane containing the opposite face lies in the closure of this face.}$$ (3.10)

Remark 5 - In 2-D the previous constraint means that internal angles of any triangle of the decomposition do not exceed $\pi/2$.

We then have the following well known discrete maximum principle for the Laplace operator [6]:

Let $x \in V_h^1$ attain its maximum at the internal node $x_j$. Then $\phi^j \in V_h^1$ be the corresponding basis function.

$$\int_{\Omega} v_x \cdot \phi^j \geq 0.$$ (3.11)

Now we are ready to prove the desired $L^\infty$-estimate.

Lemma 2 - Assume that $(H_0), (H_1), (H_\beta), (H_f), (H_{S_h})$ and (3.10) hold and in addition that

$$u_0 \in L^\infty(\Omega).$$ (3.12)

Then there exists a constant $C > 0$ independent of the approximation parameters such that

$$\max_{1 \leq n \leq N} u^n_{S_h} \leq C.$$ (3.13)
Proof. What we actually want to prove is the following estimate

$$\lim_{n \to \infty} \sup_{L(\Omega)} n \tau L \frac{1}{f} L \beta \leq c_n := \beta(0) + \frac{f_0}{L} \frac{1}{f} L \beta - 1, \text{ for } n \in \mathbb{N},$$

which obviously implies the assertion (3.13). Here $c_0$ stands for a positive constant so that $-c_0 \leq u_0 = P^0_h u_0 \leq c_0$ in $\Omega$; this is possible as a consequence of the boundedness in $L^\infty$ of the operator $P^0_h$ combined with assumption (3.12). Moreover, constant $f_0$ denotes $|f(0)|$.

The proof of (3.14) is carried out by induction. In view of (3.14), (3.13), and (3.2) the following inequalities hold in $\Omega$:

$$\beta(-c_{n-1}) \leq \beta(U^{n-1}) \leq \beta(c_{n-1})$$

$$-f_0 + L \beta(-c_{n-1}) \leq f(\beta(U^{n-1})) \leq f_0 + L \beta(c_{n-1})$$

Let $\bar{x} \in \Omega$ be a point at which $\theta^n$ attains its maximum. Since $\theta^n$ is piecewise linear, $\bar{x}$ is clearly a node of $S_h$. Let $\phi \in \phi_h$ be the corresponding shape function. Now, with the aid of the discrete equation (2.11), inequality (3.11), the definition (2.1) of the quadrature rule plus the elementary property

$$\int_{\Omega} \bar{\phi} = |\text{supp} \bar{\phi}|/(d+1),$$

we arrive at

$$\theta^n(\bar{x}) \frac{1}{d+1} \frac{|\text{supp} \bar{\phi}|}{d+1} = \langle \theta^n, \phi \rangle_h \leq \langle \beta(U^{n-1}), \phi \rangle_h + \frac{1}{\mu} \langle f(\beta(U^{n-1})), \phi \rangle_h \leq$$

$$\leq \langle \beta(c_{n-1}) + \frac{1}{\mu} [f_0 + L \beta(c_{n-1})] \rangle \frac{|\text{supp} \bar{\phi}|}{d+1}.$$

Here we have also made use of (3.15) and (3.16) to write the last inequality. The same argument produces a bound from below for $\theta^n$ when arguing with the minimum of $c^n$. Therefore
we get the estimate

\[(3.18) \quad \beta(-c_{n-1}) \left[ f_0 - L_f \beta(-c_{n-1}) \right] \leq \tau f_0 + L_f \beta(c_{n-1}), \]

Now, since

\[ U^n = U^{n-1} + \mu [P^0_n - \beta(U^{n-1})] = \alpha(U^{n-1}) + \mu P^0_n, \]

(3.17) and (3.18) yield

\[-c_{n-1} - \tau [f_0 - L_f \beta(-c_{n-1})] \leq U^n \leq c_{n-1} + \tau [f_0 + L_f \beta(c_{n-1})], \text{ in } \Omega.\]

By \((H)_\beta^1\) we can rewrite this expression as follows

\[ \|U^n\|_{L^\infty(\Omega)} \leq c_{n-1} + \tau [f_0 + L_f \max(\beta(c_{n-1}), -\beta(-c_{n-1}))] \leq \tau f_0 + c_{n-1}[1 + \tau L_f L_\beta]. \]

Finally, the claim (3.14) is an easy consequence of the inequality \(1 + \tau L_f L_\beta \leq \tau L_f L_\beta\) which leads to

\[ \tau f_0 + c_{n-1}[1 + \tau L_f L_\beta] \leq c_n, \]

where \(c_n\) was defined in (3.14).

\[ \Box \]

Remark 6 - It is easily seen from Lemma 2 that function \(\beta\) may be assumed to be only locally Lipschitz continuous, provided that \(f=0\) (see also [2, 13]). For instance, this happens for the porous medium equation for which \(\beta(s) = s|s|^{m-1}\) \((m>1)\).

Remark 7 - We now consider the particular but still relevant case \(u_0 \equiv 0\), \(f(0)=0\), which includes porous medium equations as
well as one-phase Stefan problems. Then the continuous physical unknowns $u$ and $\theta$ are clearly nonnegative. Since we wish this property to be preserved in our fully discrete scheme, let us analyze the sign of the discrete solution. For $\tau > 0$ being small enough, function $I + \frac{\tau f}{\mu}$ is nondecreasing. Therefore, assuming by induction that $U^{n-1} \geq 0$ (which holds for $n = 1$ because $U^0 = P_0^0 u_0 \geq 0$) we get $[I + \frac{\tau f}{\mu}] P_n = \beta(U^{n-1}) \geq 0$. The facts that the matrix of linear system (2.11) is an $M$-matrix and the right hand side is nonnegative imply $\theta^n \geq 0$. Finally we get $U^n = \alpha(U^{n-1}) + \mu P_n \theta^n \geq 0$, concluding the argument.

4 - ERROR ANALYSIS IN ENERGY NORMS

The primary aim of this section is to analyze the accuracy of the fully discrete nonlinear Chernoff formula (2.11), (2.12) in approximating the physical unknowns $\theta$ and $u$. The key ideas were already explained and successfully used by MAGENES, NOCHETTO & VERDI [13] in studying the discrete-time analogue of this algorithm. So the present concern is that of extending the former ideas to the fully discrete scheme ($P_{h, \tau}$). However we believe that the current results deserve a detailed presentation.

