ON COPOSITIVE MATRICES AND STRONG ELLIPTICITY FOR ISOTROPIC ELASTIC MATERIALS

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Introduction.

In this paper we establish necessary and sufficient conditions for the strong-ellipticity of the equations governing an isotropic (compressible) nonlinearly elastic material at equilibrium. Our work extends results of KNOWLES and STERNBERG [5] who obtained such conditions for both ordinary[1] and strong-ellipticity in the special case when the underlying deformations are plane.

The main tool we use in proving our results is a representation theorem for copositive matrices due to COTTLE, HABETLER and LEMKE [2]. An $n \times n$ matrix $\mathbf{M}$ is said to be *strictly copositive* if

$$\mathbf{v} \cdot \mathbf{M}\mathbf{v} > 0 \quad \text{whenever} \quad v_i \geqslant 0 \quad \text{and} \quad \mathbf{v} \neq \mathbf{0} \, .$$

We limit our attention to 3×3 matrices and use the result of [2] to obtain a more straightforward characterization of copositive matrices. We also provide an alternative proof of the above-mentioned representation theorem in the special case when $n = 3$.

1. Notation.

We let

Lin = space of all linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$

with inner product

$$\mathbf{G} \cdot \mathbf{H} = \mathrm{trace}(\mathbf{G}\mathbf{H}^T) ,$$

where $\mathbf{H}^T$ is the transpose of $\mathbf{H}$. We write

$$\mathrm{Lin}^+ = \{\mathbf{H} \in \mathrm{Lin} : \det \mathbf{H} > 0\} ,$$

$$\mathrm{Orth}^+ = \{\mathbf{Q} \in \mathrm{Lin}^+ : \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\} ,$$

where det is the determinant and $\mathbf{I}$ the identity. Any linear transformation $\mathbf{H}$ admits the unique decomposition

$$\mathbf{H} = \mathbf{E} + \mathbf{W}$$

into a symmetric tensor $\mathbf{E}$ and a skew tensor $\mathbf{W}$; in fact,

$$\mathbf{E} = \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) , \qquad \mathbf{W} = \frac{1}{2}(\mathbf{H}-\mathbf{H}^T) .$$

We call $\mathbf{E}$ and $\mathbf{W}$, respectively, the <u>symmetric</u> and <u>skew</u> parts of $\mathbf{H}$.

Given an orthonormal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ of $\mathbb{R}^3$ we denote the components of a vector $\mathbf{v}$ and the matrix of a linear transformation $\mathbf{H}$ in the usual manner: $v_i = \mathbf{v} \cdot \mathbf{e}^i$, $H_{ij} = \mathbf{e}^i \cdot \mathbf{H}\mathbf{e}^j$.

We write adj for the unique continuous function adj : Lin $\rightarrow$ Lin that satisfies

$$(\mathrm{adj}\,\mathbf{H})\mathbf{H} = (\det \mathbf{H})\mathbf{I} . \tag{1.1}$$

We note that, given any orthonormal basis, the matrix $(\mathrm{adj}\,\underset{\sim}{H})_{ij}$ is the matrix of cofactors of the matrix H_{ji}.

Given any two vectors $\underset{\sim}{a}$, $\underset{\sim}{b} \in \mathbb{R}^3$ we write $\underset{\sim}{a} \otimes \underset{\sim}{b}$ for the tensor product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$; in components

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})_{ij} = a_i b_j \quad .$$

We write ∇ for the gradient operator in $\mathbb{R}^3$: for a vector field $\underset{\sim}{u}$, $\nabla \underset{\sim}{u}$ is the tensor field with components $(\nabla \underset{\sim}{u})_{ij} = \partial u_i / \partial x_j$.

Given a function $\Phi(\underset{\sim}{a}, \underset{\sim}{b}, \ldots, \underset{\sim}{c})$ with vector or tensor arguments we write, e.g., $\partial_{\underset{\sim}{a}} \Phi(\underset{\sim}{a}, \underset{\sim}{b}, \ldots, \underset{\sim}{c})$ for the partial Frechet derivative with respect to $\underset{\sim}{a}$ holding the remaining arguments fixed. When the function has only one argument, e.g., $\Phi(\underset{\sim}{F})$ we write

$$\frac{d}{d\underset{\sim}{F}} \Phi(\underset{\sim}{F}).$$

2. Copositive Matrices.

Fix an orthonormal basis $\{e^1, e^2, e^3\}$ of $\mathbb{R}^3$. We say that a linear transformation B is *strictly copositive with respect to* $\{e^1, e^2, e^3\}$ if

$$v \cdot Bv > 0 \quad \text{whenever} \quad v_i \geqslant 0 \quad \text{and} \quad v \neq 0 \tag{2.1}$$

for i = 1, 2, 3. Since copositivity is not a property of linear transformations alone, but rather of a linear transformation together with an orthonormal basis we will fix an orthonormal basis and refer to the strict copositivity of the matrix B. We note that the copositivy of B is determined solely by its symmetric part. For convenience we define

$$E = \begin{bmatrix} a & \ell & m \\ \ell & b & n \\ m & n & c \end{bmatrix} \quad .$$

Theorem 2.1. (COTTLE, HABETLER and LEMKE [2], p. 306)[2]. Necessary and sufficient for the symmetric matrix E to be strictly copositive are the following conditions:

$$a > 0, \qquad b > 0, \qquad c > 0,$$

$$\ell + (ab)^{1/2} > 0, \quad m + (ac)^{1/2} > 0, \quad n + (bc)^{1/2} > 0, \tag{2.2}$$

and

$$\det E > 0 \quad \text{whenever} \quad (\operatorname{adj} E)_{ij} > 0, \quad i, j = 1, 2, 3.$$

For a proof of the above result see the Appendix. Of more immediate interest is an alternative characterization of copositive matrices which we have found to be more useful in application.

