THE CONVERGENCE ON THE MULTIGRID ALGORITHM FOR NAVIER-STOKES EQUATION

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THE CONVERGENCE ON THE MULTIGRID ALGORITHM FOR NAVIER - STOKES EQUATIONS

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Abstract

This paper deals with a multigrid algorithm for the numerical solution of Navier-Stokes problems. The main difficulties of the multigrid solution, besides the indefiniteness of the stiffness matrix of the corresponding discrete problems, are the lack of regularity of the solution of continuous problem and the choice of an appropriate smoother. In this paper, we give the convergence proof and the estimate of the contraction number of the multigrid algorithm.

1. Introduction

Multigrid method is a new that for solving the numerical solutions of elliptic differential equations. It briefly consists of smoothing process and coarse-grid correction procedure such that the operation work is decreased and the convergence rate improved. Multigrid Method for solving the large systems of linear equations which arise in the numerical solution of boundary value problems by finite elements, has been discussed by many authors, e.g. Astrachancev [1], Nicolaides [10], Bank and Dupont [12], Hackbusch [6],[7], Wesseling [12] and Verfuth [11]. In this paper, we discuss the convergence properties of the multigrid algorithm for nonlinear Navier-Stokes problem:

\[-\mu \nabla u + u \cdot \nabla u + \nabla p = f, \text{ in } \Omega\]
\[
\text{div } u = 0, \text{ in } \Omega
\]
\[
u = 0, \text{ on } \partial \Omega
\]

where \( u = (u_1, u_2, \ldots, u_d) \) and \( p \) are the velocity and the pressure of fluid respectively, \( u \) is its viscosity, \( \Omega \subseteq \mathbb{R}^d \) a sufficiently smooth domain, and

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\[ \mathbf{u} \cdot \mathbf{u} = \left( \sum_{j=1}^{d} u_j \frac{\partial u_j}{\partial x_j} \right) i=1,...,d \ . \]

The general structure of our convergence analysis for the multigrid procedure is similar to that of Hackbusch [6],[7], and the convergence result of the multigrid algorithm for Navier-Stokes equations is based on the convergence theorem of nonlinear multigrid methods [7]. It is known that the main sufficient conditions of the convergence of multigrid method are the smoothing properties and the approximation properties. Therefore in this paper, we first prove the smoothing properties in some discrete norms, and then give the proof of approximation property from the usual approximation assumptions in terms of Sobolev spaces, and finally discuss the convergence of multigrid algorithm and estimate the contraction number under general assumptions. We have to consider the non-linearity of equation (1.1) and the different orders of differentiability of \( u \) and \( p \) in (1.1). This will be compensated for by considering the nonlinear multigrid methods inside the neighborhood of solution \([u^*,p^*]\) and by introducing mesh-dependent norms. To simplify the analysis we present a smoothing procedure which is related to the Jacobi iteration for scalar problems [7].

2. Preliminaries

Consider Navier-Stokes equation (1.1) in a smooth enough domain \( \Omega \) and its variational formulation:

Find \([u,p] \in H^1_0(\Omega)^d \times L^2_0(\Omega)\) such that

\[
\begin{aligned}
a(u;u,v) + b(v,p) &= (f,v)_0, \quad V \ v \in H^1_0(\Omega)^d. \\
b(u,q) &= 0, \quad V \ q \in L^2_0(\Omega). 
\end{aligned}
\] (2.1)

here, \( a \) and \( b \) are trilinear bilinear form, respectively:

\[
\begin{aligned}
a(w;u,v) &= a_0(u,v) + a_1(w;u,v), \\
a_0(u,v) &= \nabla \cdot (\nabla \cdot \nabla u, \nabla v)_0, \\
a_1(w;u,v) &= \sum_{j=1}^{d} \int_{\Omega} w_j \frac{\partial u_j}{\partial x_j} v_i \ dx = ((w \nabla u) \cdot v_i, \ \\
b(u,p) &= -(\text{div} \ u, p) \ , \ L^2_0(\Omega) = \{ p \in L^2(\Omega) : \int_{\Omega} p \ dx = 0 \}.
\end{aligned}
\]
and \((\ldots)\) denotes the scalar product in \(L^2(\Omega)^m = L^2(\Omega) \times \ldots \times L^2(\Omega)\) (\(m\) positive integer).

Let

\[
X = H^1_0(\Omega)^d, \quad Y = L^2_0(\Omega), \quad Z = X \times Y
\]

The elements of them are denoted by \(u, v, \ldots, p, q, \ldots\), and \([u, p], [v, q], \ldots\), resp. They are equipped with the norms \(\| \cdot \|_0, \| \cdot \|_1\), and

\[
\| [u, p] \|_{1,0} = \left( \| u \|_1^2 + \| p \|_0^2 \right)^{1/2},
\]

respectively. If \(\Omega \subset \mathbb{R}^d\) (\(d = 2, 3\)) is simply connected, bounded domain, or the boundary \(\partial \Omega\) sufficiently smooth, \(a\) and \(b\) satisfy following continuity, coercivity and Brezzi's conditions [5]:

\[
\begin{align*}
(2.2a) \quad |a(w; u, v)| &< A \| w \|_1 \| u \|_1 \| v \|_1, \quad \forall \ w, u, v \in X \\
(2.2b) \quad a(w; v, v) &> \alpha \| v \|_1, \quad \alpha > 0, \ \forall w, v \in X
\end{align*}
\]

Where \(\alpha\) is independently of \(w, v\).

\[
\begin{align*}
(2.2c) \quad |b(u, p)| &< B \| u \|_1 \| p \|_0, \quad \forall u \in X, \ p \in Y \\
(2.2d) \quad \sup_{u \in X, \ p \neq 0} \frac{b(u, p)}{\| u \|_1 \| p \|_0} &> \beta \| p \|_0, \quad \beta > 0, \ \forall p \in Y
\end{align*}
\]

In addition we assume the following regularity assumptions of problem (1.1) & the corresponding duality problem: If \(f \in L^2(\Omega)^d\), then \([u, p], [w, q] \in H^2_0(\Omega)^d \times H^1(\Omega)\) with

\[
\begin{align*}
(2.2e) \quad \| [u, p] \|_{2,1} & = \left( \| u \|_2^2 + \| p \|_1^2 \right)^{1/2} < c \| f \|_0. \\
(2.2f) \quad \| [w, q] \|_{2,1} & = \left( \| w \|_2^2 + \| q \|_1^2 \right)^{1/2} < c \| f \|_0,
\end{align*}
\]

here \(c\) denotes a generic constant and \([w, p]\) satisfies

\[
\begin{align*}
(2.2e) \quad a(v; v, w) + b(v, q) = (f, v)_0, \quad \forall v \in X, \ v \in X \\
b(w, \lambda) = 0, \quad \forall \lambda \in Y
\end{align*}
\]
Example 2.1 \[8\] if $\Omega$ is a convex polygon or if the boundary $\partial \Omega$ is sufficiently smooth, the regularity assumptions (2.2e-f) hold.