Before establishing the precise statements let us show what the general strategy is. We first define the errors $e_\theta$ and $e_u$ by

$$e_\theta(t) := \theta(t) - \theta^n, \quad e_u(t) := u(t) - U^n, \quad \text{for } t \in I^n, \quad 1 \leq n \leq N. \quad (4.1)$$

Then, after integrating (2.9) on $I^n$, we can write the set of discrete-in-time equations satisfied by the continuous solution; namely
\[(4.2) \quad <\partial u^n, \phi> + <\nabla \theta^n, \nabla \phi> = <\tilde{I}^n(\beta(u)), \phi>, \quad \forall \phi \in H^1_0(\Omega), 1 \leq n \leq N.\]

We now take the difference between (4.2) and (3.1), sum over \(n\) from 1 to \(n\leq N\) and multiply by \(\tau\). The resulting expression is the following error equation

\[(4.3) \quad <\frac{e_t}{u} - e_0, \chi> + \int u^n \Sigma \tau(\theta^n - \theta^0), \nabla \chi> = \mu[I - P_h^0] \Sigma \theta^n, \chi> + \int \Sigma \tau[f^n(\beta(u)) - f(\beta(U^{n-1}))], \chi>, \quad \forall \chi \in V_1^h.\]

The next step is to choose a suitable test function \(\chi\). Let us take \(\chi = \tau[P_h^{1-i} - \theta^0] \in V_1^h\) and sum over \(i\) from 1 to \(m\leq N\). After reordering we get

\[(4.4) \quad \sum_{i=1}^{m} \int u(t) - u_i^t, e_\theta(t)dt + \int \Sigma \tau <u^i - U^i, [I - P_h^{1-i}] e_\theta^i> + \int <u_0 - U^0, \Sigma \tau[P_h^{1-i} - \theta^0]> + \mu \Sigma \tau <[I - P_h^0] \theta^n, P_h^{1-i} e_\theta^i > + \Sigma \tau <[f^n(\beta(u)) - f(\beta(U^{n-1}))], P_h^{1-i} e_\theta^i > = III + ... + VII.\]

We now proceed to estimate each one of these terms. To begin with, notice that

\[(4.5) \quad u = \mu \theta + a(u) \quad \text{and} \quad U^n = \mu P_h^0 \theta^n + a(U^{n-1})\]

whence

\[(4.6) \quad e_u(t) = \mu e_\theta(t) + [a(u(t)) - a(U^{n-1})] + \mu[I - P_h^0] e_\theta^n, \quad \text{for} \ t \in T^N.\]

Formula (4.5) is the connection with the phase relaxation scheme [26]. Moreover, we know that
\[(4.7) \quad e_\theta(t) = [\beta(u(t)) - \beta(U^{n-1})] - \frac{1}{\mu}[U^n - U^{n-1}] - [I - P_0^h]_0^n, \text{ for } t \in \mathbb{R}^n.\]

Then, in view of (4.6) and (4.7), term I in (4.4) can be split as follows:

\[(4.8) \quad I = \mu \|e_\theta\|^2_{L^2(0, t^m; L^2(\Omega))} + \]

\[\sum_{i=1}^{m} \int_{\Omega} <a(u(t)) - a(U^{i-1}), \beta(u(t)) - \beta(U^{i-1})> dt + \]

\[\frac{1}{\mu} \sum_{i=1}^{m} \int_{\Omega} <e_\theta(t), U^{i-1} - U^{i-1}> dt - \]

\[\frac{1}{\mu} \sum_{i=1}^{m} \int_{\Omega} <u(t), U^{i-1} - U^{i-1}> dt + \sum_{i=1}^{m} \tau [I - P_0^h][U^{i-1}]_0,i > dt \]

\[\sum_{i=1}^{m} \int_{\Omega} <a(u(t)) - a(U^{i-1}), [I - P_0^h][U^{i-1}]_0,i > dt + \]

\[\mu \sum_{i=1}^{m} \int_{\Omega} <e_\theta(t), [I - P_0^h][U^{i-1}]_0,i > dt = I_1 + I_2 + \ldots + I_8.\]

The fact that both \(a\) and \(\beta\) are increasing functions results in \(I_2 \geq 0\). Term \(I_3\) is handled by means of (3.5) and (\(H_{u_0}\)); namely

\[I_3 \geq \frac{\tau}{2\mu} \sum_{i=1}^{m} \left[ \|U^{i-1}\|_{L^2(\Omega)}^2 - \|U^{i-1}\|_{L^2(\Omega)}^2 \right] \geq -\frac{\tau}{2\mu} \|U^{i-1}\|_{L^2(\Omega)}^2 \geq -C\tau.\]

For \(I_4\) we use inequalities (3.4) and (3.6) to arrive at

\[|I_4| \leq \frac{1}{4} I_1 + \tau \sum_{i=1}^{m} \frac{1}{4} \|U^{i-1}\|_{L^2(\Omega)}^2 \leq \frac{1}{4} I_1 + C\tau.\]

By virtue of approximation property (2.5) of \(P_0^h\) and a priori estimates (3.6) we have

\[(4.9) \quad \sum_{i=1}^{m} \tau [I - P_0^h][U^{i-1}]_0,i^2 \leq Ch^2.\]

Then
\[ |I_8| \leq \frac{1}{4} I_1^m + Ch^2. \]

At the same time, using (2.6) and (3.9) we get

\[ |I_7| \leq Ch \{ \|u\|_{L^2(0,t^m;L^2(\Omega))} + \left( \sum_{i=1}^{m} \| \tau U^i \|_{L^2(\Omega)} \right)^{\frac{1}{2}} \} \leq Ch; \]

and, again by (3.6), we obtain

\[ |I_6| \leq \tau \sum_{i=1}^{m} \| U^i - U^i_{1} - 1 \|_{L^2(\Omega)}^2 + Ch^2 \leq C\tau + Ch^2. \]

The remaining term \( I_5 \) will be analyzed later on according to diverse regularity assumptions upon \( u_0 \). Let us instead bound term II in (4.4). To do so we need the following elementary identity which is an easy consequence of (3.5):

\[ \sum_{i=1}^{m} \sum_{n=1}^{i} a_i^2 \sum_{n=1}^{i} a_n^2 = \sum_{i=1}^{m} a_i^2 + \sum_{i=1}^{m} a_i^2, \quad \text{for} \ a_i \in \mathbb{R}^d, \ 1 \leq i \leq N. \]

Thus, using approximation property (2.4) of the Ritz projection \( P_h^1 \) in conjunction with the regularity \( \int_{\theta \in L^\infty(0,T;H^2(\Omega))} \) which comes from integrating the original P.D.E. (2.9) in time, we easily obtain

\[ II \geq \frac{1}{2} \| \int_0^t P_h^1 e_\theta(t) dt \|_{L^2(\Omega)}^2 \geq \frac{1}{2} \| \int_0^t e_\theta(t) dt \|_{L^2(\Omega)}^2 - C h^2 =: \Pi^m - Ch^2. \]