Theorem 2.2. Necessary and sufficient for a symmetric matrix E to be strictly copositive are the following conditions:

$$a > 0\ , \qquad b > 0\ , \qquad c > 0\ ,$$

$$L \equiv \ell + (ab)^{1/2} > 0\ , \qquad M \equiv m + (ac)^{1/2} > 0\ , \qquad N \equiv n + (bc)^{1/2} > 0\ , \tag{2.3}$$

$$c^{1/2}\ell + b^{1/2}m + a^{1/2}n + (2LMN)^{1/2} + (abc)^{1/2} > 0\ .$$

Proof. In view of the previous theorem we shall prove that if $(2.3)_{1,2}$ are satisfied then $(2.3)_3$ is equivalent to $(2.2)_3$.

We first note that by $(2.3)_1$ we can write

$$\underset{\sim}{E} = \mathrm{diag}\left\{a^{1/2}, b^{1/2}, c^{1/2}\right\} \underset{\sim}{E}^* \mathrm{diag}\left\{a^{1/2}, b^{1/2}, c^{1/2}\right\}\ ,$$

where

$$\underset{\sim}{E}^* = \begin{bmatrix} 1 & \lambda & \mu \\ \lambda & 1 & \nu \\ \mu & \nu & 1 \end{bmatrix}\ , \tag{2.4}$$

$$\lambda \equiv \ell/(ab)^{1/2}\ , \qquad \mu \equiv m/(ac)^{1/2}\ , \qquad \nu \equiv n/(bc)^{1/2}.$$

In the new variables $(2.3)_2$ reduces to

$$\lambda_1 \equiv \lambda + 1 > 0\ , \qquad \mu_1 \equiv \mu + 1 > 0\ , \qquad \nu_1 \equiv \nu + 1 > 0\ . \tag{2.5}$$

It is clear from the above factorization of $\underset{\sim}{E}$ that it suffices to prove that

$$\det \underset{\sim}{E}^* > 0 \qquad \text{whenever} \qquad (\mathrm{adj}\, \underset{\sim}{E}^*)_{ij} > 0\ , \quad i,j = 1,2,3 \tag{2.6}$$

is equivalent to

$$\Delta_+ \equiv \lambda_1 + \mu_1 + \nu_1 + (2\lambda_1\nu_1\mu_1)^{1/2} - 2 > 0 \tag{2.7}$$

whenever (2.5) is satisfied.

If we compute adj $\underset{\sim}{E}^*$ we find that

$$\text{adj}\ \underset{\sim}{E}^* = \begin{bmatrix} \nu_1(2-\nu_1) & \mu_1\nu_1 - t & \lambda_1\nu_1 - t \\ \mu_1\nu_1 - t & \mu_1(2-\mu_1) & \lambda_1\mu_1 - t \\ \lambda_1\nu_1 - t & \lambda_1\mu_1 - t & \lambda_1(2-\lambda_1) \end{bmatrix} , \tag{2.8}$$

where

$$t \equiv \lambda_1 + \mu_1 + \nu_1 - 2 .$$

Also it is easy to show that

$$\det \underset{\sim}{E}^* = 2\lambda_1\mu_1\nu_1 - t^2 = \Delta_+\Delta_- , \tag{2.9}$$

where

$$\Delta_\pm \equiv (2\lambda_1\mu_1\nu_1)^{1/2} \pm t . \tag{2.10}$$

<u>Sufficiency</u>. We now prove that (2.7) implies (2.6) whenever (2.5) is satisfied. Let $\lambda_1, \mu_1 \nu_1, \Delta_+ > 0$ and suppose that all of the components of adj $\underset{\sim}{E}^*$ are strictly positive then we must show that det $\underset{\sim}{E}^*$ is strictly positive to finish the sufficiency portion of this proof. By (2.9) we need only prove that $\Delta_- > 0$. This is a consequence of the following lemma.

<u>Lemma</u>. *If λ_1, μ_1, ν_1 and all of the components of adj $\underset{\sim}{E}^*$ are strictly positive then so is Δ_-.*

<u>Proof</u>. By hypothesis and (2.8) we find that $\mu_1\nu_1 > t$, $\lambda_1\mu_1 > t$, $\lambda_1\nu_1 > t$, and $\mu_1\lambda_1\nu_1 \in (0, 8)$. If we combine these relations we find that

$$t < (\lambda_1\mu_1\nu_1)^{2/3} < (2\lambda_1\mu_1\nu_1)^{1/2} .$$

This is the desired result. ■

Necessity. We now prove that (2.6) implies (2.7) whenever (2.5) is satisfied. Let $\lambda_1, \mu_1, \nu_1 > 0$ and suppose that (2.6) is satisfied then we must show that $\Delta_+ > 0$.

