THE COUPLING METHOD OF FINITE ELEMENTS
AND BOUNDARY ELEMENTS FOR RADIATION PROBLEMS

BY

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THE COUPLING METHOD OF FINITE ELEMENTS
AND BOUNDARY ELEMENTS FOR RADIATION PROBLEMS

He Yinnian* and Li Kaitai*

Abstract The paper presents the variational formulation and well posedness of the coupling method of finite elements and boundary elements for radiation problem. The convergence and optimal error estimates for the approximation solution and numerical experiment are provided.

1. Introduction

Many scattering and radiation problems in mathematical physics are reduced to the problem of Helmholtz equation

\[ \Delta u + K^2 u = f \quad x \in \mathbb{R}^2 \text{ (or } \mathbb{R}^3) \]

It is important in many areas of applications e.g. the design of waveguide, exploring for mineral material and the study of the biological effects of microwave radiation, and so on. The difficulty is that the domain is whole space. Many people present various method which are used for this problem, e.g. FEM. BE and infinite element ([2]-[5], [29], [10]). Each method is of its characterization. However, the coupling method of FE and BE most attracts one' attention.

2. Variational Formulation

Let us consider the following radiation problem.

\[ \Delta K + K^2 u = f \quad \text{in } \mathbb{R}^2 \]

(2.1) \[ u = 0(r^{-1}) \quad r = |x| + 1^\infty \]

\[ \frac{\partial u}{\partial n} + iku = 0(r^{-1/2}) \quad r \rightarrow +\infty \]

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where \( f(x) \) is a function on \( \mathbb{R}^2 \) with supp \( f \subseteq \Omega_1 \), \( \Omega_1 \subseteq \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \Gamma \) and \( \Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega}_1 \) is the exterior domain (see Figure 1).

The complex function \( k \) is defined by

\[
k(x) = \begin{cases} k_1(x) & x \in \Omega_1, \\ k_0 & x \in \Omega_2,
\end{cases}
\]

where \( k_0 \) is a constant, \( k_1(x) \rvert_{\Gamma} = k_0 \), \( |k^2| < M \),
\(-C_1 < \text{Im} k^2 < -C_2 \), \( M, C_1, C_2 \) are positive constants.

Let

\[
H^1_{\ast}(\mathbb{R}^2) = \{ u : u \in H^1(\mathbb{R}^2), \quad u = 0(r^{-1}), \quad \frac{\partial u}{\partial r} + ik_0 u = o(r^{-1/2}), \quad r \to \infty \}
\]

with the norm \( \lVert \cdot \rVert_{1, \mathbb{R}^2} \) of \( H^1(\mathbb{R}^2) \), which is also a Hilbert space.

In fact, let \( \{v_n\} \subseteq H^1_{\ast}(\mathbb{R}^2) \) which converge to an element \( v \in H^1_{\ast}(\mathbb{R}^2) \), that is

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} |\nabla v_n - \nabla v|^2 \, dx + \int_{\mathbb{R}^2} |v_n - v|^2 \, dx = 0
\]

Recalling F. Riesz theorem, \( \{v_n\} \) contains a subsequence \( \{v_{nk}\} \) such that

\[
\lim_{k \to \infty} v_{nk}(x) = v(x) \quad \text{a.e.} \quad x \in \mathbb{R}^2
\]

Since \( v_{nk} \in H^1(\mathbb{R}^2) \) and \( v \in H^1(\mathbb{R}^2) \), thus

\[
\lim_{k \to \infty} \frac{\partial}{\partial r} v_{nk}(x) = \frac{\partial}{\partial r} v(x) \quad \text{a.e.} \quad x \in \mathbb{R}^2
\]

This implies

\[
v = 0(r^{-1}) \quad r \to \infty
\]

\[
\frac{\partial v}{\partial r} + ik_0 v = o(r^{-1/2}) \quad r \to \infty
\]
by
\[ \nu_n = O(r^{-1}) \quad r \to \infty \]
\[ -\frac{3}{\alpha} V_n + i k_0 \nu_n = O(r^{-1/2}) \quad r \to \infty \]

Hence \( \nu \in H_*(\mathbb{R}^2) \)

Then, variation problem associated with (2.1) is

find \( u \in H^1_*(\mathbb{R}^2) \) such that

(2.4)
\[ A(u,v) = (f,v) \quad \forall v \in H^1_*(\mathbb{R}^2). \]

by means of Green's formula, where

(2.5)
\[ A(u,v) = -\int_{\mathbb{R}^2} \nabla u \cdot \nabla \overline{v} \, dx + \int_{\mathbb{R}^2} k^2 u \overline{v} \, dx, \]

(2.6)
\[ (f,v) = \int_{\mathbb{R}^2} f \overline{v} \, dx. \]

\( \overline{v} \) denotes complex conjugate function of \( v \).