Treating III requires a duality argument between \( H^{-1}(\Omega) \) and \( H_0^1(\Omega) \). The fact that \( e_\theta \) is uniformly bounded in \( H_0^1(\Omega) \), i.e. \( \| e_\theta \| \leq C \) which results from (2.6) and (3.6),

and again the regularity property (2.6) lead to

\[ |III| = \left| \sum_{i=1}^{m} \int_0^t \int_{\Omega} \frac{\partial u}{\partial t} ds e_\theta(t) dt \right| \leq \]

\[ \leq \tau \| \frac{\partial u}{\partial t} \|_{L^2(0,t^m;H^{-1}(\Omega))} \| e_\theta \|_{L^2(0,t^m;H_0^1(\Omega))} \leq C \tau. \]
The other term IV is analyzed by making use of the approximation property (2.4) of $P_h^1$ and of (2.6) and (3.9). Indeed we have

$$|IV| \leq C \| u \|_{L^2(0,t^m;L^2(\Omega))} + \left[ \sum_{i=1}^{m} \frac{\tau}{L^2(\Omega)} \right]^\frac{1}{2} \left( \sum_{i=1}^{m} \frac{\| (I-P_h^1) e_i \|_{L^2(\Omega)}^2}{L^2(\Omega)} \right)^\frac{1}{2} \leq C h.$$ 

Using now $(h_{u_0})$ and approximation property (2.4) we easily obtain

$$|V| \leq C h \| u_0 \|_{L^2(\Omega)} \int_0^t \| P_h^1 e_\theta(t) \|_{L^2(\Omega)}^2 \leq \frac{1}{2} h^2 + Ch^2.$$ 

In order to get a bound from above for VI we first rewrite this term in a suitable form; namely

$$VI = \mu \sum_{i=1}^{m} \sum_{n=1}^{\infty} \tau [\theta^n, P_h^1 e_\theta] + \mu \sum_{i=1}^{m} \sum_{n=1}^{\infty} \tau [\theta^n, P_h^1 e_\theta] = VI_1 + VI_2.$$ 

(4.11)

The estimate for $VI_1$ proceeds along the same lines as for III, with the only difference that now we exploit the superconvergence error estimate (2.5). Thus

$$|VI_1| \leq C h^2 \sum_{i=1}^{m} \sum_{n=1}^{\infty} \| e_i - \theta_i \|_{L^2(\Omega)}^2 \leq C \frac{h^2}{\tau},$$

(4.12)

where again the a priori estimates (2.6) and (3.6) have been used. The other term instead is an error due to the quadrature rule; so with the aid of (2.3) we can control its contribution as follows

$$|VI_2| \leq C h^2 \sum_{i=1}^{m} \sum_{n=1}^{\infty} \| e_i - \theta_i \|_{L^2(\Omega)}^2 \leq C \frac{h^2}{\tau}.$$ 

(4.13)

It only remains to estimate the source term VII, which obviously satisfies

$$VII = \sum_{i=1}^{m} \sum_{n=1}^{\infty} \tau [\theta^n (\beta(u)) - f(\beta(u^{n-1})), e_\theta(t) + [P_h^1 - I] e_i].$$

(4.14)
In view of (3.4), (H), (4.7), the approximation property (2.4) and (4.9) we can write

\[ |\mathcal{V} I| \leq \frac{1}{4} I_1^m + \| I - P_h^{1/2} \| L^2(0, T; H^1(\Omega)) + \]

\[ C \sum_{i=1}^{m} \tau \| f_n(\beta(u)) - f(\beta(U^{n-1})) \| L^2(\Omega) \]

\[ \leq C \sum_{i=1}^{m} \| I - P_h^{1/2} \| L^2(\Omega) \]

where we have also employed (2.6) and the discrete $H^{1/2}$-bound in (3.6).

Collecting all the previous estimates and inserting them into (4.4) implies:

\[ \frac{\mu}{4} \| e_0 \| L^2(0, T; H^1(\Omega)) + \frac{1}{4} \| \int_0^t e_0(t) dt \| L^2(\Omega) \]

\[ + C \sum_{i=1}^{m} \| e_i \| L^2(0, T; H^1(\Omega)) \]

\[ \leq C[\tau + h^{1/2}] + \frac{1}{\mu} \sum_{i=1}^{m} \int_0^T <u(t), u_i - u_{i-1}> dt. \]

The last term will be shown to be $O(\tau^{1/2})$, where $0 < \nu < 1/2$ depends on the regularity assumed upon $u_0$ and $\beta(u_0)$ (see Theorems below). Then, applying the discrete Gronwall inequality yields

\[ \| e_0 \| \leq \int_0^T \| e_0 \| + \int_0^T \| e_0 \| L^\infty(0, T; H^1(\Omega)) \]

\[ \leq C[\tau^{1/2} + h^{1/2}] = o(h, \tau). \]

These estimates lead to the following $H^{-1}$-error bound for the unknown $u$

\[ \| u \| L^\infty(0, T; H^{-1}(\Omega)) \leq C o(h, \tau). \]
In order to prove this result let us introduce the Green operator \( G : H^{-1}(\Omega) \rightarrow H^1_0(\Omega) \) associated with the Laplacian; so it is defined by

\[
(4.17) \quad \langle \nabla G \zeta, \nabla \phi \rangle = \langle \zeta, \phi \rangle, \quad \forall \phi \in H^1_0(\Omega), \ \zeta \in H^{-1}(\Omega).
\]

The corresponding discrete Green operator \( G_h : H^{-1}(\Omega) \rightarrow V^1_h \) is defined by

\[
(4.18) \quad \langle \nabla_h \zeta, \nabla \chi \rangle = \langle \zeta, \chi \rangle, \quad \forall \chi \in V^1_h, \ \zeta \in H^{-1}(\Omega).
\]

These operators enjoy the following properties:

\[
(4.19) \quad \| \nabla G \zeta \|_{L^2(\Omega)} \leq \| \nabla \zeta \|_{L^2(\Omega)} = \| \zeta \|_{H^{-1}(\Omega)} = \langle \zeta, G \zeta \rangle^{\frac{1}{2}},
\]

\[
(4.20) \quad \| [G-G_h] \zeta \|_{L^2(\Omega)} \leq Ch^2 \| \zeta \|_{L^2(\Omega)}.
\]