Introduce a Navier-Stokes' operator

\[(2.3a)\quad L = -v\Delta + (I \cdot \nabla) \quad \text{grad}\]
\[\quad \text{div} \quad 0\]

where $I$ denotes an identity operator. Then (2.1) is equivalent to

\[(2.3b)\quad L(v) = f ,\]

here $v = (u, p)^T$, $F = (f, 0)^T$, or equivalently

\[(2.4)\quad L(u) = f\]

Obviously, the linearisation of $L$ equals

\[(2.5)\quad \begin{pmatrix} L_{11} + \delta L_{11}(u) & L_{12} \\ L_{12} & 0 \end{pmatrix}\]

where $(\delta L_{11}(u, v))_{i=1, \ldots, d} = \sum_{j=1}^{d} (u_j \frac{\partial v_j}{\partial x_j} + v_j \frac{\partial u_j}{\partial x_j})$, $L_{11} = -\mu \Delta$, $L_{12} = \text{grad}$, $L_{12}^* = -\text{div}$.

The principal part of $L(u)$ is thus a stokes' operator, and $\delta L_{11}(u)$ a lower order term.

Let $X_\ell \subset X$ and $Y_\ell \subset Y$ ($\ell > 0$) be two families of finite element subspaces with

$h_0 > h_1 > \ldots > h_{\ell-1} > h_\ell > \ldots$, $x_{\ell-1} \subset x_\ell$, $y_{\ell-1} \subset y_\ell$,

which satisfy the usual approximation assumptions and inverse inequality:

\[(2.6a)\quad \inf_{v \in X_\ell} \| v - v_\alpha \|_\ell^\beta \leq \epsilon h_\beta v_\alpha \| v \|_\beta , \forall \, v \in H^\beta(\Omega)^d , 0 < \alpha < 1 < \beta < 2 , v \in X_\ell^\alpha\]

\[(2.6b)\quad \inf_{p \in Y_\ell} \| p - p_\alpha \|_\ell^\beta \leq \epsilon h_\beta p_\alpha \| p \|_\beta , \forall \, p \in H^\beta(\Omega) , 0 < \alpha < \beta < 1 . p \in Y_\ell^\alpha\]

\[(2.6c)\quad \| v_\ell \|_1 \leq \epsilon h_\alpha \| v_\ell \|_0 , \quad \forall \, v \in x_\ell .\]
The spaces \( x_\ell \) and \( y_\ell \) have to fit together such that the discrete analogue of (2.2d) holds:

\[
\sup_{u_\ell \in x_\ell, \ p_\ell \neq 0} \frac{b(u_\ell, p_\ell)}{\|u_\ell\|_1} > r \|p_\ell\|_0, \quad \forall p_\ell \in y_\ell,
\]

where \( r \) stands for a constant independent of \( h_\ell \).

Example 2.2 [3]. Let \( \Omega \subset \mathbb{R}^2 \) be a polygonal domain. Denote by \( S_\ell \) the space of continuous, piecewise linear finite elements on a regular triangulation of \( \Omega \). Then the families \( x_\ell = S_{\ell/2}^2 \subset x \) and \( y_\ell = S_{\ell} \subset y \) satisfy assumptions (2.6a-c)

Put the product space

\[
U_\ell = x_\ell \times y_\ell
\]

equipped with the discrete norm

\[
\| [u_\ell, p_\ell] \|_U = \left( h_\ell^{-2} \|u_\ell\|_0^2 + \|p_\ell\|_0^2 \right)^{1/2}.
\]

Define a matrix

\[
H_\ell = \begin{pmatrix}
  h_\ell I & 0 \\
  0 & 1
\end{pmatrix}
\]

here \( I \) is a \( d \times d \) identity matrix. Then we get

\[
\| [u_\ell, p_\ell] \|_U = \|H^{-1}_\ell [u_\ell, p_\ell] \|_0,
\]

where \( \|*, *\|_0 = (\|*\|_0^2 + \|*\|_0^2)^{1/2} \)

Consider the discrete analogue of (2.1) in \( u_\ell : \)

Find \( [u_\ell, p_\ell] \in U_\ell \) such that

\[
a(u_\ell; u_\ell, v_\ell) + b(v_\ell, p_\ell) = (f, v_\ell)_0, \quad \forall \ v_\ell \in x_\ell,
\]

\[
b(u_\ell, q_\ell) = 0, \quad \forall \ q_\ell \in y_\ell.
\]

If a basis is given in \( \overline{u}_\ell \), denoted by \( [\phi_i, \psi_j] \), then \( [u_\ell, p_\ell] \) has an expression...
\[ [u_\ell, p_\ell] = \sum_{ijx} [u_{ijx}, p_{ijx}] [\Phi_i, \Psi_j], \]

where \([u_{ijx}, p_{ijx}]_{ijx}\) denotes the \((ijx)th\) component. With the intention of not overloading the presentation with notations, we again denote by \([u_\ell, p_\ell]\) the vector consisting of its components. Then according to Galerkin's method, problem (2.10) has an equivalent matrix-vector system

\[ (2.11) \quad L_\ell(u_\ell) = f_\ell \]

here \(U_\ell = (u_\ell, p_\ell)^T\), \(F_\ell = (f_\ell, 0)^T\).

Thanks to above conditions, (2.1) or (2.10) at least has a solution \([u^*, p^*]\), or \([u_\ell^*, p_\ell^*]\).

Corresponding to the non-linear operator \(L_\ell(*)\) on (2.11), there exists the linearization form

\[ (2.13) \quad L_\ell(U_\ell) = \begin{pmatrix} L_{\ell,11} + \delta L_{\ell,11}(y_\ell) & L_{\ell,12} \\ \ell^* & 0 \end{pmatrix} \]

where \(L_{\ell,11}, \delta L_{\ell,11}(u_\ell), L_{\ell,12}\) are the discrete analogues of \(L_{11}, \delta L_{11}(u), L_{12}\) resp. It is clear that

\[ \begin{pmatrix} L_{\ell,11} & L_{\ell,12} \\ \ell^* & 0 \end{pmatrix} \]

is equals to discrete analogue of Stokes' operator.