**Theorem 2.1.** Let \( f \in H^{-1}(_*(\mathbb{R}^2)) \) with \( \text{supp} f \subset \Omega_1 \). Then, variational problem (2.4) has a unique solution \( u \in H^1_*(\mathbb{R}^2) \) and

(2.7)
\[ \|u\|_{1, \mathbb{R}^2} \leq C \|f\|_{-1, \mathbb{R}^2}, \]

where \( C > 0 \) is constant.

**Proof.** It is obvious that \( A(\cdot, \cdot) : H^1_*(\mathbb{R}^2) \times H^1_*(\mathbb{R}^2) \to \mathbb{R} \) is continuous sesquilinear form. It is sufficient to prove that \( A(\cdot, \cdot) \) is \( H^1_*(\mathbb{R}^2) \) - coercive by Lax-Milgram theorem.

In fact

(2.8)
\[ A(v, v) = (|v|^2_{1, \mathbb{R}^2} + \int_{\Omega_1} k_0^2 |v|^2 \, dx + k_0^2 \|v\|_{0, \Omega_2}^2) \]

so
\[ |A(v,v)|^2 = (-|v|_{1, R}^2 + \text{Re} k_0^2 \|v\|_{0, \Omega_2}^2 + \int_{\Omega_1} \text{Re} k_1^2 |v|^2 \, dx)^2 + (\text{Im} k_0^2 \|v\|_{0, \Omega_2}^2 + \int_{\Omega_1} \text{Im} k_1^2 |v|^2 \, dx)^2. \]

Set \(0 < \varepsilon < 1\), using young inequality, we obtain

\[ |A(v,v)|^2 > (1-\varepsilon)|v|_{1, R}^4 + (1-\varepsilon^{-1})(R_k_0^2 \|v\|_{0, \Omega_2}^2 + \int_{\Omega_1} \text{Re} k_1^2 |v|^2 \, dx)^2 + (\text{Im} k_0^2 \|v\|_{0, \Omega_2}^2 + \int_{\Omega_1} \text{Im} k_1^2 |v|^2 \, dx)^2 > (1-\varepsilon)|v|_{1, R}^4 + (1 - \varepsilon^{-1})M^2 + C_2^2)|v|_{0, R}^4 \]

Taking \(\varepsilon > 0\) such that \(\varepsilon^{-1} < (1, 1 + C_2^2/M^2)\), then, there exists \(C_3 > 0, C_4 > 0\) such that

\[ 1 + C_2^2/M^2 - \frac{1}{\varepsilon} > C_3, \quad \frac{1}{\varepsilon} - 1 > C_4. \]

Therefore, we have

\[ |A(v,v)|^2 > C_4 \varepsilon |v|_{1, R}^4 + C_3 M^2 \|v\|_{0, R}^4 > C_5 \|v\|_{1, R}^4 \]

where \(C_5\) depends on \(C_3\) and \(C_4\). Hence, it is proved that \(A(\ldots)\) is \(H^1_\varepsilon(R^2)\) coercive.

However, the problem (2.1) can alternatively be formulated as follows:

\[ \begin{align*}
(2.10a) & \quad \Delta u_1 + k_1^2 u_1 = f \quad \text{in } \Omega_1. \\
(2.10b) & \quad \Delta u_2 + k_0^2 u_2 = 0 \quad \text{in } \Omega_2, \\
(2.10c) & \quad u_1 = u_2 \quad \text{on } \Gamma,
\end{align*} \]
\[
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \lambda \quad \text{on } \Gamma,
\]

\[
u_2 = 0(r^{-1}), \quad \frac{\partial u_2}{\partial r} + ik_0 u_2 = 0(r^{-\frac{1}{2}}), \quad r \rightarrow \infty,
\]

where \( u_1 = u|_{\Omega_1}, \quad i = 1,2, \) and \( \partial / \partial n \) denotes the outward normal derivative to \( \Gamma \).

The equations (2.10a) and (2.10b) signify a decomposition into two problems in the separate domains \( \Omega_1 \) and \( \Omega_2 \), while (2.10c) and (2.10d) reflect the appropriate coupling of these two problems.

Let us now give a variational formulation of (2.10). Since \( \Delta u + k_1^2 u = f \) in \( \Omega_1 \), we find, using Green's formula, that

\[
a(u,v) + \langle \gamma_0 v, \lambda \rangle = (f,v) \quad \forall v \in H^1(\Omega_1)
\]

where \( \gamma_0 \) is 0-order trace operator on \( \Gamma \) and

\[
\lambda = -\frac{\partial u}{\partial n} \big|_{\Gamma}, \quad \langle \gamma_0 v, \lambda \rangle = \int_{\Gamma} v \lambda ds,
\]

\[
a(u,v) = -\int_{\Omega_1} uvdx + \int_{\Omega_1} k_1^2 uvdx.
\]