Taking \( \chi := G_h(u^i-u^i) \) as test function in (4.3) allows one to express the desired \( H^{-1} \)-bound as follows

\[
\begin{align*}
\| e^{i}_{u} \|_{H^{-1}(\Omega)}^2 &= \| u^i - u^i \|_{H^{-1}(\Omega)}^2 = \langle e^{i}_{u}, G e^{i}_{u} \rangle = \\
&= \langle e^{i}_{u}, [G-G_h] e^{i}_{u} \rangle + \langle \nabla \sum_{n=1}^{\infty} \tau[\theta^n - \theta^0], V G e^{i}_{u} \rangle + \\
&\quad + \langle \nabla [I-P^0_h] \sum_{n=1}^{\infty} \theta^n, G e^{i}_{u} \rangle + \langle e^0_u, G e^{i}_{u} \rangle + \\
&\quad + \langle \sum_{n=1}^{\infty} \tau[f^n(\beta(u)) - f(\beta(u)^n)], G e^{i}_{u} \rangle =: \text{VIII} + \ldots + \text{XII}.
\end{align*}
\]

Using the a priori estimates (2.6) and (3.9) together with (4.20), it is easily seen that \( |\text{VIII}| = O(h^2) \). In view of (4.15) and (4.19), next term is bounded by
\[ |IX| \leq \left\| \nabla \mathbf{e}^i h u \right\|_{L^2(\Omega)} \leq \frac{1}{5} \left\| e^i u \right\|_{H^{-1}(\Omega)}^2 + C_0(h, \tau). \]

Arguing as in (4.12) and (4.13) with term \( X \) gives

\[ |X| \leq Ch^2 \sum_{n=1}^{\infty} \left\| \nabla g \right\|_{H^1(\Omega)} \left\| \mathbf{e}^i h u \right\|_{L^2(\Omega)} \leq \frac{1}{5} \left\| e^i u \right\|_{H^{-1}(\Omega)}^2 + C_0(h, \tau). \]

because we can assume without loss of generality that \( \sigma(h, \tau) \leq 1, \) thus \( h^2/\tau \leq 1. \) By virtue of (\( H_0^1 \)) and (2.4), term XI is easily bounded by

\[ \left| \left< e^0 u, G^i h u \right> \right| \leq \frac{1}{5} \left\| e^i u \right\|_{H^{-1}(\Omega)}^2 + Ch^2. \]

The same technique applied in (4.14) can now be used combined with Poincaré inequality and (3.6) to arrive at

\[ |XII| \leq C \left\| \nabla g \right\|_{H^1(\Omega)} \sum_{n=1}^{\infty} \int_0^t \beta(u(t)) - \beta(u^{n-1}) \right\|_{L^2(\Omega)} \frac{dt}{tn} \leq \frac{1}{5} \left\| e^i u \right\|_{H^{-1}(\Omega)}^2 + C_0(h, \tau). \]

Concluding we have proved the error bound

\[ \max_{1 \leq i \leq N} \left\| u^i - u^i \right\|_{H^{-1}(\Omega)} \leq C_0(h, \tau). \]

The assertion (4.16) follows from the fact that \( u \in H^1(0,T; H^{-1}(\Omega)) \subset C^0, \frac{1}{2}(0,T; H^{-1}(\Omega)). \)

Now we are in a position to state the main result of this paper.

**Theorem 1** - Let \( (H_0^1), (H_1^\beta), (H_f) \) and \( (H_h^S) \) hold. Assume that

\[ (4.21) \quad u_0 \in L^\infty(\Omega), \quad \Delta \beta(u_0) \in L^1(\Omega) \]

and in addition that
\begin{equation}
\max_{1 \leq n \leq N} \| u^n \|_{L^\infty(\Omega)} \leq C.
\end{equation}

Let \( \tau \) be chosen so that \( \tau = C^* h \) for \( C^* \) being an arbitrary positive constant. Then we have

\begin{equation}
\| e^\theta \|_{L^2(Q)} + \int_0^t \| e^\theta \|_{L^\infty(0,T;H^1(\Omega))} + \| u \|_{L^\infty(0,T;H^{-1}(\Omega))} = O(h^{\frac{1}{2}}).
\end{equation}

**Remark 8** - The a priori estimate (4.22) holds, for instance, provided that the triangulation \( S_h \) is of acute type (see Lemma 2).

**Remark 9** - The assumption \( \Delta \beta(u_0) \in L^1(\Omega) \) can be slightly weakened according to Remark 1, because the only property we need is \( \frac{\partial u}{\partial t} \in L^\infty(0,T;M(\Omega)) \).

**Proof of Theorem 1.** In view of estimates (4.15) and (4.16) it remains to demonstrate the bound

\begin{equation}
\left| \sum_{i=1}^m \int_{I_i} <u(t),u^i-u^{i-1}> dt \right| \leq C \tau.
\end{equation}

To this aim we need the following summation by parts formula

\[
\sum_{i=1}^m a_i[b_i - b_{i-1}] = a_m b_m - a_0 b_0 - \sum_{i=1}^m b_{i-1}[a_i - a_{i-1}].
\]

We can then rewrite the previous term as follows

\[
\sum_{i=1}^m \int_{I_i} <u(t),u^i-u^{i-1}> dt = \sum_{i=1}^m \tau <\tilde{u}^i, u^i-u^{i-1}> = \tau <\tilde{u}^m, U^m> - \tau <u_0, U^0> - \sum_{i=1}^m \tau <\tilde{u}^i - \tilde{u}^{i-1}, u^{i-1}>,
\]

where \( \tilde{u}^0 := u_0 \). The assumption \( u_0 \in L^\infty(\Omega) \) together with (4.22) and (2.6) implies that the first and middle terms are \( O(\tau) \).

For the last term we note that
\[
-u^{-i-1} = \frac{1}{\tau} \int \int [u(t) - u(t-\tau)] dt = \frac{1}{\tau} \int \int_{t-\tau}^{t} \frac{\partial u}{\partial s}(s) ds dt
\]
and the fact that \( \frac{\partial u}{\partial s} \in L^{\infty}(0,T;M(\Omega)) \) because \( \Delta \beta(u_0) \in L^{1}(\Omega) \) (see (R2)). So what we have in mind is to use a duality argument between \( M(\Omega) \) and \( C^{0}(-\Omega) \), which unfortunately fails because \( U^{i} \not\in C^{0}(-\Omega) \). The remedy consists of regularizing \( U^{i} \), say by convolution, in such a way that we get continuous functions \( U^{i}_{\varepsilon} \) having the properties:

\[
\max_{1 \leq i \leq N} \| U^{i}_{\varepsilon} \|_{L^{\infty}(\Omega)} \leq C,
\quad U^{i}_{\varepsilon} \to U^{i} \quad \text{strongly in } L^{2}(\Omega).
\]

Thus

\[
|\langle u^{-i-1}, U^{-i-1} \rangle | \leq \| u^{-i-1} \|_{\infty} \| U^{-i-1} - U_{\varepsilon}^{-i-1} \|_{L^{2}(\Omega)} \leq C_{i} + C \| u_{\varepsilon} \|_{L^{\infty}(0,T;L^{2}(\Omega))} \| U^{-i-1} - U_{\varepsilon}^{-i-1} \|_{L^{2}(\Omega)}
\]

Finally, taking the limit as \( \varepsilon \to 0 \) and using the a priori estimate (2.6) yields the assertion (4.24).