Suppose, for the sake of contradiction, that $\Delta_+ \leq 0$. If follows from (2.10) that $t < 0$ and hence that all of the off diagonal components of adj $\underset{\sim}{E}$ are strictly positive. In addition, $\lambda_1, \mu_1, \nu_1 > 0$ and $t < 0$ implies that $\lambda_1, \mu_1, \nu_1 \in (0, 2)$ and hence that all of the diagonal components of adj $\underset{\sim}{E}^*$ are strictly positive. Thus

$$(\text{adj}\, \underset{\sim}{E}^*)_{ij} > 0 \ , \qquad i, j = 1, 2, 3 \ . \tag{2.11}$$

If we now combine the last with (2.6) and (2.9) we conclude that

$$0 < \det \underset{\sim}{E}^* = \Delta_+ \Delta_- \ . \tag{2.12}$$

But, by (2.5), (2.11) and the lemma $\Delta_- > 0$ and hence by (2.12), $\Delta_+ > 0$. This is a contradiction. ∎

3. The constitutive relation. Isotropic materials.

We consider a body $\mathcal{B}$ and we identify $\mathcal{B}$ with the regular region of $\mathbb{R}^3$ it occupies in a fixed reference configuration. A deformation f of $\mathcal{B}$ is a member of the space

$$\mathrm{Def} = \left\{ f \in C^1(\mathcal{B}, \mathbb{R}^3) : \det \nabla f > 0 \right\} .$$

We assume that the body is elastic with response function $S : \mathrm{Lin}^+ \times \mathcal{B} \longrightarrow \mathrm{Lin}$. S gives the (Piola-Kirchhoff) stress

$$S\left(\nabla f(x), x\right)$$

at $x \in \mathcal{B}$ when the body is deformed by f. We shall henceforth fix an arbitrary $x_0 \in \mathcal{B}$ and suppress the dependence of the function S upon the point of the body. Writing F for $\nabla f(x_0)$ we assume that

$$\begin{aligned} QS(F) &= S(QF) \text{ for all } Q \in \mathrm{Orth}^+ , \\ S(F)F^T &= FS(F)^T , \end{aligned} \tag{3.1}$$

for all $F \in \mathrm{Lin}^+$. The restriction $(3.1)_1$ is a consequence of invariance under change of observer, while $(3.1)_2$ follows from balance of moments.

We assume that the body is isotropic so that

$$S(F)Q = S(FQ) \tag{3.2}$$

for any $Q \in \mathrm{Orth}^+$ and $F \in \mathrm{Lin}^+$. A well-known[3] consequence of (3.1) and (3.2) is that there exist functions $\hat{\sigma}_i : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$, $i = 1, 2, 3$, such that

$$S(F) = \sigma_1 F + \sigma_2 F F^T F + \sigma_3 (\det F) F^{-T} , \tag{3.3}$$

where

$$\sigma_i = \hat{\sigma}_i \left(\frac{1}{2} F \cdot F, \ \frac{1}{4} FF^T \cdot FF^T, \ \det F \right) .$$

We shall assume that each $\hat{\sigma}_i$ is C^1 and hence that $\underset{\sim}{S}$ is C^1 on its domain of definition.

Remark. We have not assumed that the body is hyperelastic, i.e., that there is a function $\tilde{\sigma}: \mathrm{Lin}^+ \times \mathcal{B} \longrightarrow \mathbb{R}$ such that

$$S(\underset{\sim}{F}, \underset{\sim}{x}) = \partial_{\underset{\sim}{F}} \tilde{\sigma}(\underset{\sim}{F}, \underset{\sim}{x}) .$$

If the body is both hyperelastic and isotropic then in addition there is a function $\hat{\sigma}: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

$$\tilde{\sigma}(\underset{\sim}{F}) = \hat{\sigma}\left(\frac{1}{2} \underset{\sim}{F} \cdot \underset{\sim}{F}, \ \frac{1}{4} \underset{\sim}{F}\underset{\sim}{F}^T \cdot \underset{\sim}{F}\underset{\sim}{F}^T, \ \det \underset{\sim}{F}\right).$$

Our notation is constructed so that $\hat{\sigma}_{,i} = \hat{\sigma}_i$.

The linear transformation $\underset{\sim}{A}(\underset{\sim}{F}): \mathrm{Lin} \longrightarrow \mathrm{Lin}$ defined by

$$\underset{\sim}{A}(\underset{\sim}{F}) = \partial_{\underset{\sim}{F}} S(\underset{\sim}{F})$$

is called the *elasticity tensor*. We say that the elasticity tensor is *strongly-elliptic* at $\underset{\sim}{F} \in \mathrm{Lin}^+$ if

$$\underset{\sim}{H} \cdot \underset{\sim}{A}(\underset{\sim}{F})\underset{\sim}{H} > 0$$

whenever $\underset{\sim}{H} = \underset{\sim}{a} \otimes \underset{\sim}{b}$ with $\underset{\sim}{a} \neq \underset{\sim}{0}$, $\underset{\sim}{b} \neq \underset{\sim}{0}$.