Moreover, since \( \Delta u + k_0^2 u = 0 \) in \( \Omega_2 \), we find, using Green's formula and (2.10e), that (cf[8]),

\[
\frac{1}{2} u(x) = \int_{\Gamma} u(y)G_0(x,y)ds_y - \int_{\Gamma} \lambda(y)G(x,y)ds_y \quad x \in \Gamma,
\]

\[
u(x) = \int_{\Gamma} u(y)G_0(x,y)ds_y - \int_{\Gamma} \lambda(y)G(x,y)ds_y \quad x \in \Omega_2,
\]

where

\[
G(x,y) = -\frac{1}{4\pi} H^{(2)}_0(k_0 |x-y|) \quad x \neq y
\]

is the fundamental solution associated with the two-dimensional Helmholtz equation and
\[ G_n(x,y) = \frac{\partial}{\partial n_y} G(x,y) \quad x \neq y, \quad y \in \Gamma \]

with \( n_y \) being the outward unit normal to \( \Gamma \) at \( y \in \Gamma \). \( H_0^{(2)}(z) \) is the second kind and 0-order Hankel function.

Now, taking complex conjugate for (2.12) and formally multiplying (2.12) by the function \( \mu(x) \in H^{-1/2}(\Gamma) \) and integrating over \( \Gamma \), we obtain

\[ 2b(\lambda, \mu) - \langle \gamma_0 u, \mu \rangle + 2\langle G_n u, \mu \rangle = 0 \quad \forall \mu \in H^{-1/2}(\Gamma). \tag{2.16} \]

where

\[ b(\lambda, \mu) = -\int_{\Gamma} \int_{\Gamma} \overline{\chi(y)} \overline{G(x,y)} \mu(x) ds_x ds_y, \tag{2.17} \]

\[ G_n u = \int_{\Gamma} G_n(x,y)u(y)ds_y \quad x \in \Gamma. \tag{2.18} \]

Combining (2.11) and (2.16), variational formulation for the coupling method of FE and BE can be rewritten as following: Find \((u, \lambda) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma)\) such that

\[ a(u, v) + \langle \gamma_0 v, \lambda \rangle = (f, v) \quad \forall v \in H^1(\Omega_1), \tag{2.19} \]

\[ 2b(\lambda, \mu) - \langle \gamma_0 u, \mu \rangle + 2\langle G_n u, \mu \rangle = 0 \quad \forall \mu \in H^{-1/2}(\Gamma). \]

3. The Well-Posedness of (2.19)

First of all, we prove that sesquilinear form \( b(\ast, \ast) \) is continuous and coercive on \( H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \), and \( \langle \gamma_0 u, \mu \rangle - 2\langle G_n u, \mu \rangle \) is a continuous linear functional on \( H^{-1/2}(\Gamma) \) for \( u \in H^1(\Omega_1) \). Hence the variational problem:

\[ 2b(\lambda, \mu) - \langle \gamma_0 u, \mu \rangle + 2\langle G_n u, \mu \rangle = 0 \quad \forall \mu \in H^{-1/2}(\Gamma), \]

has a unique solution \( \lambda = \lambda(\gamma_0 u) \).

Next, we consider problem: Find \( u \in H^1(\Omega_1) \) such that
(3.1) \[ a(u,v) + \langle v_0, \lambda(v_0u) \rangle = (f,v) \quad \forall v \in H^1(\Omega_1). \]

Let

(3.2) \[ B(u,v) = a(u,v) + \langle v_0, \lambda(v_0u) \rangle \]

Fortunately, sesquilinear form \( B(\cdot, \cdot) \) on \( H^1(\Omega_1) \times H^1(\Omega_1) \) is continuous and \( H^1(\Omega_1) \) - coercive. Therefore, the coupling variational problem (2.19) is well-posedness.

1° The Properties of \( b(\cdot, \cdot) \)

First, we consider an auxiliary problem:

\[
\begin{align*}
\Delta u + k^2 u &= 0 & \text{in } \mathbb{R}^2, \\
u &= u_0 & \text{on } \Gamma, \\
u &= 0 & \text{on } \partial \Omega, \\
- \frac{\partial u}{\partial r} + i k_0 u &= 0 & \text{as } r \to \infty, \\
\end{align*}
\]

(3.3)

where \( u_0 \in H^{1/2}(\Gamma) \).

From section 2 and reference [12]. We can prove

**Lemma 3.1.** Let \( u_0 \in H^{1/2}(\Gamma) \), then there exists a unique solution \( u \in H^1_*(\mathbb{R}^2) \) of (3.3) and

(3.4) \[ C_1 \| u \|_{1,\mathbb{R}^2} \leq \| u_0 \|_{1/2,\Gamma} \leq C_2 \| u \|_{1,\mathbb{R}^2} \]

where \( C_1 \) and \( C_2 \) are positive constants.