Let us now extend the previous error estimate to a weaker situation.

**Theorem 2** Let \( (H^{0}_{h}), (H^{1}_{h}), (H^{2}_{h}) \) and \( (H^{3}_{h}) \) hold. Let \( \tau \) be chosen so that \( \tau = C*h^{4/3} \) for any positive constant \( C* \). Then we have

\[
(4.25) \quad \| e^{\theta} \|^2 + \int_{0}^{\tau} \| e^{\theta} \|^2 + \| e^{\theta} \|^2 \| L^{\infty}(0,T;L^{2}(\Omega)) \| = O(h^{1/3}).
\]

**Proof.** The present task is that of proving the estimate

\[
(4.26) \quad \| \sum_{i=1}^{m} \langle u^{-i}, U^{-i} - U^{-i-1} \rangle \| \leq C_{i}^{\frac{1}{2}}.
\]
which implies that $\nu = 1/4$ in (4.15). Indeed, the relation between $\tau$ and $h$ as well as the final rate of convergence are trivial consequences of (4.15) and (4.16). In order to derive (4.26) we make use of the a priori estimates (2.6) and (3.6) which lead to

\[
\sum_{i=1}^{m} \left| \sum_{j<i} u_{j} u_{i} u_{i-1} \right| \leq C \tau^{i} \| u \|_{L^2(0,T;L^2(\Omega))} \quad \sum_{i=1}^{m} \left| u_{i} - u_{i-1} \right|^{2} \leq C \tau^{i/2},
\]

and complete the theorem.

As one can easily check, the dominating terms on the right hand side of (4.15) are $\tau^{1/4}$ and $h^{1/4}$ rather than $h^{1/2}$. If we are interested in getting a better balance between these terms we can simply modify the discrete scheme as follows. Let us now look for functions $U_{n} \in V_{h}^{0}$ and $\phi_{n} \in V_{h}^{1}$ ($1 \leq n \leq N$) so that for all $\chi \in V_{h}^{1}$ we have

\[
U^{0} := P_{h}^{0} u_{0}, \quad \phi^{0} := P_{h}^{0} \beta(u_{0})
\]

(4.27) \( \langle \phi^{n}, \chi \rangle_{h} + \sum_{i=1}^{m} \langle \phi^{n}, \nabla \chi \rangle_{h} = \langle \beta(U^{n-1}) + \frac{1}{2} f(\beta(U^{n-1})), \chi \rangle_{h} + \langle [I-P_{h}^{0}]\phi^{n-1}, \chi \rangle_{h} \)

\[
U_{n} := U^{n-1} + h \mu \sum_{n=1}^{N} \langle [P_{h}^{0}I] \phi^{n-1}, \phi^{n} \rangle_{h}.
\]

This scheme is still stable; indeed the proof follows the same lines as that of Lemma 1 except for the term in (3.8) which is replaced by

\[
\sum_{n=1}^{N} \langle [P_{h}^{0}I][\phi^{n-1} - \phi^{n}], \phi^{n} \rangle_{h} = \mu \sum_{n=1}^{N} \langle [P_{h}^{0}I][\phi^{n-1} - \phi^{n}], [I-P_{h}^{0}]\phi^{n} \rangle_{h} = \mu \sum_{n=1}^{N} \langle [P_{h}^{0}I][\phi^{n-1} - \phi^{n}], [P_{h}^{0}I] \phi^{n} \rangle_{h}
\]

\[
= - \frac{\mu}{2} \langle [P_{h}^{0}I] \phi^{m}, [P_{h}^{0}I] \phi^{m} \rangle_{h} + \mu \langle [P_{h}^{0}I] \phi^{0}, [P_{h}^{0}I] \phi^{0} \rangle_{h}
\]

\[
- \frac{\mu}{2} \sum_{n=1}^{N} \langle [P_{h}^{0}I][\phi^{n-1} - \phi^{n}], [P_{h}^{0}I][\phi^{n-1} - \phi^{n}] \rangle_{h} \geq 0,
\]

where we have used the equality (3.5).
Finally, we have the following improved result which reproduces for the fully discrete Chernoff approximation that one already shown by VERDI & VISINTIN for the phase relaxation scheme [26].

**Theorem 3** Let \( (H_n), (H_\beta), (H_f) \) and \( (H_{\Sigma}) \) hold. Let \( \{\theta^n, U^n\}_{n=1}^N \) stand for the discrete solutions associated with the scheme (4.27). Then, for \( \tau = C*h^2 \) where \( C* \) denotes any positive constant, we get

\[
(4.28) \quad \|e_0\|_{L^2(Q)} + \int_0^t \|e_\theta\|_{L^2(0,T;H^1(\Omega))} + \|e_u\|_{L^\infty(0,T;H^{-1}(\Omega))} = O(h^{3/2}).
\]

**Proof.** We only have to estimate the contribution due to term VI in (4.4), which now becomes

\[
VI = \mu \sum_i \tau \langle [I-P_h^0] \theta, P_h^{1-i} \theta \rangle_h = \mu \sum_{i=1}^m \tau \langle [I-P_h^0] \theta, P_h^{1-i} \theta \rangle_h + \mu \sum_{i=1}^m \tau \langle \theta, P_h^{1-i} \theta \rangle_h - \langle \theta, P_h^{1-i} \theta \rangle_h = VI_1 + VI_2.
\]

Then arguing similarly to (4.12) and (4.13) yields

\[
|VI_1|, |VI_2| \leq Ch^2 \sum_{i=1}^{m} \tau \|\theta-i\|_{L^2(\Omega)} \leq Ch^2, \quad \|\theta-i\|_{L^2(\Omega)} \leq Ch^2.
\]

which implies the desired result.

Let us conclude this section by making some comments.