Proposition 3.1. For any $\underset{\sim}{F} \in \mathrm{Lin}^+$ and $\underset{\sim}{H} \in \mathrm{Lin}$

$$\begin{aligned} \underset{\sim}{A}(\underset{\sim}{F})\underset{\sim}{H} = \sigma_1 \underset{\sim}{H} &+ \sigma_2\left[\underset{\sim}{H}\underset{\sim}{F}^T\underset{\sim}{F} + \underset{\sim}{F}\underset{\sim}{H}^T\underset{\sim}{F} + \underset{\sim}{F}\underset{\sim}{F}^T\underset{\sim}{H}\right] \\ &+ \sigma_3(\det \underset{\sim}{F})\left[(\underset{\sim}{F}^{-T} \cdot \underset{\sim}{H})\, F^{-T} - \underset{\sim}{F}^{-T}\underset{\sim}{H}^T\underset{\sim}{F}^{-T}\right] \\ &+ \sum_{i,j=1}^{3} (\underset{\sim}{G}^i \cdot \underset{\sim}{H})\, \underset{\sim}{G}^j \ \sigma_{i,j} \ , \end{aligned} \qquad (3.4)$$

where

$$G^1 = F, \qquad G^2 = FF^TF, \qquad G^3 = (\det F)F^{-T}.$$

Proof. If we differentiate (3.3) with respect to F in the direction $H \in \mathrm{Lin}$ and note that

$$\frac{d}{dF}(FF^T \cdot FF^T) = 4FF^TF,$$

$$\frac{d}{dF}(\det F) = (\det F)F^{-T},$$

$$\frac{d}{dF}(F^{-T})[H] = -F^{-T}H^TF^{-T}$$

we arrive at equation (3.4). ∎

Let $F \in \mathrm{Lin}^+$ and suppose that $F = VR$ is the polar decomposition of F so that $R \in \mathrm{Orth}^+$ and V is symmetric and positive definite. Then by the spectral theorem there is an orthonormal basis $\{e^1, e^2, e^3\}$ such that

$$V = \sum_{i=1}^{3} \lambda_i e^i \otimes e^i \quad . \tag{3.5}$$

The scalars λ_i are called the local principal stretches (of the deformation f at the point x_0), while the vectors e^i are called the principal axes of strain (in the deformed state).

The Cauchy Stress T is given by the relation

$$T(F) = S(F)F^T/\det F \; . \tag{3.6}$$

We note that as a consequence of $(3.1)_2$, T is symmetric. The eigenvalues of T are called the local principal stresses. For an isotropic material we can combine (3.3), (3.5) and (3.6) to arrive at

$$\underset{\sim}{T}(\underset{\sim}{F}) = \sum_{i=1}^{3} t_i \underset{\sim}{e}^i \otimes \underset{\sim}{e}^i ,$$

where the principal stresses t_i are given by

$$t_i = (\lambda_i^2 \sigma_1 + \lambda_i^4 \sigma_2)/\lambda_1\lambda_2\lambda_3 + \sigma_3 . \qquad (3.7)$$

The BAKER-ERICKSEN [1] inequality is the requirement that the principal stresses have the same order as the principal stretches:

$$(t_i - t_j)(\lambda_i - \lambda_j) > 0 , \qquad \lambda_i \neq \lambda_j .$$

In view of (3.7) slightly stronger than this is the requirement

$$BE_i \equiv \sigma_1 + \sigma_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_i^2) > 0 \qquad (3.8)$$

(even if $\lambda_i = \lambda_j$).

The *tension-extension* inequality is the requirement that each principal stress is a strictly increasing function of the corresponding principal stretch. Slightly stronger than this is the requirement

$$TE_i \equiv (\lambda_1\lambda_2\lambda_3\lambda_i^{-1}) \frac{\partial t_i}{\partial \lambda_i} > 0 , \qquad i = 1,2,3 . \qquad (3.9)$$

For a more detailed discussion of constitutive inequalities see TRUESDELL and NOLL [7], pp 153-162.

Proposition 3.2. *Let $\underset{\sim}{F} \in \mathrm{Lin}^+$ with polar decomposition $\underset{\sim}{F} = \underset{\sim}{V}\underset{\sim}{R}$. Then the elasticity tensor is strongly-elliptic at $\underset{\sim}{F}$ if and only if it is strongly-elliptic at $\underset{\sim}{V}$.*

Proof. Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$ with $\underset{\sim}{a} \neq \underset{\sim}{0}$, $\underset{\sim}{b} \neq 0$ and define

$$\underset{\sim}{K} \equiv \underset{\sim}{a} \otimes \underset{\sim}{b}\underset{\sim}{R} = \underset{\sim}{a} \otimes \underset{\sim}{R}^T \underset{\sim}{b} . \qquad (3.10)$$

If we take the derivative of (3.2) with respect to $\underset{\sim}{F}$ in the direction $\underset{\sim}{K}$ and let $\underset{\sim}{Q} = \underset{\sim}{R}^T$ we discover that

$$\underset{\sim}{A}(\underset{\sim}{F})[\underset{\sim}{K}]\underset{\sim}{R}^T = A(\underset{\sim}{V})[\underset{\sim}{K}\underset{\sim}{R}^T] .$$

Taking the inner product of the last equation with $\underset{\sim}{K}\underset{\sim}{R}^T$ we find, with the aid of (3.10), that

$$(\underset{\sim}{a} \otimes \underset{\sim}{R}^T \underset{\sim}{b}) \cdot \underset{\sim}{A}(\underset{\sim}{F})[\underset{\sim}{a} \otimes \underset{\sim}{R}^T \underset{\sim}{b}] = (\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{A}(\underset{\sim}{V})[\underset{\sim}{a} \otimes \underset{\sim}{b}] .$$

The desired result follows immediately. ∎

4. Necessary and sufficient conditions for strong-ellipticity.

Let $\mathbf{F} \in \text{Lin}^+$ with polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$. Then for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we define

$$SE(\mathbf{a}, \mathbf{b}) \equiv (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{A}(\mathbf{V})[\mathbf{a} \otimes \mathbf{b}]. \tag{4.1}$$

In view of Proposition 3.2 our objective is to establish conditions on the principal stretches λ_i and on the functions $\hat{\sigma}_i$ which are equivalent to SE being strictly positive for all $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$.