Assume

(3.5) \[ K = \{ v; v \in H^1_*(\mathbb{R}^2), \supp (\Delta v + k_0^2 v) \subset \Gamma \}. \]

Thus, (3.3) defines an extensional operator \( E \) of \( H^{1/2}(\Gamma) \) onto \( K \) such that

(3.6) \[ u = Eu_0 \]

Obviously, \( E : H^{1/2}(\Gamma) \to K \) is an isomorphism and
(3.7) \[ u_0 = E^{-1}u = \gamma_0 u. \]

Next, we define an operator \( L \) on \( H^{-1/2}(\Gamma) \) by

(3.8) \[ L(q) = \int \int \int G(x,y)q(y)dydx. \quad \text{for} \quad x \in \mathbb{R}^2 \]

We have the following result for the operator \( L \).

**Lemma 3.2.** The operator \( L \) is an isomorphism from \( H^{-1/2}(\Gamma) \) onto \( K \) and the correspondence of \( q \in H^{-1/2}(\Gamma) \) and \( u \in K \) is defined by

(3.9) \[ u = L(q), \]
\[ q = \left[ \frac{\partial u}{\partial n} \right] = -\frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+}, \]

and

(3.10) \[ C_1 \| u \|_{1,R^2} < \| q \|_{-1/2, \Gamma} < C_2 \| u \|_{1,R^2}, \]

where \( \left[ \frac{\partial u}{\partial n} \right] \) denotes the jump crossing \( \Gamma \) and

\[ \frac{\partial n}{\partial n^-} = \frac{\partial}{\partial n^+} (u|_{\Omega_1})|_{\Gamma}, \quad \frac{\partial u}{\partial n^+} = \frac{\partial}{\partial n^-} (u|_{\Omega_1})|_{\Gamma} \]

**Proof** For any \( q \in H^{-1/2}(\Gamma) \), we consider the variational problem

(3.11) \[ A_0(u,v) = -\langle \gamma_0 v, q \rangle \quad \forall v \in H^1(R^2). \]

(3.12) \[ A_0(u,v) = -\int_{R^2} u \overline{v} \, dx + \int_{R^2} k_0^2 u^2 v \, dx. \]

So, \( A_0(\cdot, \cdot) \) has the properties as \( A(\cdot, \cdot) \) and \( -\langle \gamma_0 v, q \rangle \) is linear bounded functional on \( H^1(R^2) \) by trace theorem. Thus (3.11) has a unique solution \( u \in H^1(R^2) \) such that

(3.13) \[ C_1 \| u \|_{1,R^2} < \| q \|_{-1/2, \Gamma} < C_2 \| u \|_{1,R^2}. \]
However, using abstract Green's formula, we have

\begin{align*}
A_0(u,v) &= - \int_{\Omega_1} \nu u \bar{\nu} \text{d}x + \int_{\Omega_2} k_0^2 \nu u \bar{\nu} \text{d}x - \int_{\Omega_1} \nu u \bar{\nu} \text{d}x + \int_{\Omega_2} k_0^2 \nu u \bar{\nu} \text{d}x \\
&= - \int_{\Gamma} \left[ \frac{\partial u}{\partial n} \right] \nu \text{d}x + \int_{\Omega_1} (\Delta u + k_0^2 u) \bar{\nu} \text{d}x + \int_{\Omega_2} (\Delta u + k_0^2 u) \bar{\nu} \text{d}x.
\end{align*}

From (3.11), we obtain

\begin{equation}
\Delta u + k_0^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma.
\end{equation}

and

\begin{equation}
q = \left[ \frac{\partial u}{\partial n} \right].
\end{equation}

So, \( u \in K. \)

Since \( \Delta u + k^2 u = 0 \) in \( \mathbb{R}^2 \backslash \Gamma, \) we have

\begin{align*}
\int_{\Gamma} \frac{\partial u}{\partial n} (y) G(x,y) \text{d}s_y - \int_{\Gamma} G_n(x,y) u(y) \text{d}s_y \\
&\quad = \begin{cases} 
\frac{1}{2} \ u(x) & x \in \Gamma, \\
0 & x \in \Omega_2,
\end{cases}
\end{align*}

and

\begin{align*}
- \int_{\Gamma} \frac{\partial u}{\partial n^+} (y) G(x,y) \text{d}s_y + \int_{\Gamma} G_n(x,y) u(y) \text{d}s_y \\
&\quad = \begin{cases} 
0 & x \in \Omega_1, \\
\frac{1}{2} \ u(x) & x \in \Gamma, \\
u(x) & x \in \Omega_2.
\end{cases}
\end{align*}
Combining (3.17) and (3.18) we have

\begin{equation}
(3.19) \quad u(x) = \int \frac{\partial u}{\partial n}(y) G(x,y) \, dsy = L(q).
\end{equation}

Conversely, let \( u(x) \in K \), using the discussion as above, we have

\begin{equation}
(3.20) \quad u(x) = L(q),
\end{equation}

and

\begin{equation}
(3.21) \quad q = [ \frac{\partial u}{\partial n} ].
\end{equation}

Hence \( L \) is an isomorphism of \( H^{-1/2}(\Gamma) \) onto \( K \) and the proof is complete.