**Remark 10** - From equation (2.12) and the a priori estimates (3.6) it follows that

\[
\sum_{n=1}^{N} \tau \|P_h^{0} U^n - \beta(U^n)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{N} \|\beta(U^n) - \beta(U_{n-1}^n)\|_{L^2(\Omega)}^2 \leq C_1.
\]
By using (4.9) we get \( \| e_\beta(u) \|_{L^2(Q)} \leq C_0(h, \tau) \), where \( e_\beta(u)(t) := \theta(t) - \beta(u^n) \) for \( t \in I^n \).

**Remark 11** - The error estimates shown above hold also for different situations. Indeed, assume that \( \beta \in C^{1,1} \) (\( \beta \) is no longer convex!) and that either a nonhomogeneous Dirichlet condition or a linear flux condition is imposed on \( \partial \Omega \), i.e.

\[
(4.29) \quad \frac{\partial}{\partial \nu} \beta(u) + p(x) \beta(u) = g(x, t) \quad \text{a.e. on } \partial \Omega,
\]

where \( 0 \leq p(x) \leq P \) for a.e. \( x \in \partial \Omega \). In this case the domain cannot be decomposed exactly and one is forced to analyze the layer \( \Omega \setminus \Omega_h \). The appropriate techniques to handle such a case were presented by NOCHETTO & VERDI in [20] and are omitted here.

**Remark 12** - Assume that a mixed boundary condition is imposed on the boundary. Thus there is a lack of regularity which makes the standard \( L^2 \)-duality argument fail. However, the error bound \( \| (I-P_h)z \| \leq C h^{1/2 - \delta} \| z \|_{H^1_0(\Omega)} \) still holds. The current analysis applies in this case provided the surface of separation between Dirichlet and Neumann conditions is regular enough, giving in the weakest situation of Theorem 2 a rate of convergence \( O(h^{1/4 - \delta}) \) under the mild relation \( \tau = C h \).

5 - **NUMERICAL RESULTS**

In this section we first describe the implementation of our scheme and then the results of some numerical tests. These experiments were performed in collaboration with G. Sacchi. Further discussions comparing performances of the present algorithm and other techniques in approximating both solutions and interfaces will appear elsewhere. [21].
5.1 - Implementation of the scheme

To begin with, we rewrite the discrete problem in matrix form. Denoting by \( \{\phi^j_h\}_{j=1}^J \) and \( \{\psi^k_h\}_{k=1}^K \) the canonical basis of \( V^1_h \) and \( V^0_h \) respectively, we define the following matrices

\[
M := \{(\phi^j_h, \phi^l_h)_{j,l=1}^J\}, \quad K := \{(\psi^j_h, \psi^l_h)_{j,l=1}^J\}, \quad A := M + \frac{1}{\mu} K,
\]
\[
P := \{(\phi^j, \psi^k)_{j=1}^J, k=1}^K\).
\]

Moreover, let us denote by \( y_k \) the barycenter of \( S_k \subseteq S_h \) and by \( \{x_j\}_{j=1}^J \) the nodes of \( S_h \). For any \( \phi \in V^1_h \) and \( \psi \in V^0_h \) we also denote by \( \phi = \{\phi_j\}_{j=1}^J \) and \( \psi = \{\psi_k\}_{k=1}^K \) the vectors with the nodal values of \( \phi \) and the barycentral values of \( \psi \), respectively.

Setting \( B^{n-1} := \sum_{k=1}^K \beta(U_k^{n-1}) \psi^k \) and \( F^{n-1} := \sum_{k=1}^K f(\beta(U_k^{n-1})) \psi^k \), and noting that \( P_0^h \phi = \sum_{k=1}^K \phi(y_k) \psi^k \) for \( \phi \in V^1_h \), the discrete problem \( (P_h, \tau) \) can be equivalently written as follows

\[
(5.1) \quad A^n = P[\mu F^{n-1} + \frac{1}{\mu} P^{n-1}]
\]
\[
(5.2) \quad U_k^n = U_k^{n-1} + \mu[\phi^n(y_k) - \beta(U_k^{n-1})], \quad k=1, \ldots, K.
\]

Since \( A \) is a symmetric and positive definite matrix, linear system (5.1) admits one and only one solution. The crucial point in the performance of the method is the efficient resolution of linear system (5.1) because equation (5.2) is just a set of scalar algebraic corrections.
Our algorithm has been implemented for general polygonal domains in two dimensions, by using an automatic grid code to decompose the domain into triangles. A Cholesky factorization of the matrix $\mathbf{A}$ is recommended whenever the band-width is "small"[1]. Indeed, since matrix $\mathbf{A}$ remains unchanged at each time step, the factorization is made once at the beginning. Nevertheless, when the use of an automatic grid code is required, for instance in decomposing a general domain, the band-width might be "large". Then iterative methods seem to perform better than direct ones. In particular, since $\mathbf{A}$ is positive definite, an incomplete Cholesky factorization for preconditioning matrix $\mathbf{A}$ combined with conjugate gradient iterations are recommended for solving linear system (5.1)[1, 15;21]. Moreover, matrix $\mathbf{A}$ is actually strictly diagonally dominant in the experiments below, making easier the preconditioning [1,15;21].

Let us now turn our attention to the approximation of the free boundary. The location of this crucial unknown as zero level set of the solution $\{\theta^n\}_{n=1}^N$ is rough for two-phase problems and simply impossible for one-phase problems. This is due to the behavior of operator $[I - \frac{1}{\mu}]^{-1}$ which smooths out singularities or, in other words, which replaces a movement of the interface at finite velocity by a diffusion at infinite speed. Moreover, the higher the interface velocity is the bigger is the artificial diffusion added by $[I - \frac{1}{\mu}]^{-1}$. This heuristic explanation tell us two things. First, we have to seek the discrete interface as a $\sigma$-level set for $\sigma \neq 0$; this is in the spirit of [19]. Second, the level $\sigma$ should be related to the expected speed of the free boundary rather than being kept fixed.

5.2 - Numerical tests

The goals of our numerical experiments are to evaluate the actual order of convergence of both variables $u$ and $\theta$ as
well as the approximation of the free boundary. If \( u^n \) and \( \theta^n \) are the discrete solutions associated with a mesh-size \( h \), we set \( E^h := \{ \tau \sum_{n=1}^{N} \sum_{k=1}^{K} \int_{S_k} \Pi_h ([\theta^n - \theta^m]^2) \, dx \}^{1/2} \) and analogously for \( E^h_u \), with obvious changes where \( u \) is discontinuous. Then assuming the relations \( E^h_u = C^h \Pi_u \) and \( E^h_\theta = C^h \Pi_\theta \) it follows that

\[
P_u = \frac{h_1}{\log(h_1/h_2)}, \quad P_\theta = \frac{h_1}{\log(h_1/h_2)}.
\]

The space domains in all examples below are rectangles, which are decomposed uniformly. The discrete initial data are chosen according to Remark 2. Examples 1, 2, 3 are concerned with the two-phase Stefan problem for which function \( \beta \) is:

\[
\beta(u) = \begin{cases} 
  c_1 u & \text{if } u < 0, \\
  0 & \text{if } 0 \leq u \leq 1, \\
  c_2 (u - 1) & \text{if } u > 1.
\end{cases}
\]

Example 4 is the Barenblatt-Pattle solution of the porous medium equation for \( \beta(u) = |u| \). We also point out that, even if mixed boundary conditions are prescribed in our examples, both approximation property (2.4) and error estimates (4.23) still hold; indeed, the regularity of the associated Green operator follows from applying reflection techniques. All experiments were performed on an IBM 4361, using the relation \( \tau = C^* h \).