For the remainder of this section we fix the orthonormal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ obtained from the spectral decomposition of $\mathbf{V}$ (the principal axes of strain) and we take components of vectors and linear transformations with respect to this basis.

If we let $\mathbf{H} = \mathbf{a} \otimes \mathbf{b}$ we find, with the aid of (3.4), (3.5), (4.1) and the fact that $H_{ii}H_{jj} = H_{ij}H_{ji}$, that

$$SE = \sigma_1 \sum_{i,j} H_{ij}^2 + \sigma_2 \sum_{i,j} \left[H_{ij}^2(\lambda_i^2 + \lambda_j^2) + H_{ij}H_{ji}\lambda_i\lambda_j\right]$$

$$+ \sum_{i,j} (\mathbf{M}^T \boldsymbol{\Sigma} \mathbf{M})_{ij} H_{ii} H_{jj} \quad , \tag{4.2}$$

where

$$\mathbf{M} \equiv \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \\ \lambda_2\lambda_3 & \lambda_1\lambda_3 & \lambda_1\lambda_2 \end{bmatrix}$$

and

$$\Sigma_{ij} \equiv \hat{\sigma}_{i,j} \quad .$$

We again note that $H_{ii}H_{jj} = H_{ij}H_{ji}$ and rewrite (4.2) in the form

$$SE = \sum_{i \neq j} H_{ij}^2 \left[\sigma_1 + \sigma_2(\lambda_i^2 + \lambda_j^2)\right] + \sum_{i=j} H_{ij}^2\left[\sigma_1 + \sigma_2(\lambda_i^2 + \lambda_j^2)\right]$$

$$+ \sigma_2 \sum_{i,j} H_{ii}H_{jj}\lambda_i\lambda_j + \sum_{i,j} (\underset{\sim}{M}^T \underset{\sim}{\Sigma} \underset{\sim}{M})_{ij} H_{ii}H_{jj} \; . \tag{4.3}$$

If we now let $\delta_i = \pm 1$ for $i = 1, 2, 3$ we can write the first term in (4.3) as

$$\sum_{i>j} \left[\sigma_1 + \sigma_2(\lambda_i^2 + \lambda_j^2)\right] \left[H_{ij} - \delta_i\delta_j H_{ji}\right]^2 + \sum_{i \neq j} \left[\sigma_1 + \sigma_2(\lambda_i^2 + \lambda_j^2)\right] \delta_i\delta_j H_{ij}H_{ji} \; . \tag{4.4}$$

Finally, we combine (4.3) and (4.4) to arrive at

$$SE = BE_3(H_{12} - \delta_1\delta_2H_{21})^2 + BE_2(H_{13} - \delta_1\delta_3H_{31})^2 + BE_1(H_{23} - \delta_2\delta_3H_{32})^2$$

$$+ \sum_{i,j} Q(\delta_1, \delta_2, \delta_3)_{ij} \, \delta_i H_{ii} \delta_j H_{jj} \, , \tag{4.5}$$

where

$$Q(\delta_1, \delta_2, \delta_3)_{ij} \equiv \sigma_1 + \sigma_2(\lambda_i^2 + \delta_i\delta_j\lambda_i\lambda_j + \lambda_j^2) + (\underset{\sim}{M}^T \underset{\sim}{\Sigma} \underset{\sim}{M})_{ij} \delta_i\delta_j \tag{4.6}$$

and BE_i are defined by (3.8).

Theorem 4.1. Necessary and sufficient for the elasticity tensor to be strongly-elliptic at $\underset{\sim}{F} \in \mathrm{Lin}^+$ are the following conditions:

$$BE_i > 0 \;, \qquad i = 1,2,3 \;, \tag{4.7}$$

$$\underset{\sim}{Q}(\delta_1, \delta_2, \delta_3) \text{ \underline{strictly copositive for all eight choices} } \delta_1, \delta_2, \delta_3 = \pm 1 \,. \tag{4.8}$$

Proof. Sufficiency. Suppose (4.7) and (4.8) are satisfied. Then for any $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$ with $\underset{\sim}{a} \neq \underset{\sim}{0}$, $\underset{\sim}{b} \neq \underset{\sim}{0}$ put $\underset{\sim}{H} = \underset{\sim}{a} \otimes \underset{\sim}{b}$ and $\delta_i = \operatorname{sgn} H_{ii}$ (if $H_{ii} = 0$, put $\delta_i = +1$) for each $i = 1,2,3$. It is then clear from (4.5), (4.7), (4.8) and the definition of copositivity that SE is strictly positive. This is the desired result.