**Theorem 3.1.** The sesquilinear form \( b(q_1, q_2) \) is continuous on \( H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \) and \( H^{-1/2}(\Gamma) - \) coercive, i.e.

\begin{equation}
(3.22) \quad b(q_1, q_2) < C \| q_1 \|_{-1/2, \Gamma} \| q_2 \|_{-1/2, \Gamma} , \quad \forall q_1, q_2 \in H^{-1/2}(\Gamma),
\end{equation}

\begin{equation}
(3.23) \quad b(q_1, q_2) > C \| q_1 \|_{-1/2, \Gamma}^2 , \quad \forall q_1 \in H^{-1/2}(\Gamma).
\end{equation}

**Proof** Let \( q_1, q_2 \in H^{-1/2}(\Gamma) \), recalling (2.17) and (3.8), we obtain

\begin{equation}
(3.24) \quad b(q_1, q_2) = - \langle \gamma_0 L(q_1), q_2 \rangle = - \langle \gamma_0 u_1, q_2 \rangle = A_0(u, v)
\end{equation}

Hence

\[ |b(q_1, q_2)| = |\langle \gamma_0 u_1, q_2 \rangle| < C \| \gamma_0 u_1 \|_{-1/2, \Gamma} \| q_2 \|_{-1/2, \Gamma}. \]

Notice

\begin{equation}
(3.25) \quad b(q_1, q_1) = - \langle \gamma_0 u_1, q_1 \rangle
\end{equation}

and

\begin{equation}
(3.26) \quad A_0(u_1, v) = - \langle \gamma_0 v, q_1 \rangle \quad \forall \, v \in H^{1}(R^2),
\end{equation}
It is easy to get
\[ |b(q_1, q_1)| = |\gamma_0 u_1, q_1| = |A_0(u_1, u_1)| > C \| u_1 \|_{L^2}^2 > C \| q_1 \|_{-\frac{1}{2}}^2, \gamma \]
by (3.9). Thus, theorem 3.1 is proved.

2° The well-posedness of (2.19)

Let \( u \in H^1(\Omega_1) \), we prove that \( \gamma < G_n u, u > - \gamma_0 u, u \) is a bounded linear functional on \( H^{-\frac{1}{2}}(\Gamma) \). To this end, we first prove the following lemma.

**Lemma 3.3.** Let \( u_0 \in H^{\frac{1}{2}}(\Gamma) \), then \( G_n u_0 \in H^{\frac{1}{2}}(\Gamma) \) and
\[ \|G_n u_0\|_{\frac{1}{2}, \Gamma} < C \| u_0 \|_{\frac{1}{2}, \Gamma}. \]

**Proof.** We consider an auxiliary problem
\[
(3.28) \quad \Delta w + k^2 w = 0 \quad \text{in} \quad \Omega_1 \\
\quad w = u_0 \quad \text{on} \quad \Gamma
\]
where \( u_0 \in H^{\frac{1}{2}}(\Gamma) \).

Recalling lemma 3.1, (3.28) has a unique solution \( w \in H^1(\Omega_1) \). Using Green's theorem, we obtain
\[ \frac{1}{2} u_0(x) = - G_n u_0 + \gamma_0 L(\frac{\partial w}{\partial n}) \quad x \in \Gamma. \]

Since \( \omega \in H^1(\Omega_1) \), then \( \frac{\partial \omega}{\partial n} \in H^{-\frac{1}{2}}(\Gamma) \). We know \( \gamma_0 L(\frac{\partial \omega}{\partial n}) \in H^{\frac{1}{2}}(\Gamma) \) by Lemma 3.2 and the trace theorem. From (3.29), \( G_n u_0 \in H^{\frac{1}{2}}(\Gamma) \) and
\[ \|G_n u_0\|_{\frac{1}{2}, \Gamma} < \frac{1}{2} \| u_0 \|_{\frac{1}{2}, \Gamma} + \| \gamma_0 L(\frac{\partial \omega}{\partial n}) \|_{\frac{1}{2}, \Gamma} \]
\[ < \frac{1}{2} \| u_0 \|_{\frac{1}{2}, \Gamma} + \| \frac{\partial \omega}{\partial n} \|_{-\frac{1}{2}, \Gamma} < C \| u_0 \|_{\frac{1}{2}, \Gamma}. \]

On the other hand, we find, applying Lemma 3.3, that
\[ |2\langle G_n u, u \rangle - \langle \gamma_0 u, u \rangle| < |2\langle G_n u, u \rangle + \langle \gamma_0 u, u \rangle| \]

\[ < 2 \|G_n u\|_1, \|u\|_{-1/2, \Gamma} + \|\gamma_0 u\|_{1/2, \Gamma} \|u\|_{-1/2, \Gamma} < C \|\gamma_0 u\|_{1/2, \Gamma} \|u\|_{-1/2, \Gamma} < C \|u\|_1, \|u\|_{-1/2, \Gamma} \cdot \]