**Notation:**
- \( N \): number of time steps,
- \( N_t \): number of triangles,
- \( N_n \): number of nodes,
- \( CPU \): CPU time in seconds.

**Example 1** ([2, 20, 24]) - Let \( \Omega = [0, 0.5] \times [0, 0.25], \quad 0 < t < T = 0.25; \)

\( c_1 = c_2 = 1 \). The exact enthalpy is

\[
u(x, y, t) = \begin{cases} 
  2[e^{\phi(x, y, t)} - 1] + 1, & \text{if } \phi(x, y, t) \geq 0, \\
  e^{\phi(x, y, t)} - 1, & \text{if } \phi(x, y, t) < 0.
\end{cases}
\]
where \( \phi(x,y,t) := -x - y + 2t + 0.1 = 0 \) is the free boundary. Neumann conditions are assigned on the sides \( x = 0.5 \) and \( y = 0.25 \), and Dirichlet data are prescribed on the sides \( x = 0 \) and \( y = 0 \). We obtained the results summarized in Table 1 and Figures 1 and 2.

<table>
<thead>
<tr>
<th>Nel</th>
<th>Nn</th>
<th>N</th>
<th>( E_u \cdot 10^2 )</th>
<th>( E_\theta \cdot 10^3 )</th>
<th>( p_u )</th>
<th>( p_\theta )</th>
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<tbody>
<tr>
<td>100</td>
<td>66</td>
<td>25</td>
<td>5.29</td>
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<td>0.62</td>
<td>9</td>
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<td>196</td>
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<td>4.77</td>
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<td>30</td>
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<tr>
<td>400</td>
<td>231</td>
<td>50</td>
<td>4.48</td>
<td>3.88</td>
<td>0.25</td>
<td>0.58</td>
<td>107</td>
</tr>
<tr>
<td>900</td>
<td>496</td>
<td>75</td>
<td>4.05</td>
<td>3.06</td>
<td></td>
<td></td>
<td>457</td>
</tr>
</tbody>
</table>

**Example 2** ([20, 24]): Let \( \Omega := -0.22, 0.18[x, 0, 0.2], \quad 0 < t < T := 0.4; \)
\( C_1 = 1/2, \quad C_2 = 1/3 \). The exact enthalpy is

\[
\begin{align*}
  u(x,y,t) := & \begin{cases} 
    6\phi(x,y,t) + 1, & \text{if } \phi(x,y,t) \geq 0 \\
    2\phi(x,y,t), & \text{if } \phi(x,y,t) < 0 
  \end{cases} 
\end{align*}
\]

where \( \phi(x,y,t) := x^2 + y^2 - \frac{e^{-4t}}{4e^{2.4}} = 0 \) is the interface. Dirichlet data are assigned on the sides \( x = -0.22, x = 0.18 \) and \( y = 0.2 \), and a vanishing flux condition is prescribed on \( x = 0 \). Table 2 contains the computational results. Discrete and continuous isothermal curves are shown in Figures 3-5, as well.

<table>
<thead>
<tr>
<th>Nel</th>
<th>Nn</th>
<th>N</th>
<th>( E_u \cdot 10^2 )</th>
<th>( E_\theta \cdot 10^4 )</th>
<th>( p_u )</th>
<th>( p_\theta )</th>
<th>CPU</th>
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<td>64</td>
<td>45</td>
<td>20</td>
<td>5.51</td>
<td>12.46</td>
<td>0.28</td>
<td>0.44</td>
<td>4</td>
</tr>
<tr>
<td>144</td>
<td>91</td>
<td>30</td>
<td>4.93</td>
<td>10.43</td>
<td>0.16</td>
<td>0.52</td>
<td>16</td>
</tr>
<tr>
<td>324</td>
<td>190</td>
<td>45</td>
<td>4.62</td>
<td>8.46</td>
<td></td>
<td></td>
<td>67</td>
</tr>
<tr>
<td>784</td>
<td>435</td>
<td>70</td>
<td>4.06</td>
<td>7.00</td>
<td>0.29</td>
<td>0.43</td>
<td>328</td>
</tr>
</tbody>
</table>
Example 3 ([20,26]) - Let \( \Omega = ]0,0.85[ ]0,0.1[, \ 0 < t < T = 1; \ C_1 = C_2 = 1. \) The exact enthalpy, which does not depend on \( y, \) is

\[
u(x,y,t) := \begin{cases} 
4[s_2(t)-x] & \text{if } x \leq s_2(t), \text{ if } 0 < t \leq 0.25; \\
-[x-s_2(t)]^2 & \text{if } x > s_2(t) \\
&s_1(t)^2-x^2+1 & \text{if } x \leq s_1(t) \\
4[s_2(t)-x] & \text{if } s_1(t) < x \leq s_2(t), \text{ if } 0.25 < t \leq 0.75; \\
-[x-s_2(t)]^2 & \text{if } x > s_2(t) \\
&s_1(t)^2-x^2+1 & \text{if } x \leq s_1(t), \text{ if } 0.75 < t < 1 \\
-[x-s_1(t)]^2-2[x-s_1(t)][t-0.75] & \text{if } x > s_1(t)
\end{cases}
\]

where \( s_1(t) := t - 0.25 \) and \( s_2(t) := 0.5[t+0.25] \) determine the mushy region. A Dirichlet condition is assigned on \( x = 0.85 \) and homogeneous Neumann conditions are imposed on the sides \( x = 0 \) and \( y = 0, \ y = 0.1. \) We obtained the results contained in Table 3.