Necessity. Let SE be strictly positive for every choice of $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$ with $\underset{\sim}{a} \neq \underset{\sim}{0}$, $\underset{\sim}{b} \neq \underset{\sim}{0}$. If we put $\underset{\sim}{a} = (1,0,0)^T$, $\underset{\sim}{b} = (0,1,0)^T$ then (4.7) reduces to

$$0 < SE = BE_3 \,.$$

Similarly BE_1 and BE_2 are positive. Thus we arrive at (4.5).

To prove (4.8) let $\underset{\sim}{x} \in \mathbb{R}^3$ with $\underset{\sim}{x} \neq \underset{\sim}{0}$ and $x_i \geqslant 0$ for $i = 1,2,3$. For any choice of $\delta_1, \delta_2, \delta_3 = \pm 1$ define

$$\underset{\sim}{a} = \begin{pmatrix} \delta_1 x_1^{1/2} \\ \delta_2 x_2^{1/2} \\ \delta_3 x_3^{1/2} \end{pmatrix} , \qquad \underset{\sim}{b} = \begin{pmatrix} x_1^{1/2} \\ x_2^{1/2} \\ x_3^{1/2} \end{pmatrix} .$$

In this case (4.5) reduces to

$$0 < SE = \sum_{i,j} Q(\delta_1, \delta_2, \delta_3)_{ij} x_i x_j = \underset{\sim}{x} \cdot \underset{\sim}{Q}(\delta_1, \delta_2, \delta_3) \underset{\sim}{x} \,.$$

Thus we arrive at (4.8). ∎

Let us now define[4]

$$TE_i \equiv \sigma_1 + 3\lambda_i^2 \sigma_2 + \sum_{k,j} M_{ki}\, \sigma_{k,j}\, M_{ji} \quad ,$$

$$X_1 \equiv \lambda_2\lambda_3\sigma_2 + \frac{1}{2}\sum_{k,j} M_{k2}(\sigma_{k,j} + \sigma_{j,k})M_{j3} \quad ,$$

$$X_2 \equiv \lambda_1\lambda_3\sigma_2 + \frac{1}{2}\sum_{k,j} M_{k1}(\sigma_{k,j} + \sigma_{j,k})M_{j3} \quad ,$$

$$X_3 \equiv \lambda_1\lambda_2\sigma_2 + \frac{1}{2}\sum_{k,j} M_{k1}(\sigma_{k,j} + \sigma_{j,k})M_{j2} \quad .$$

Then the symmetric part of $\underset{\sim}{Q}(\delta_1, \delta_2, \delta_3)$ can be written as (cf (4.6))

$$\begin{bmatrix} TE_1 & BE_3 + X_3\delta_1\delta_2 & BE_2 + X_2\delta_1\delta_3 \\ BE_3 + X_3\delta_1\delta_2 & TE_2 & BE_1 + X_1\delta_2\delta_3 \\ BE_2 + X_2\delta_1\delta_3 & BE_1 + X_1\delta_2\delta_3 & TE_3 \end{bmatrix} \quad .$$

We are now ready to state our main theorem.

Theorem 4.2. Necessary and sufficient for the elasticity tensor to be strongly-elliptic at $\underset{\sim}{F} \in \mathrm{Lin}^+$ are the following conditions:[5]

$$BE_i > 0 \;, \qquad TE_i > 0 \;, \qquad i = 1, 2, 3 \;, \tag{4.9}$$

$$E_i \equiv BE_i + (TE_1TE_2TE_3TE_i^{-1})^{1/2} > |X_i| \;, \qquad i = 1, 2, 3, \tag{4.10}$$

$$TE_1^{1/2}(BE_1 + \theta_1X_1) + TE_2^{1/2}(BE_2 + \theta_2X_2) + TE_3^{1/2}(BE_3 + \theta_1\theta_2X_3)$$
$$+ \left[2(E_1 + \theta_1X_1)(E_2 + \theta_2X_2)(E_3 + \theta_1\theta_2X_3)\right]^{1/2} + (TE_1TE_2TE_3)^{1/2} > 0 \tag{4.11}$$

for all four choices of $\theta_1, \theta_2 = \pm 1$.

Remark. If $X_1X_2X_3 \leqslant 0$ then the four inequalities in (4.11) reduce to the single inequality obtained by choosing θ_1, θ_2 such that

$$X_1\theta_1 \leqslant 0 , \qquad X_2\theta_2 \leqslant 0 , \qquad X_3\theta_1\theta_2 \leqslant 0 ,$$

since all of the terms involving the θ's have the lowest possible values.

If $X_1X_2X_3 > 0$ then three of the inequalities are independent while the fourth, obtained by choosing θ_1, θ_2 so that

$$X_1\theta_1 > 0, \qquad X_2\theta_2 > 0 , \qquad X_3\theta_1\theta_2 > 0 , \tag{4.12}$$

is implied by the other three. To see that the fourth inequality is not independent note that the choice of θ_1, θ_2 that satisfies (4.12) maximizes each term in (4.11). Thus any other choices of θ_1, θ_2 will lower the value of such terms and cause the inequality to become more stringent.

To see that the remaining three inequalities are independent define

$$\beta_i \equiv |X_i| \, TE_i^{1/2}/T , \qquad \eta_i \equiv 1 - \beta_i + BE_i TE_i^{1/2}/T , \tag{4.13}$$

where

$$T \equiv (TE_1 TE_2 TE_3)^{1/2} .$$

A simple computation now shows that (4.11) is equivalent to

$$\eta^+ - 2 + 2\beta_i + \left[2\eta^\times \eta_i^{-1}(\eta_i + 2\beta_i)\right]^{1/2} > 0 , \tag{4.14}$$

where

$$\eta^+ = \eta_1 + \eta_2 + \eta_3 , \qquad \eta^\times = \eta_1\eta_2\eta_3 .$$

We note that by (4.10) and (4.13) the variables β_i, η_i are strictly positive. The independence of the formulas now follows from the complete symmetry of (4.14).