Let \(F_\ell\) be the vector space of the right-hand grid functions \(F_\ell\) on (2.11), equipped with the norm

\[ (2.14) \quad [F_\ell, F_\ell^2] = \left\{ h^2 \|F_\ell\|_0^2 + \|F_\ell^2\|_0^2 \right\}^{1/2}, \]

here \(F_\ell = (F_{\ell 1}, F_{\ell 2})^T\). Then from the definition of \(H_\ell\) in (2.8), it follows that

\[ (2.15) \quad [F_\ell, F_\ell^2] = H_\ell [F_\ell, F_\ell^2]_0 \]
For the vector forms $[u, P]$, and $[F_{x1}, F_{x2}]$ on the equation (2.11), we again denote $\| \cdot \|_u$ and $\| \cdot \|$ their norms, respectively;

$$\| [u, P] \|_u = \left( h^{-2} \| u \|_u^2 + \| P \|_2^2 \right)^{1/2},$$

$$\| [F_{x1}, F_{x2}] \| = \left( h^{-2} \| F_{x1} \|_2^2 + \| F_{x2} \|_2^2 \right)^{1/2},$$

where $\| \cdot \|$ denotes the Euclidean norm.

Notice that for the standard finite element case, the following inequalities appear to hold [4]:

$$c \| u \|_0 < \| u \|_0 < c_1 \| u \|, \quad \forall u \in X, \ c, c_1 > 0$$

$\| u \|_0$ denotes the $L^2$-norm for function $u$, but $\| u \|$ the Euclidean norm of vector $u$. The fact is similar to $V_x$.

3. A Nonlinear Multigrid Algorithm.

We consider a nonlinear multigrid iteration for equation (2.12) inside the neighborhood of the solution $[u^*, P^*]$. Let $\epsilon_\omega$ be a positive number such that $L_\omega (\cdot \cdot)$ is a homeomorphism from

$$U_\omega (\epsilon_\omega) = \{ u_\omega : \| u_\omega - u^*_\omega \|_u < \epsilon_\omega \}$$

to

$$F_\omega (\epsilon_\omega) = \{ f_\omega : L_\omega(u_\omega) = u_\omega \in U_\omega (\epsilon_\omega) \}.$$
Do steps 1 and 2 for \( i=1,2,\ldots \).

1. Smoothing

\[ (3.2b) \quad u_{\ell}^{i} = \Phi_{\ell}(u_{\ell}, 0) ; \]

2. Coarse-grid correction.

\[ (3.2c) \quad d_{\ell-1} = \gamma \hat{u}_{\ell}^{i} \]

\[ (3.2d) \quad S = \sigma(\ell-1, d_{\ell-1}) ; \]

\[ (3.2e) \quad \hat{d}_{\ell-1} = F_{\ell-1} - Sd_{\ell-1} \]

Compute an approximate solution \( u_{\ell-1} \) of the defect equation on \( \Omega_{\ell-1} \):

\[ (3.2f) \quad L_{\ell-1}(u_{\ell-1}) = \hat{d}_{\ell-1} \]

by performing \( r \), \( r > 2 \), iterations of \((\ell-1)\) grid scheme to (3.2f) with starting value \( u_{\ell-1} \).

\[ (3.2g) \quad u_{\ell}^{i+1} = u_{\ell}^{i} + \frac{p(u_{\ell-1}^{i} - u_{\ell-1})}{s} \]

In above algorithm, \( s \) denotes a parameter which is defined as follows:

\[ (3.3) \quad S = \frac{\sigma_{\ell-1}}{\|d_{\ell-1}\|_{F}} , \quad d_{\ell-1} \neq 0 , \]

\[ \begin{align*}
S & = 0 , & d_{\ell-1} = 0 , \\
\end{align*} \]

here \( \sigma_{\ell-1} \) is some positive number, \( r \) and \( P \) stand for the restriction and the prolongation operator:

\[ (3.4) \quad r = \text{blockdiag}\{r_{1}, r_{2}\} , \quad p = \text{blockdiag} \{P_{1}, P_{2}\} , \]

respectively.

At the lowest level \( \ell = 0 \), we have to solve equations \( L_{0}(v_{0}) = \hat{d}_{0} \) on the coarsest grid. Assume that these equations are solved by an iteration

\[ u_{0}^{j+1} = \Phi_{0}(u_{0}^{j}, d_{0}) \]
converging to $L_0^{-1}(d_0)$ whose contraction number is denoted by $\phi_0$.

In order to analyse the convergence of above algorithm applied to equation (2.11) by the convergence theorem of the nonlinear multigrid methods [7], we first discuss Stokes equations and the linearization that of Navier – Stokes equation (see below) in section 4.5 and 6.


To give the analysis of the smoothing properties of the smoother (3.1), we shall now discuss the linear stokes equations.

Consider Stokes equations in the domain $\Omega$:

$$
-\nabla + \text{grad } p = f, \quad \text{in } \Omega, \\
\text{div } u = 0, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \Omega.
$$

The saddle point problem of (4.1) reads:

Seek $[u, p] \in Z$ such that

$$
a_0(u, v) + b(v, p) = (f, v)_0, \quad \forall v \in X, \\
b(u, q) = 0, \quad \forall q \in Y.
$$

with $a_0(u, v) = (\nabla u, \nabla v)_0$. Assume $L_{11} = -\Delta$, $L_{12} = \text{grad}$, and $L^*_1 = \text{div}$. Then (4.2) is equivalent to the following equation

$$
L u = f
$$

with

$$
L = \begin{pmatrix} L_{11} & L_{12} \\ L^*_1 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} u \\ p \end{pmatrix}, \quad f = \begin{pmatrix} f \\ 0 \end{pmatrix}
$$

$L_{11}, L_{12}$ and $L^*_1$ are the operators associated with the respective forms $a$ and $b$.

The discrete analogue corresponding to (4.2) in $U_h$ can be written as follows:
Find \([u_\ell, p_\ell] \in U_\ell\) such that
\[
a_o(u_\ell, v_\ell) + b(v_\ell, p_\ell) = (f, v_\ell)_0, \quad \forall v_\ell \in X_\ell, \\
b(u_\ell, q_\ell) = 0 \quad \forall q_\ell \in Y_\ell.
\]

(4.4)

Applying the basis \([\phi_i, \psi_j]\), problem (4.4) becomes the following equivalent matrix-vector form

(4.5)
\[
L_\ell u_\ell = f_\ell
\]

here
\[
L_\ell = \begin{pmatrix}
L_{\ell,11} & L_{\ell,12} \\
L_{\ell,12}^* & 0
\end{pmatrix}, \quad u_\ell = \begin{pmatrix}
u_\ell \\
p_\ell
\end{pmatrix}, \quad f_\ell = \begin{pmatrix}
f_\ell \\
o
\end{pmatrix}.
\]