Thus, \( 2 \langle G_n u, u \rangle - \langle \gamma_0 u, u \rangle \) is a linear bounded functional on \( H^{-1/2}(\Gamma) \). By virtue of properties of \( b(\cdot, \cdot) \), we have

**Theorem 3.2.** Let \( u \in H^1(\Omega_1) \), then, the variational problem: Find \( \lambda \in H^{-1/2}(\Gamma) \) such that

\[
(3.30) \quad b(\lambda, u) = \frac{1}{2} \langle \gamma_0 u, u \rangle - \langle G_n u, u \rangle, \forall u \in H^{-1/2}(\Gamma)
\]

has a unique solution \( \lambda = \lambda(\gamma_0 u) \) and

\[
(3.31) \quad \|\lambda\|_{-1/2, \Gamma} < C \|\gamma_0 u\|_{1/2, \Gamma} < C \|u\|_1, \Omega_1
\]

Moreover, we need consider the variation problem:

Find \( u \in H^1(\Omega_1) \) such that

\[
(3.32) \quad B(u, v) = (f, v), \forall v \in H^1(\Omega_1).
\]

To this end, we introduce an auxiliary exterior problem:

\[
\Delta \omega + k_0^2 \omega = 0 \quad \text{in} \quad \Omega_2
\]

\[
(3.33) \quad \omega = g \quad \text{on} \quad \Gamma
\]

\[
\omega = 0(r^{-1}) \quad r \to \infty
\]

\[
\frac{\partial \omega}{\partial r} + ik_0 \omega = 0(r^{-1/2}) \quad r \to \infty
\]
It is well-known (see [1]) that (3.33) has a unique solution and the normal derivative \( \partial u / \partial n \) is well defined. If \( g \in H^{1/2}(\Gamma) \), then \( \partial u / \partial n \in H^{-1/2}(\Gamma) \), we let \( F \) be the mapping defined by \( Fg = \partial u / \partial n \). Then, \( F \) is a bounded mapping from \( H^{1/2}(\Gamma) \) into \( H^{-1/2}(\Gamma) \).

Recalling the equation (2.10) of section 2, \( u_2 \) satisfies problem (3.33) when \( g = \gamma_0 u_1 \). Set \( u_1 = u \) in \( \Omega_1 \), then, we have

\[
(3.34) \quad \lambda(\gamma_0 u) = \frac{\partial u_2}{\partial n} = F(\gamma_0 u).
\]

Thus

\[
(3.35) \quad B(u,v) = a(u,v) + \langle \gamma_0 v, F(\gamma_0 u) \rangle.
\]

Set \( u,v \in C^\infty(\Omega_1) \), extend \( u,v \) to \( \hat{u}, \hat{v} \in C^\infty(\mathbb{R}^2) \) which are defined by

\[
\hat{u}|_{\Omega_1} = u, \quad \hat{u}|_{\Omega_2} = 0,
\]

and \( \hat{u}|_{\Omega_2}, \hat{v}|_{\Omega_2} \) satisfy problem (3.33) when \( g = \gamma_0 u, \gamma_0 v \). Using Green's formula in \( \Omega_2 \), we have

\[
(3.36) \quad \langle \gamma_0 v, F(\gamma_0 u) \rangle = -\int_{\Omega_2} \hat{u} \cdot \nabla \hat{v} \, dx + \int_{\Omega_2} k_2^2 \hat{u} \hat{v} \, dx.
\]

By (3.35) and (3.36), it is easy to see

\[
(3.37) \quad B(u,v) = a(u,v) + \langle \gamma_0 v, F(\gamma_0 u) \rangle = A(\hat{u}, \hat{v}).
\]

Since \( C^\infty(\Omega_1) \) and \( C^\infty(\mathbb{R}^2) \) are dense in \( H^1(\Omega_1) \) and \( H^1(\mathbb{R}^2) \) respectively, thus, we obtain

\[
(3.38) \quad B(u,v) = A(\hat{u}, \hat{v}) \quad \forall u,v \in H^1(\Omega_1).
\]

Therefore,

\[
|B(v,v)| = |A(\hat{v}, \hat{v})| > c \| \hat{v} \|_{L^2}^2 \| R^2 > c \| v \|_{L^1, \Omega_1}^2.
\]
\[ |B(v,v)| = |a(u,v) + \langle \gamma_0 v, F(\gamma_0 u) \rangle | \]

\[ \langle C u \parallel_1, \Omega, v \rangle_1, \Omega + \parallel \gamma_0 v \parallel_{1/2, \Gamma}^n F(\gamma_0 u) \parallel_{-1/2, \Gamma} \rangle < C \parallel u \parallel_1, \Omega, v \parallel_1, \Omega. \]

Hence the sesquilinear form \( B(\cdot, \cdot) \) is continuous on \( H^1(\Omega_1) \times H^1(\Omega_1) \) and \( H^1(\Omega_1) \) - coercive. Thus, we have