Proof of Theorem 4.2. If we apply Theorem 2.2 to $Q(\delta_1, \delta_2, \delta_3)$ we get necessary and sufficient conditions for Q to be copositive. The second inequality in (4.9) is a direct consequence of $(2.3)_1$ while (4.11) follows from $(2.3)_3$ with $\theta_1 = \delta_2\delta_3$, $\theta_2 = \delta_1\delta_3$ and hence $\theta_1\theta_2 = \delta_1\delta_2$ (remember that $\delta_i = \pm 1$). To deduce (4.10) we note that $(2.3)_2$ and the various choices of δ_i yield, e.g.,

$$BE_1 + X_1 + (TE_2TE_3)^{1/2} > 0 \ ,$$

$$BE_1 - X_1 + (TE_2TE_3)^{1/2} > 0 \ ,$$

and similar equations in the other variables. Equation (4.10) is immediate. This concludes the proof. ∎

Appendix.

We now present an alternative proof to COTTLE, HABETLER and LEMKE's theorem (Theorem 2.1) in the case $n = 3$.

Necessity. Let $\underset{\sim}{E}$ be strictly copositive. If we take $\underset{\sim}{v} = \underset{\sim}{e}^i$ in the definition of copositivity we arrive at $(2.2)_1$ (all the diagonal elements of $\underset{\sim}{E}$ must be strictly positive). To prove the first inequality in $(2.2)_2$ take $\underset{\sim}{v} = x\underset{\sim}{e}^1 + y\underset{\sim}{e}^2$ in (2.1) so that

$$0 < \underset{\sim}{v} \cdot \underset{\sim}{E}\underset{\sim}{v} = ax^2 + 2\ell xy + by^2 = (a^{1/2}x - b^{1/2}y)^2 + 2\left[\ell + (ab)^{1/2}\right]xy \quad ,$$

from which the result is clear. The rest of $(2.2)_2$ follows in a similar manner.

Finally, to prove $(2.2)_3$ we suppose that all of the components of adj $\underset{\sim}{E}$ are strictly positive. As a consequence, not only is adj $\underset{\sim}{E}$ strictly copositive, but adj$\underset{\sim}{E}$ maps positive vectors into positive vectors. Therefore, for any vector $\underset{\sim}{v}$ satisfying $v_i \geqslant 0$, $i = 1,2,3$ and $\underset{\sim}{v} \neq \underset{\sim}{0}$ we find, with the aid of (1.1), that

$$0 < (\text{adj}\,\underset{\sim}{E}\,\underset{\sim}{v}) \cdot \underset{\sim}{E}(\text{adj}\,\underset{\sim}{E}\,\underset{\sim}{v}) = \underset{\sim}{v} \cdot (\text{adj}\,\underset{\sim}{E}\,\underset{\sim}{v}) \det \underset{\sim}{E} \quad .$$

Since adj $\underset{\sim}{E}$ is also copositive we conclude from the above equation that $\det \underset{\sim}{E} > 0$. This is the desired result.

Sufficiency. Let (2.2) be satisfied. Then by the argument used at the beginning of the proof of Theorem 2.2 it suffices to prove that $\underset{\sim}{E}^*$ (as given by (2.4)) is strictly copositive whenever (2.5) and (2.6) are satisfied. Thus if we let $\underset{\sim}{v} = x\underset{\sim}{e}^1 + y\underset{\sim}{e}^2 + z\underset{\sim}{e}^3$ we want to show that

$$0 < \underset{\sim}{v} \cdot \underset{\sim}{E}^*\underset{\sim}{v} = x^2 + y^2 + z^2 + 2\lambda xy + 2\mu yz + 2\nu yz \tag{A.1}$$

whenever $x, y, z \geqslant 0$ and $x^2 + y^2 + z^2 \neq 0$.

Case I. One of the diagonal components of adj $\underset{\sim}{E}^*$ is negative. Without loss of generality let $(\mathrm{adj}\,\underset{\sim}{E})_{11} \leqslant 0$. Then by (2.5) and (2.8) we find that $\nu \geqslant 1$. If we rewrite (A.1) we conclude

$$\underset{\sim}{v} \cdot \underset{\sim}{E}^* \underset{\sim}{v} = (-x+y+z)^2 + 2(\lambda+1)xy + 2(\mu+1)xz + 2(\nu-1)yz \ .$$

Equation (2.5) and $\nu \geqslant 1$ then yield the desired result, (A.1).

Case II. All of the diagonal components of adj $\underset{\sim}{E}^*$ are positive, but one of the off-diagonal components is negative. Without loss of generality let $(\mathrm{adj}\,\underset{\sim}{E})_{12} \leqslant 0$. Then by (2.5) and (2.8) we find that $\lambda \geqslant \mu\nu$ and $\lambda, \mu, \nu \in (-1, 1)$. If we once more rewrite (A.1) we discover

$$\underset{\sim}{v} \cdot \underset{\sim}{E}^* \underset{\sim}{v} = (\mu x+\nu y+z)^2 + (1-\mu^2)x^2 + (1-\nu^2)y^2 + 2(\lambda-\mu\nu)xy \ .$$

The relations $\lambda \geqslant \mu\nu$ and $\lambda, \mu, \nu \in (-1, 1)$ then yield (A.1).