Let

(4.6)
\[
\hat{L}_\ell = H_\ell L_\ell H_\ell
\]

with \(H_\ell\) from (2.8). The elements of matrix \(\hat{L}_\ell\) are \(\hat{L}_{\ell,11} = h_\ell^2 L_{\ell,11}\), \(\hat{L}_{\ell,12} = h_\ell L_{\ell,12}^*\), \(L_{\ell,12} = h_\ell L_{\ell,12}\) and \(\hat{L}_{\ell,22} = 0\). Assume
\[
S_\ell = L_{\ell,12}^{-1} L_{\ell,11}^{-1} L_{\ell,12}.
\]

Obviously,
\[
\hat{S}_\ell = S_\ell.
\]

**Lemma 4.1 [6].** Let

(4.7a)
\[
\|L_{\ell,11}\|_{-1+1} < c,
\]
(4.7b)
\[
\|L_{\ell,11}^{-1}\|_{1+1} < c,
\]
(4.7c)
\[
\|L_{\ell,12}\|_{-1+0} < c,
\]

here \(c\) denotes generic constant, and \(\|\cdot\|_{-1+1}\) the matrix norm \(\|\cdot\|_{H^{-1}(\Omega) \oplus H^1(\Omega)}\), similarly \(\|\cdot\|_{-1+0}\), etc. Then there exists a constant \(C\) such that
\[
\|S_\ell\|_{0+0} < c.
\]

In addition assume

(4.7e)
\[
\|S_\ell^{-1}\|_{0+0} < c.
\]
Then the following inequalities hold

$$
\|L^1 \|_{(-1,0) \times (1,0)} < c, \quad \|L^{-1} \|_{(1,0) \times (-1,0)} < c,
$$

where \( \| \cdot \|_{(-1,0) \times (1,0)} \) denotes the matrix norm \( \| \cdot \|_{H^{-1}(\Omega) \times L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)} \), etc.

Lemma 4.2 [7]. The discrete Brezzi's condition (2.6c) is equivalent to inequality (4.7e).

Remark 4.3. Using \( S = \hat{S}_{\ell} \) and \( S^{-1} = \hat{S}^{-1}_{\ell} \), and inequalities (4.7a-d), it follows that there exist constants \( C_{\ell} \) and \( C'_{\ell} \) so that

$$
\| \hat{L}_{\ell} \| < C_{\ell}, \quad \| \hat{S}^{-1}_{\ell} \| < C'_{\ell},
$$

here \( \| \cdot \| \) denotes spectral norm.

Remark 4.4. From the continuity, coercivity and discrete Brezzi's conditions of \( a \) and \( b \), we obtain the inequalities (4.7a-e).

(4.8) \( 1 / \omega \leq \| \hat{L}_{\ell} \| < C_{\ell} \).

Then we define the smoother for equation 4.5):

(4.9a) \( \phi_{\ell}(u, f) = u - 2 \omega \hat{H}_{\ell}^2 \hat{L}_{\ell} \hat{H}_{\ell}^2 (L_{\ell} u - f) \)

whose iterate matrix equals

(4.9b) \( S_{\ell} = I - 2 \omega \hat{H}_{\ell}^2 \hat{L}_{\ell} \hat{H}_{\ell}^2 \hat{L}_{\ell} \).

Lemma 4.5 [7]. Let \( U \) be a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle_u \) and norm \( \| u \|_u = \sqrt{\langle u, u \rangle_u} \), \( u \in U \) and \( A \) a matrix such that \( 0 < A = A^* < 1 \).

Then

$$
\| (I - A)^\nu \|_{U \times U} < \eta_0(\nu),
$$

where

$$
\eta_0(\nu) = \frac{\nu^\nu}{(1 + \nu)^{1+\nu}}
$$
Theorem 4.6 [7]. The smoother (4.9a-b) satisfies the smoothing property:

\[ \| L_\ell S_\ell^\nu \|_{F+U} \leq \eta(\nu) = C_\ell \sqrt{\eta_0(2\nu)} \], \text{ for all } \nu > 0, \ell > 0 \]

with \( C_\ell \) from Remark 4.3

Proof. Note that

\[ S_\ell = I - \omega_\ell^2 \ell H_\ell \ell^* H_\ell H_\ell = I - \ell X_\ell H_\ell^{-1} \],

\[ \hat{S}_\ell = H_\ell^{-1} S_\ell H_\ell = I - \omega_\ell^2 \ell \ell^* H_\ell = I - X_\ell, \]

where

\[ X_\ell = \omega_\ell^2 \ell \ell^* \ell \ell \text{ with} \]

\[ 0 < X_\ell < I. \]

So, using (2.9), (2.15) and Lemma 4.5, it follows that

\[ \| L_\ell S_\ell^\nu H_\ell^{-1} S_\ell^\nu H_\ell \|_{F+U}^2 \leq \| H_\ell^{-1} S_\ell^\nu H_\ell \|_{F+U}^2 \leq \| \ell \ell^* H_\ell \|_{F+U}^2 = \| \ell \hat{S}_\ell \|_{F+U}^2 = \| \omega_\ell^2 \ell X_\ell (1 - X_\ell)^2 \|_{F+U} \leq \eta_0(2\nu) \]

which is the desired theorem.

From (4.11) and (4.12), we have the following result:

**Corollary 4.7.** There exists a constant \( C \) such that

\[ \| S_\ell \|_{U+U} < C, \quad \| S_\ell^{(\nu)} \|_{U+U} < C. \]

5. A Linearization Equation.

Consider one more general linear equation

\[ -\Delta u + c \cdot \nabla u + d \cdot u + \nabla \cdot \text{grad } p = f, \quad \text{in } \Omega, \]

\[ \text{div } u = 0, \quad \text{in } \Omega, \]

\[ u = 0, \quad \text{on } \partial \Omega, \]

where \( c \) and \( d \) denote two sufficiently smooth vector functions in \( \Omega \).
\[ c = (c_1, \ldots, c_d), \quad d = (d_1, \ldots, d_d), \]
\[
c \cdot \varphi u = \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i},
\]
\[
d u = \left( \sum_{j=1}^{d} u \frac{\partial d_i}{\partial x_j} \right) i = 1, \ldots, d.
\]

The equivalent saddle point problem of (5.1) reads:

Find \([u, p] \in \mathbb{Z}\) such that

\[
a_0(u, v) + a_1(u, v) + b(v, p) = (f, v)_0, \quad \forall v \in \mathbb{X},
\]

(5.2)

\[
b(u, q) = 0, \quad \forall q \in \mathbb{Y},
\]

with

(5.3a) \quad a_0(u, v) = \mu(\varphi u, \varphi u)_0,

(5.3b) \quad a_1(u, v) = (uc + du, v)_0

where

\[
uc = \left( \sum_{j=1}^{d} c_j \frac{\partial u}{\partial x_j} \right) i = 1, \ldots, d.
\]

The problem (5.2) is equivalent to

(5.4a) \quad Lu = f

with

(5.4b) \quad L = \mu \Delta + c \cdot \nabla + d \quad \text{grad} \quad u = f, \quad \text{div} \quad 0, \quad p, \quad 0.