**Theorem 3.3.** If \( f \in H^{-1}(\mathbb{R}^2) \) and \( \text{supp} f \subseteq \Omega_1 \). Then, problem (3.32) has a unique solution \( u \in H^1(\Omega_1) \) and

\[ (3.39) \quad \parallel u \parallel_1, \Omega \leq C \parallel f \parallel_{-1, \mathbb{R}^2}. \]

Further, problem (2.19) has a unique solution \( (u, \lambda) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma) \) and

\[ (3.40) \quad \parallel (u, \lambda) \parallel_{H^1(\Omega_1) \times H^{-1/2}(\Gamma)} = \parallel u \parallel_1, \Omega + \parallel \lambda \parallel_{-1/2, \Gamma} < C \parallel u \parallel_1, \Omega < C \parallel f \parallel_{-1, \mathbb{R}^2}. \]

4. The Error Estimate of the Approximation Solution

We let \( V_h \subseteq H^1(\Omega_1) \) and \( H_h \subseteq H^{-1/2}(\Gamma) \) be finite element subspaces depending on parameter \( h > 0 \). We also assume that \( V_h \) and \( H_h \) satisfy the following approximate hypothesis:

\( (H_1) \) For any \( v \in H^m(\Omega_1) \), there exists a constant \( C \) and a function \( \pi_h v \in V_h \) such that

\[ (4.1) \quad \parallel v - \pi_h v \parallel_{1, \Omega} < C h^s \parallel v \parallel_{s+1, \Omega} \quad (0 < s < m-1) \]

\( (H_2) \) For any \( v \in H^{m-3/2}(\Gamma) \), there exists a constant \( c_1 \) and a function \( P_h u \in H_h \) such that

\[ (4.2) \quad \parallel u - P_h u \parallel_{-1/2, \Gamma} < c_1 h^s \parallel u \parallel_{S-1/2, \Gamma} \quad (0 < s < m-1) \]

where \( \pi_h : H^1(\Omega_1) \to V_h \) and \( P_h : H^{-1/2}(\Gamma) \to H_h \) are projectors.
The approximate space $V_h$ and $H_h$ can be constructed by classical method. Let $T_h = \{T\}$ be a regular triangulation of $\Omega_1$ with maximal diameter at most $h$ and $S_h = \{s\}$ be the sides of $G$ with maximal side length at most $h$. $V_h$ is the set of all continuous functions which are the polynomials in each $T$, the order of which is less than $m-1$ and $H_h$ is the set of all continuous functions which are the polynomials in each $s$, the order of which is less than $m-2$. So, $V_h$ and $H_h$ satisfy hypothesis (H$_1$) and (H$_2$) (see reference [12]).

Now, let us consider the approximate problem of (2.19):

Find $(u_h, \lambda_h) \in V_h \times H_h$ such that

$$a(u_h, v) + \gamma_0 \langle v, \lambda_h \rangle = (f, v) \quad \forall v \in V_h,$$

(4.3)

$$2b(\lambda_h, \mu) + 2G_n(u_h, \mu) - \gamma_0(u_h, \mu) = 0 \quad \forall \mu \in H_h.$$  

Due to $V_h \subset H^1(\Omega_1)$ and $H_h \subset H^{-1/2}(G)$, the approximate problem (4.3) also has a unique solution $(u_h, \lambda_h) \in V_h \times H_h$ by the discussion of section 3.

Subtracting (4.3) from (2.19), we have

$$a(u - u_h, v) + \gamma_0 \langle v, \lambda - \lambda_h \rangle = 0 \quad \forall v \in V_h,$$

(4.4)

$$2b(\lambda - \lambda_h, \mu) + 2G_n(u - u_h, \mu) - \gamma_0(u - u_h, \mu) = 0 \quad \forall \mu \in H_h.$$  

Let $u_h \in H_h$, from (4.4), we have

$$2b(u_h - \lambda_h, \mu) = \gamma_0(u - u_h, \mu) - 2G_n(u - u_h, \mu) - 2b(\lambda - \lambda_h, \mu) \quad \forall \mu \in H_h.$$  

(4.5)

Taking $\mu = u_h - \lambda_h \in H_h$ in (4.5), in view of coerciveness of $b(\cdot, \cdot)$, we obtain

$$c_1 \|u_h - \lambda_h\|_{1/2, G} \leq \|b(u_h - \lambda_h, u_h - \lambda_h)\| \leq c_2(\|\gamma_0(u - u_h)\|_{1/2, G} + \|\lambda - \lambda_h\|_{-1/2, G}) \|u_h - \lambda_h\|_{-1/2, G}.$$
Hence

\[(4.6) \quad \| \mu_h - \lambda_h \|_{L^2, \Gamma} \leq C(\| u - u_h \|_{L^2, \Omega_1} + \| \lambda - \lambda_h \|_{L^2, \Gamma}).\]

By triangle inequality, final result is obtained

\[(4.7) \quad \| \lambda - \lambda_h \|_{L^2, \Omega_1} \leq C(\| u - u_h \|_{L^2, \Omega_1} + \| \lambda - \lambda_h \|_{L^2, \Gamma}).\]