Case III. All of the components of adj $\underset{\sim}{E}^*$ are strictly positive. Then by hypothesis, $(2.2)_3$, det $\underset{\sim}{E}^*$ is also positive. If we now use the spectral theorem we discover that

$$\underset{\sim}{E}^* = \omega_1 \underset{\sim}{f}^1 \otimes \underset{\sim}{f}^1 + \omega_2 \underset{\sim}{f}^2 \otimes \underset{\sim}{f}^2 + \omega_3 \underset{\sim}{f}^3 \otimes \underset{\sim}{f}^3 ,$$

$$\mathrm{adj}\ \underset{\sim}{E}^* = \omega_2\omega_3 \underset{\sim}{f}^1 \otimes \underset{\sim}{f}^1 + \omega_1\omega_3 \underset{\sim}{f}^2 \otimes \underset{\sim}{f}^2 + \omega_1\omega_2 \underset{\sim}{f}^3 \otimes \underset{\sim}{f}^3 ,$$

where $\omega_1 \geqslant \omega_2 \geqslant \omega_3$ and the orthonormal vectors $\underset{\sim}{f}^i$ are chosen so that not all of their components are negative with respect to the basis $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \underset{\sim}{e}^3\}$.

If $\omega_1, \omega_2, \omega_3 > 0$ we are done. Since det $\underset{\sim}{E}^* > 0$ the only other possibility is that $\omega_1 > 0$, $\omega_2 < 0$, $\omega_3 < 0$. We now apply Perron's Theorem[6] to adj $\underset{\sim}{E}^*$ to conclude that all of the components of $\underset{\sim}{f}^1$ are strictly positive. Let $\mathcal{O}_1$ be the (open) first octant in $\mathbb{R}^3$, the set of vectors with strictly positive components. It is clear that

$$\underset{\sim}{f}^1 \in \mathcal{O}_1 \, ,$$

$$\operatorname{span}\{\underset{\sim}{f}^2, \underset{\sim}{f}^3\} \cap \mathcal{O}_1 = \emptyset \, . \qquad \text{(A.2)}$$

Let $\underset{\sim}{v} \in \mathcal{O}_1$ with $\underset{\sim}{v}$ not parallel to $\underset{\sim}{f}^1$ and define $\underset{\sim}{w} : [0,1] \longrightarrow \mathbb{R}^3$, $h : [0,1] \longrightarrow \mathbb{R}$ by

$$\underset{\sim}{w}(t) \equiv \underset{\sim}{v} + (t-1)(\underset{\sim}{v} \cdot \underset{\sim}{f}^1)\underset{\sim}{f}^1 ,$$

$$h(t) \equiv \frac{1}{2}\, \underset{\sim}{w}(t) \cdot \underset{\sim}{E}^* \underset{\sim}{w}(t) \, .$$

It follows, with the aid of $(A.2)_2$, that

$$\dot{h}(t) = \omega_1 t (\underset{\sim}{v} \cdot \underset{\sim}{f}^1)^2 \geqslant 0 \, ,$$

$$\underset{\sim}{w}(1) = \underset{\sim}{v} \in \mathcal{O}_1 \, , \qquad \text{(A.3)}$$

$$\underset{\sim}{w}(0) \notin \mathcal{O}_1 \, .$$

In view of the continuity of $\underset{\sim}{w}$, the last two equations yield the existence of a $t_0 \in [0,1]$ such that

$$\underset{\sim}{w}(t_0) \in (\partial \mathcal{O}_1) \setminus \{\underset{\sim}{0}\}$$

(remember $\underset{\sim}{v} \not\parallel \underset{\sim}{f}^1$). We note that by hypothesis, $(2.2)_{1,2}$, $\underset{\sim}{E}^*$ is positive definite on the set $\partial \mathcal{O}_1 \setminus \{\underset{\sim}{0}\}$. Thus $h(t_0) > 0$ and hence by $(A.3)_1$ we find that

$$h(1) = \underset{\sim}{v} \cdot \underset{\sim}{E}^* \underset{\sim}{v} > 0 \, .$$

This concludes the proof. ∎

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Footnotes

1 STEPHENSON [5] has recently obtained necessary and sufficient conditions for the ordinary ellipticity of the full three-dimensional equations.

2 The theorem in [2] is valid for $n \times n$ matrices. We are only concerned with the case when $n = 3$.

3 Cf., e.g., GURTIN [4] or TRUESDELL and NOLL [7]. The functions $\hat{\sigma}_i$ are not actually defined on the indicated set, but only on the subset which defines possible simultaneous values of the principal invariants. We are assuming that the functions $\hat{\sigma}_i$ have C^1 extensions to the entire first octant.

4 A simple computation shows that this definition of TE_i is equivalent to the definition given in equation (3.9).

5 Conditions (4.9) are well known, cf., e.g., TRUESDELL and NOLL [7]. A condition similar to (4.10) was obtained by KNOWLES and STERNBERG [5] for plane deformations.

6 Cf., e.g., GANTMACHER [3], Vol. II Chapter XII.