In contrast with (2.5), we get \(c = d = u\) with \(u\) from (2.5).

Put a general form

\[
L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^* & 0 \end{bmatrix} = L^1 + L^{11},
\]

where

\[
L_{11} = -\mu \Delta + c \cdot \nabla + d, \quad L_{12} = \text{grad}, \quad L_{12}^* = \text{div},
\]

\[
L' = \begin{bmatrix} -\mu \Delta & 0 \\ 0 & \text{grad} \end{bmatrix}, \quad c \cdot \nabla + d = 0
\]

\[
\text{div} \quad 0, \quad L^{11} = 0, \quad 0.
\]
Obviously, if $u = 1$, $L'$ is a principal term of $L$ and the Stokes' operator.

Let

\[ (5.5) \quad a(u, v) = a_0(u, v) + a_1(u, v) \]

if $c, d$ and the boundary $\partial \Omega$ are smooth enough, $a$ satisfies conditions (2.2a-b) with $a$ independent of $c, d$.

Example 5.1 [5] If $\Omega \subset \mathbb{R}^d$ (d=2,3) is a simply connected, bounded domain, and $c$ and $d$ belong to $V$, then $a$ and $b$ satisfy (2.2a-d) where

\[ V = \{ v \in X : \text{div} \ v = 0 \}. \]

The discrete analogue of (5.2) in $U_\ell$ reads:

Seek $[u_\ell, p_\ell] \in U_\ell$ such that

\[ (5.6) \quad a(u_\ell, v_\ell) + b(v_\ell, p_\ell) = (f_\ell, v_\ell)_0, \quad \forall v_\ell \in X_\ell, \]

or the slightly more general problem

\[ (5.7) \quad a(u_\ell, v_\ell) + b(v_\ell, p_\ell) + b(u_\ell, q_\ell) = F_\ell(v_\ell, q_\ell), \quad \forall [v_\ell, q_\ell] \in U_\ell. \]

$F_\ell$ is a linear functional on $U_\ell$. In particular on the finest grid:

\[ F_\ell(v_\ell, q_\ell) = (f, v_\ell)_0, \quad \forall [v_\ell, q_\ell] \in U_\ell. \]

In the case of a basis given in $U_\ell$, (5.6) has a corresponding matrix-vector

\[ (5.8a) \quad L_\ell u_\ell = f_\ell \]

with

\[ (5.8b) \quad L_\ell = \begin{bmatrix} L_{\ell,11} & L_{\ell,12} & u_\ell & f_\ell \\ L_{\ell,21} & L_{\ell,22} & p_\ell & 0 \end{bmatrix}, \quad U_\ell = \begin{bmatrix} u_\ell \\ p_\ell \end{bmatrix}, \quad f_\ell = \begin{bmatrix} f_\ell \\ 0 \end{bmatrix} \]

The discrete analogues of $L'$ and $L''$ are denoted by $L'_\ell$ and $L''_\ell$ resp.
Theorem 5.1  Define the smoothing iteration (4.9a-b) for problem (5.8a). Then the \( L_\varepsilon S_\varepsilon^V \) again satisfies the smoothing property (4.10) with \( L_\varepsilon \) from (5.8b) and \( \eta(v) = (1 + \varepsilon) \sqrt{\eta_0(2v)} \) here \( \varepsilon \) is any positive number:

Proof. Let

\[
S_\varepsilon' = I - \omega_\varepsilon L_\varepsilon^2 H_\varepsilon \left[ \begin{array}{ccc} L_\varepsilon & 0 \\ 0 & L_\varepsilon \end{array} \right],
\]

\[
L_\varepsilon = H_\varepsilon^T \varepsilon \varepsilon H_\varepsilon, \quad \omega_\varepsilon^{-1} = \| L_\varepsilon \| < c_\varepsilon,
\]

\[
S_\varepsilon'' = 0
\]

Then it follows from Theorem 4.6 that \( L_\varepsilon S_\varepsilon^V \) satisfies (4.10) with

\[
\eta'(v) = c_\varepsilon \sqrt{\eta_0(2v)}.
\]

Since

\[
\| L_\varepsilon [u_\varepsilon, p_\varepsilon] \|_F = \| L_\varepsilon [u_\varepsilon,11, u_\varepsilon] \|_F = h_\varepsilon \| [u_\varepsilon,11, u_\varepsilon] \|,
\]

\[
\| L_\varepsilon [u_\varepsilon,11, u_\varepsilon] \|_1 = (L''_\varepsilon,11, u_\varepsilon, L''_\varepsilon,11, u_\varepsilon) = a_1(u_\varepsilon, L''_\varepsilon,11, u_\varepsilon) < c \| u_\varepsilon \|_0 \| L''_\varepsilon,11, u_\varepsilon \|,
\]

we have

\[
\| L''_\varepsilon [u_\varepsilon, p_\varepsilon] \|_F < c \| u_\varepsilon \|_0.
\]

Consequently, we obtain

\[
\| L''_\varepsilon \|_F = \sup_{u} \frac{\| L''_\varepsilon [u_\varepsilon, p_\varepsilon] \|_F}{\| [u_\varepsilon, p_\varepsilon] \|_u} < c \| u_\varepsilon \|_0 / (h^{-1}_\varepsilon \| u_\varepsilon \|_0) = ch_\varepsilon (L_{\varepsilon+\varepsilon}).
\]

Then by Corollary 4.7, \( S_\varepsilon'' = 0 \) and the criterion of smoothing property [7, Criterion 6.2.7], we can prove the desired result.
6. The Approximation Property.

It has been shown that the smoothing property (4.10) holds for problem (5.8a) in Section 5. We will now further discuss the approximation for the system (5.8a-b) in this section so that the convergence of the algorithm (3,2) will be obtained.