<table>
<thead>
<tr>
<th>Nel</th>
<th>Nn</th>
<th>N</th>
<th>( E_u \cdot 10^2 )</th>
<th>( E_\theta \cdot 10^3 )</th>
<th>( p_u )</th>
<th>( p_\theta )</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>26</td>
<td>31</td>
<td>5.38</td>
<td>8.30</td>
<td>0.35</td>
<td>0.79</td>
<td>4</td>
</tr>
<tr>
<td>96</td>
<td>75</td>
<td>62</td>
<td>4.22</td>
<td>4.81</td>
<td>0.33</td>
<td>0.78</td>
<td>42</td>
</tr>
<tr>
<td>384</td>
<td>245</td>
<td>124</td>
<td>3.35</td>
<td>2.81</td>
<td>0.31</td>
<td>0.75</td>
<td>439</td>
</tr>
<tr>
<td>1536</td>
<td>873</td>
<td>248</td>
<td>2.69</td>
<td>1.66</td>
<td></td>
<td></td>
<td>4581</td>
</tr>
</tbody>
</table>

The evolution in time of discrete and continuous free boundaries is shown in Figure 6, whereas discrete enthalpies for several time steps are drawn in Figure 7 as functions of the space variable.

Example 4 - Let \( \Omega = ]2,3[ ]0,0.1[, \ 0 < t < T = 1. \) The exact density is

\[
u(x,y,t) := \frac{1}{s(t)} \left[1 - \left(\frac{x}{s(t)}\right)^2\right] \text{ if } x < s(t), \quad := 0 \text{ if } x \geq s(t),
\]
where \( s(t) = [12(t+1)]^{1/3} \) determines the interface. The solution does not depend on \( y \); hence we prescribe a homogeneous flux on \( y = 0 \) and \( y = 0.1 \). Dirichlet data are imposed on \( x = 2 \) and \( x = 3 \). Table 4 sums up the computational results.

<table>
<thead>
<tr>
<th>Nel</th>
<th>Nn</th>
<th>N</th>
<th>( E_u \cdot 10^3 )</th>
<th>( E_\theta \cdot 10^4 )</th>
<th>( p_u )</th>
<th>( p_\theta )</th>
<th>CPU</th>
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</thead>
<tbody>
<tr>
<td>40</td>
<td>33</td>
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<td>6.97</td>
<td>8.43</td>
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<td>0.70</td>
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<tr>
<td>90</td>
<td>64</td>
<td>30</td>
<td>5.51</td>
<td>6.36</td>
<td>0.66</td>
<td>0.79</td>
<td>26</td>
</tr>
<tr>
<td>250</td>
<td>156</td>
<td>50</td>
<td>3.92</td>
<td>4.38</td>
<td>0.69</td>
<td>0.85</td>
<td>162</td>
</tr>
<tr>
<td>640</td>
<td>369</td>
<td>80</td>
<td>2.84</td>
<td>2.93</td>
<td>0.69</td>
<td>0.85</td>
<td>825</td>
</tr>
</tbody>
</table>

Figures 8 and 9 illustrate the evolution of the true and approximate interfaces as well as of the density \( u \) for various time steps.

The actual rates of convergence of the variable \( \theta \) agree with the theoretical one shown in Theorem 1. Moreover the algorithm seems to approximate the variable \( u \) in \( L^2(Q) \), as well. Even though this is not suggested by theoretical results the observed rate is almost \( O(h^{1/4}) \) for the Stefan problem.

The numerical evidence indicates that the present algorithm is not capable of locating interfaces without any further treatment of the discrete solution; this is due to the artificial diffusion added by operator \( [I - \frac{T}{\mu} \Delta ]^{-1} \). Since the location of level sets away from the free boundary is quite precise, the correct remedy might be a sort of local postprocessing of the discrete solution; this is the subject of current research. However, problems having a smooth free boundary which moves along a prescribed direction suggested by the initial datum can be simply handled as follows. The discrete free boundary at time \( t^n \) is
to be sought as $\sigma_n$-level set of variable $\Theta^n$, where $\sigma_n$ depends on
the discrete interface velocity. More precisely, we propose the empirical relation

$$
\sigma_n := \text{Ch}^{1/2} \text{ for } n = 1, 2 ; \quad \sigma_n := \frac{\theta_{s^n-1}}{\theta_{s^{n-2}} \sigma_{n-1}} \text{ for } 3 \leq n \leq N
$$

where $\theta_{s^n}$ denotes the average change of the discrete interface location from time $t^{n-1}$ to $t^n$ and $C > 0$ is a constant to be determine according to the initial velocity.

We finally present some pictures illustrating the success of the above trick to approximate the exact free boundaries of our four numerical tests. Discrete and continuous isothermal curves are depicted, as well.
EXAMPLE 1: $N_{e1}=900$, $N_n=496$, $N=75$;

$h=2.357\times10^{-2}$, $\tau=3.333\times10^{-3}$, $\sigma_1=\sigma_2=3.93\times10^{-2}$.

Figure 1: Discrete and continuous (dotted lines) interfaces at times $t=0.05$, $0.1$, $0.15$, $0.2$, $0.25$.

Figure 2: Discrete and continuous (dotted lines) interfaces and $\sigma$-isothermal curves for $t=0.15$ and $\sigma=-0.1$, $-0.2$, $0.2$, $0.5$. 
EXAMPLE 2: \(N_{el}=784, \ N_{n}=435, \ N=70;\)
\[h=2.02 \times 10^{-2}, \ \tau=5.71 \times 10^{-3}, \ \sigma_1=\sigma_2=2.02 \times 10^{-2}.\]

Figures 3-5: Discrete and continuous (dotted lines) interfaces and \(\sigma\)-isothermal curves for \(t=0.08, 0.24, 0.4\) and \(\sigma=0.03, 0.06\).
EXAMPLE 3: $N_e=1536$, $N_n=873$, $N=248$;

$h=1.53 \times 10^{-2}$, $\tau=4.03 \times 10^{-3}$, $\sigma_1^1=\sigma_2^1=2.04 \times 10^{-2}$, $\sigma_1^2=\sigma_2^2=0$.

Figure 6: Discrete and continuous (dotted lines) interfaces for $y=\text{constant}$.

Figure 7: Discrete enthalpy for $t=0., 0.25, 0.5, 0.75, 1.$ and $y=\text{constant}$.
EXAMPLE 4: \( \text{Ne} = 640, \text{Nn} = 369, \text{N} = 80; \)

\[ h = 2.79 \times 10^{-2}, \tau = 1.25 \times 10^{-2}, \sigma_1 = \sigma_2 = 2.54 \times 10^{-3}. \]

Figure 8: Discrete and and continuous (dotted line) interfaces for \( y = \text{constant}. \)

Figure 9: Discrete and continuous (solid lines) density for \( t = 0.0, 0.25 \]
\( 0.5, 0.75, 1. \) and \( y = \text{constant}. \)