Let \( \Omega \) be smooth enough so that the regularity assumptions (2.2e-f) hold for the saddle point problem (5.2) and the following corresponding duality that:

Find \([w, q] \in Z\) such that

\[
a(v,w) + b(v,q) = (f,v)_0, \quad \forall \; v \in X, \\
b(w,\lambda) = 0, \quad \forall \; \lambda \in Y.
\]

(6.1)

Define an affine space

\[
Z_\delta(0) = \{v_\delta \in X_\delta : b(v_\delta, u_\delta) = 0, \forall \; u_\delta \in Y_\delta\}.
\]

Lemma 6.1. Let \([u, P]\) and \([u_\delta, p_\delta]\) are the solutions of problem (5.2) and (5.6) resp. Then there exist constants \(C_1\) and \(C_2\) so that

\[
\|u - u_\delta\|_1 \leq C_1 \inf_{v_\delta \in Z_\delta(0)} \|u - v_\delta\|_1 + C_2 \inf_{u_\delta \in Y_\delta} \|P - u_\delta\|_1.
\]

Proof. Assume

\[
w_\delta = v_\delta - u_\delta;
\]

then

\[
a(w_\delta, w_\delta) = a(v_\delta - u, w_\delta) + a(u - u_\delta, w_\delta)
\]

By (5.2) and (5.6), we get

\[
a(u - u_\delta, w_\delta) = b(w_\delta, p_\delta - P)
\]
Note that for all \( v_\ell \in Z_\ell(0) \),

\[
b(v_\ell - u_\ell, u_\ell) = 0, \quad \forall \ u_\ell \in Y_\ell.
\]

So

\[
a(u - u_\ell, v_\ell - u_\ell) = b(v_\ell - u_\ell, u_\ell - p), \quad \forall \ u_\ell \in Y_\ell, \ v_\ell \in Z_\ell(0).
\]

From the coercivity and continuity conditions of \( a \) and \( b \), it follows that

\[
\alpha \| v_\ell - u_\ell \|_1^2 \leq a(v_\ell - u_\ell, v_\ell - u_\ell) = a(v_\ell - u, v_\ell - u) + b(v_\ell - u_\ell, u_\ell - p)
\]

\[
< A \| v_\ell - u \|_1 \| v_\ell - u \|_1 + B \| v_\ell - u_\ell \|_1 \| u_\ell - p \|_0.
\]

Consequently,

\[
\| v_\ell - u_\ell \|_1 \leq \frac{A}{\alpha} \| v_\ell - u \|_1 + \frac{B}{\alpha} \| u_\ell - p \|_0.
\]

Combining this with

\[
\| u - v_\ell \|_1 \leq \| u - v \|_1 + \| v - v_\ell \|_1,
\]

**Lemma 6.2.** There exists a constant \( c \) such that

\[
\inf_{v_\ell \in Z_\ell(0)} \| u - v_\ell \|_1 \leq \inf_{w_\ell \in X_\ell} \| u - w_\ell \|_1.
\]

**Proof.** It suffices that for all \( w_\ell \in X_\ell \), there exists the corresponding \( v_\ell \in Z_\ell(0) \) such that

\[
\| u - v_\ell \|_1 \leq c \| u - w_\ell \|_1.
\]

Let \( w_\ell \in X_\ell \), and \( \{ y_\ell, v_\ell \} \in U_\ell \) satisfy the following equation:

\[
(y_\ell, v_\ell)_1 + b(v_\ell, v_\ell) = 0, \quad \forall \ v_\ell \in X_\ell
\]

\[
b(y_\ell, u_\ell) = b(u - w_\ell, u_\ell), \quad \forall \ u_\ell \in Y_\ell.
\]

Using the (2.2d) and (2.6), it follows that

\[
\| v_\ell \|_0 < 1/r \| y_\ell \|_1, \quad \| y_\ell \|_1 < B/r \| u - w_\ell \|_1.
\]
Let $\nu_{\ell} = y_{\ell} + w_{\ell}$. Then

$$b(\nu_{\ell}, u_{\ell}) = b(u - w_{\ell}, u_{\ell}) + b(w_{\ell}, u_{\ell}) = b(u, u_{\ell}),$$

therefore $\nu_{\ell} \in Z_{\ell}(0)$; consequently,

$$\|u - \nu_{\ell}\|_1 \leq \|u - w_{\ell}\|_1 + \|w_{\ell}\|_1 < (1 + B/r) \|u - w_{\ell}\|_1,$$

which we wanted to prove.

Lemma 6.3. The following error estimate holds

$$\|p - p_{\ell}\|_0 \leq c(\|u - u_{\ell}\|_1 + \inf_{\mu_{\ell} \in Y_{\ell}} \|p - \mu_{\ell}\|_0)$$

with constant $C$.

Proof. By

$$b(v_{\ell}, p_{\ell} - u) = b(v_{\ell}, p_{\ell} - p) + b(v_{\ell}, p - u_{\ell})$$

and (2.6c), it follows that

$$r \|p_{\ell} - u_{\ell}\|_0 \leq \sup_{v_{\ell} \in X_{\ell}} b(v_{\ell}, p_{\ell} - u_{\ell}) / \|v_{\ell}\|_1 < A \|u - u_{\ell}\|_1 + B \|p - u_{\ell}\|_0$$

By combining the inequality and

$$\|p - p_{0}\| \leq \|p - u_{\ell}\|_0 + \|p_{\ell} - u_{\ell}\|_0,$$

We obtain the desired estimate.

Lemma 6.4. Let $u_{\ell} \in X_{\ell}$ and $p_{\ell} \in Y_{\ell}$ be orthogonal to $X_{\ell - 1}$ and $Y_{\ell - 1}$, resp., with respect to $(\cdot, \cdot)_0$, and $[v_{\ell}, q_{\ell}]$ the solution of

$$a(v_{\ell}, z) + b(z, q_{\ell}) + b(v_{\ell}, S) = (u_{\ell}, z)_0 + h_{\ell}^2 (p_{\ell}, S)_0, \quad [z, s] \in u_{\ell}.$$

Then

$$\|v_{\ell}, q_{\ell}\|_u < ch_{\ell}^2 \|u_{\ell}, p_{\ell}\|_u.$$

(6.2)

(6.3)
Proof. Let \([w, z], [v, s] \in U_\ell\) be the solution of

\[
(6.4) \quad a(w, z) + b(z, r) + b(w, s) = (u, z)_0, \quad \forall [z, s] \in U_\ell.
\]

The continuous analogue of this is as follows:

Find \([w, r] \in Z\) such that

\[
(6.5) \quad a(w, z) + b(z, r) + b(w, s) = (u, s)_0, \quad \forall [z, s] \in Z.
\]

Note that

\[
(6.6) \quad \|v, q\|_U < \|v, w, q - Y\|_U + \|w, w, v - r\|_U + \|w, r\|_U.
\]

We estimate that each term on the right-hand side of (6.6). Denote by \(\pi^0_\ell\) the orthogonal projection of \(Y\) onto \(Y_\ell\) and by \(\pi_\ell\) the orthogonal projection of \(x\) onto \(x_\ell\). It is clear that

\[
\pi^0_{\ell-1} U_\ell = \pi^0_{\ell-1} P_\ell = 0.
\]

From (6.2), (6.4) and (2.2a-d), (2.6c), we have

\[
(6.7) \quad \sup_{\|z\|_1} \frac{b(z, r - q)}{\|z\|_1} \leq A_{11} \|v, w\|_1,
\]

and

\[
\|v, w\|_2 < a(v, w, v, w) = b(v, w, v, w) < h_\ell^2 \|v, w\|_0 \|r, q\|_0.
\]

Let \([\xi, n] \in Z\) be the solution of

\[
a(\xi, \xi) + b(z, n) + b(\xi, s) = (v, w, z)_0, \quad \forall [z, s] \in Z.
\]

Then by (6.2), (6.4), (2.2e-f) and (2.6a), it follows that

\[
(6.8) \quad \|v, w\|_0 < a(v, w, \ell, \pi^{1}_{\ell-1} \xi) + b(\xi, \pi^{1}_{\ell-1} \xi, q, r) < \langle \chi \|u, u\|_0 \rangle
\]

\[
+ b(v, w, n, \pi^{0}_{\ell-1} n)
\]

\[
< \chi \|v, w\|_1 + \|r, q\|_0 \|v, w\|_0.
\]
From (6.4), (6.5), (2.2e-f), (2.6a) and Lemma 1-3, we immediately get

\[ \|w - w_{x, r - r_{x}}\|_{1,0} \leq c \inf_{(z_{x}, s_{x})} \|w - z_{x}, r - s_{x}\|_{1,0} \leq ch_{x} \|u_{x}\|_{0} \]

Combining this with a standard duality argument [4], we get

(6.9) \[ \|w - w_{x}, r - r_{x}\|_{U} \leq ch_{x} \]

From (2.2a-d), (2.6a) and (6.5), we have

(6.10) \[ \sup_{\|z\|_{1} \neq 0} \frac{\langle u_{x}, z - \pi_{x}^{0} w_{x} \rangle_{0} - a(w, z) \|z\|_{1}}{\|z\|_{1}} \leq ch_{x} \|u_{x}\|_{0} + A\|w\|_{1} \]

and

(6.11) \[ \alpha\|w\|^{2}_{1} \leq a(w, w) = \langle u_{x}, w - \pi_{x}^{0} w_{x}\rangle_{0} < ch_{x} \|u_{x}\|_{0} \|w\|_{1} \]

An argument similar to estimating \[ \|v_{x} - w_{x}\|_{0} \], yields

(6.12) \[ \|w\|^{2}_{0} \leq ch_{x} \|u_{x}\|_{0} \|w\|_{0} \]

Combining (6.6-12), the lemma is proved.

Lemma 6.5. Let \[ [w_{x}, r_{x}] \] be any element of \[ U_{x}, \hat{w}_{x}^{-1} \in X_{x}^{-1} \] and \[ r_{x}^{-1} \in Y_{x}^{-1} \] denote the orthogonal projection of \[ w_{x} \] onto \[ X_{x}^{-1} \] and of \[ r_{x} \] onto \[ Y_{x}^{-1} \], respectively, with respect to \[ \langle \cdot, \cdot \rangle_{0} \].

Then

\[ \|w_{x} - w_{x}^{-1}, r_{x} - r_{x}^{-1}\|_{U}^{2} \leq h_{x}^{-2} \|L_{x}[w_{x}, r_{x}]\|_{F} \|v_{x}, q_{x}\|_{U} \]

with \[ [v_{x}, q_{x}] \] being the solution of
(6.13) \[ a(v_{\ell}, z) + b(z, q_{\ell}) + b(v_{\ell}, s) = (w_{\ell} - w_{\ell-1}, z)_{0} + h_{\ell}^{2}(r_{\ell} - r_{\ell-1}, s)_{0}, \]
\[ V[z, s] \in U_{\ell}. \]

Proof. In equality (6.13), let \([z, s] = [w_{\ell}, r_{\ell}]\). Then by the orthogonality from this lemma and the symmetry of \(L_{\ell, 11}\), the right-hand side of (6.13) equals
\[ h_{\ell}^{2} \| [w_{\ell} - w_{\ell-1}, r_{\ell} - r_{\ell-1}] \|_{U}. \]
and the left-hand one
\[ a(v_{\ell}, w_{\ell}) + b(w_{\ell}, q_{\ell}) + b(v_{\ell}, r_{\ell}) \]
\[ = (L_{\ell, 11} v_{\ell}, w_{\ell}) + (L_{\ell, 12} w_{\ell}, q_{\ell}) + (L_{\ell, 12} v_{\ell}, r_{\ell}) \]
\[ = (L_{\ell, 11} w_{\ell} + L_{\ell, 12} v_{\ell}, v_{\ell}) + (L_{\ell, 12} w_{\ell}, q_{\ell}) \]
\[ < \| L_{\ell, 11} w_{\ell} + L_{\ell, 12} v_{\ell} \|_{u} \| v_{\ell} \|_{u} + \| L_{\ell, 12} w_{\ell} \|_{u} \| q_{\ell} \|_{u} \]
\[ < (h_{\ell}^{2} \| L_{\ell, 11} w_{\ell} + L_{\ell, 12} r_{\ell} \|^{2} + \| L_{\ell, 12} r_{\ell} \|^{2})^{1/2} (R_{\ell}^{2} \| v_{\ell} \|^{2} + \| q_{\ell} \|^{2})^{1/2} \]
\[ = \| L_{\ell}[w_{\ell}, r_{\ell}] \|_{F} \| [v_{\ell}, q_{\ell}] \|_{u}. \]

Therefore, we get
\[ \| [w_{\ell} - w_{\ell-1}, r_{\ell} - r_{\ell-1}] \|_{u} < h_{\ell}^{-2} \| L_{\ell}[w_{\ell}, r_{\ell}] \|_{F} \| [v_{\ell}, q_{\ell}] \|_{u}. \]

which is the desired estimate.

**Theorem 6.6.** There exists a constant \( C_{A} \) so that
\[ \| L_{\ell}^{-1} - p L_{\ell-1}^{-1} r \|_{U+F} < C_{A} \]

with \( r = \text{blockdiag}(r_{1}, r_{2}) \) and \( p = \text{blockdiag}(p_{1}, p_{2}) \) from (3,4).

Proof. Let \([w_{\ell}, r_{\ell}] \in U_{\ell}\) with \([w_{\ell-1}, r_{\ell-1}]\) from Lemma 6.5, and \([u_{\ell-1}, p^{*}]\) be the solution of
(6.14) \[ L_{\ell-1}[u_{\ell-1}, p_{\ell-1}] = r L_{\ell}[w_\ell, r_\ell] - r[f_\ell, 0] \]

An argument similar to estimating \( \|w - w_\ell, r - r_\ell\|_F \) in Lemma 6.4 yields

(6.15) \[ \|w_\ell - p_1 u_{\ell-1} - p_1 L_{\ell-1} r_1 f_\ell, r_\ell - p_2 p_{\ell-1}\|_F \leq c \|L_{\ell}[w_\ell, r_\ell]\|_F \leq \|v_\ell, q_\ell\|_F \]

From Lemma 6.4-5, it follows that

\[ \|w_\ell - \hat{w}_{\ell-1}, \hat{r}_{\ell-1}\|_F^2 \leq h_{\ell}^{-2} \|L_{\ell}[w_\ell, r_\ell]\|_F \leq \|v_\ell, q_\ell\|_F \]

So by combining this with (6.15), we obtain

(6.16) \[ \|w_\ell - p_1 u_{\ell-1} - p_1 L_{\ell-1} r_1 f_\ell, r_\ell - p_2 p_{\ell-1}\|_F \leq c \|L_{\ell}[w_\ell, r_\ell]\|_F \]

From (6.14), we get

\[ [u_{\ell-1}^*, p_{\ell-1}^*] = L_{\ell-1}^{-1} r L_{\ell}[w_\ell, r_\ell] - L_{\ell-1}^{-1} r[f_\ell, 0]. \]

By this, it follows that

\[ [w_\ell, r_\ell] - p[u_{\ell-1}^*, p_{\ell-1}^*] = [w_\ell, r_\ell] - p L_{\ell-1} r L_{\ell}[w_\ell, r_\ell] - p L_{\ell-1}^{-1} r[f_\ell, 0]. \]

Consequently,

\[ [w_\ell - p_1 u_{\ell-1} - p_1 L_{\ell-1} r f_\ell, r_\ell - p_2 p_{\ell-1}] = [L_{\ell}^{-1} - p L_{\ell-1} r] L_{\ell} [w_\ell, r_\ell]. \]

Substituting this into (6.16) yields

\[ \|L_{\ell}^{-1} - p L_{\ell-1} r\|_F \leq c \|L_{\ell}[w_\ell, r_\ell]\|_F. \]

Since \([w_\ell, r_\ell]\) is arbitrary, we finally have

\[ \|L_{\ell-1}^{-1} - p L_{\ell-1} r\|_F \leq c A = c. \]
which we wanted to prove,

7. The Convergence Theorem.

We will now consider the convergence theorem about the algorithm (3.2) in Section 3. Before this, we will give the restriction to some qualities in algorithm (3.2) so that it is well-defined.

Let \( \tilde{U}_k \) from (3.2a) satisfy

\[
\tilde{U}_k \in U_k (\rho_k / 6), \ k = 0, 1, \ldots, \ell - 1
\]

with \( 0 < \rho_k < \varepsilon_k (0 < k < \ell - 1) \), and \( \sigma_k \) from (3.3), \( \rho_k / \varepsilon_k (0 < k < \ell) \) and \( \phi_0 \) be sufficiently small. Then if \( U_0 \in U_0 (\rho_0) \), we obtain the following convergence theorem:

**Theorem 7.1.** Consider Navier-Stokes equations (1.1) in a domain smooth enough to satisfy the regularity assumptions (2.2e-f). Let \( X \in C^1, Y \in Y^0 \) be two families of finite element subspaces so that the assumptions (2.6a - c) hold. Then there exists a number \( \nu \) so that the multigrid algorithm (3.2) converges for \( \nu > \nu \), and its contraction number is bounded by \( c \sqrt{\eta_0 (2\nu)} \) with \( \eta_0 (\nu) \) from Lemma 4.5.

**Proof.** From (2.12), (2.13) and the smoother (3.1), we have

\[
L_\ell(u_\ell^*) = \lambda L_\ell(u_\ell^*) / a u_\ell^*,
\]

\[
S_\ell = 3 \lambda \phi_\ell(u_\ell^*, 0) a u_\ell = I - \omega^2 L_\ell(u_\ell^*) H^2 L_\ell(u_\ell^*) H^2 L_\ell(u_\ell^*).
\]

Then by Corollary 4.7, Theorem 5.1 and 6.6, and the convergence theorem of nonlinear multigrid methods [7, Theorem 9.5.12], it shows that the desired theorem holds.

From above, it is shown that the multigrid methods can successfully be used to solve Navier-Stokes equations (1.1), and for the smoothing iteration (3.1), the contraction number of the algorithm is bounded by \( c \sqrt{\eta_0 (2\nu)} \).
Appendix Construction of Restrict and Prolongation Operator

Suppose $S_\ell$ be finite element subspace at $\ell$-level, $\ell$ is an integer, $S_{\ell-1} \subset S_\ell$. $\phi_\ell(x) = \sum_{j=1}^{N_{\ell-1}} \phi_{\ell-1}(x)(x) \phi_{\ell}(x) = \sum_{j=1}^{N_{\ell-1}} \phi_{\ell-1}(x)(x) \phi_{\ell}(x), \ i = 1,2,\ldots,N_{\ell-1}$

Let us consider finite element interpolation of $U$ in $S_\ell$

$$U_\ell(x) = \sum_{i=1}^{N_\ell} U(x_i) \phi_\ell_i(x)$$

$$U_{\ell-1}(x) = \sum_{i=1}^{N_{\ell-1}} U(x_i) \phi_{\ell-1}(x)$$

$$= \sum_{j=1}^{N_\ell} \left( \sum_{i=1}^{N_{\ell-1}} U(x_i) \phi_{\ell-1}(x)(x) \right) \phi_{\ell}(x)$$

Denote

$$U_\ell^T = \{U(x_1), u(x_2), \ldots, U(x_{N_\ell})\}$$

$$U_{\ell-1}^T = \{U(x_1), U(x_2), \ldots, U(x_{N_{\ell-1}})\}$$

Then

$$U_\ell = [\phi_\ell(x)] \ U_{\ell-1}$$

Therefore

$$p = [\phi_{\ell-1}(x_1)] j=1,2,\ldots,N_{\ell-1}, \ i = 1,2,\ldots,N_{\ell}$$

i.e.

$$U_\ell = p \ U_{\ell-1}$$

On the other hand, we have

$$U_{\ell-1} = p^T \ U_\ell$$

i.e. $p, p^T$ are prolongation and restrict operator respectively.
For example, $S_\ell$ is a finite element subspace generated by linear element of triangular, and $S_{\ell+1}$ be generated from $S_\ell$ by dividing each triangular as usual into four congruent sub triangles, then

$$p = \left[ \phi_i(\ell-1)(x_j) \right] = \begin{cases} 1/2 & i \neq j \\ 1 & i = j \end{cases}$$

For Quadratic element of triangles.

$$p = \begin{cases} 1/4 & i \neq j \\ 1 & i = j \end{cases}$$

References