On the other hand, from (4.4) and (3.34), we have

\[(4.8) \quad B(u - u_h, \nu) = a(u - u_h, \nu) + \gamma_0 \nu, F(\gamma_0(u - u_h)) = 0.\]

Taking \(\nu = \nu_h - u_h\), (4.8) tends to that

\[(4.9) \quad B(\nu_h - u_h, \nu_h - u_h) = B(u - u_h, V_h - u_h).\]

Using the properties of \(B(\cdot, \cdot)\), there exist \(C_1 > 0\) and \(C_2 > 0\) such that

\[(4.10) \quad C_1 \| \nu_h - u_h \|_{L^2, \Omega_1}^2 \leq |B(\nu_h - u_h, \nu_h - u_h)|
\]

\[= |B(u - u_h, \nu_h - u_h)| \leq C_2 \| u - u_h \|_{L^2, \Omega_1} \| \nu_h - u_h \|_{L^2, \Omega_1}.\]

Then

\[(4.11) \quad \| \nu_h - u_h \|_{L^2, \Omega_1} \leq C \| u - u_h \|_{L^2, \Omega_1}.\]

Thus

\[(4.12) \quad \| u - u_h \|_{L^2, \Omega_1} \leq \| u - \nu_h \|_{L^2, \Omega_1} + \| \nu_h - u_h \|_{L^2, \Omega_1} \leq C \| u - \nu_h \|_{L^2, \Omega_1}.\]

Since (4.7) and (4.12) hold for any \(u_h \in H_h\) and \(\nu_h \in V_h\), we have

\[(4.13) \quad \| u - u_h \|_{L^2, \Omega_1} + \| \lambda - \lambda_h \|_{L^2, \Gamma} \leq C(\inf_{V_h \subseteq V_h} \| u - \nu_h \|_{L^2, \Omega_1} + \inf_{\lambda \in H_h} \| \lambda - \lambda_h \|_{L^2, \Gamma}).\]

By the regularity properties of elliptic equation in subdomain \(\Omega_1\) of \(\Omega\) (see 18), then, \(u \in H^m(\Omega_1)\) if \(f \in H^{m-2}(R^2)\). Hence, recalling the hypothesis \((H_1)\) and \((H_2)\) and (4.13), we have

**Theorem 4.1.** Let \(f \in H^{m-2}(R^2)\) with supp \(f \subseteq \Omega_1\), \(V_h\) and \(H_h\) are constructed as above, \((u, \lambda)\) and \((u_h, \lambda_h)\) are solutions of (2.19) and (4.3) respectively. Then, the error estimate \((u_h, \lambda_h)\) and \((u, \lambda)\) is that
\[ (4.14) \quad \|u - u_h\|_{1, \Omega_1} + \|\lambda - \lambda_h\|_{-1/2, \Gamma} \leq \text{ch}^{m-1}(\|u\|_{m, \Omega_1} + \|\lambda\|_{m-3/2, \Gamma}). \]

5. The Example of Numerical Computation

The numerical example is given for testing the correctness of theoretical analysis. Let \( f(x,y) = \delta(x) \delta(y), k^2 = 1-\ii, k = 2^{1/2}\exp(-\ii\pi/8), \) thus, the solution of (2.1) is

\[ u_1 = -\frac{1}{4\pi} H_0^{(2)}(k_0 r), \quad r = (x^2 + y^2)^{1/2}. \]

In this section, we find the approximate solution \( u_h \) by means of the coupling method and compute the error of \( u_1 \) and \( u_h \) in \( \Omega_1 \).

Let \( \Omega_1 = \{(x,y) \in \mathbb{R}^2; (x^2+y^2)^{1/2} < R_0\} \) with \( R_0 = 5 \) and \( T_h = \{e\} \) be a regular triangulation of \( \Omega_h \) and \( s_h = \{s_i\} \) be the sides of \( T_h = \partial \Omega_h \), where \( \Omega_h \) is a approximate domain of \( \Omega \), see Figure 2.
The approximate space \( V_h \) and \( H_h \) are

\[
V_h = \{ v \in H^1(\Omega_h); v|_e \text{ is a linear function, } v|_s \in T_h \},
\]

\[
H_h = \{ \mu \in L^2(\Omega_h); \mu|_s \text{ is a constant, } \forall s \in S_h \}.
\]

So, \( V_h \) and \( H_h \) satisfy the hypothesis \((H_1)\) and \((H_2)\) with \( m = 2 \). Thus, the convergence order of the approximate solution is \( O(h) \). The base functions \( \phi_1, \ldots, \phi_N \) of \( V_h \) are linear continuous functions which satisfy

\[
\phi_i(p_j) = \delta_{ij}, \ i, j = 1, \ldots, N
\]

where \( p_j \) is the nodes in \( \Omega_h \). The base functions \( \psi_1, \ldots, \psi_m \) of \( H_h \) satisfy

\[
\psi_i(x) = \begin{cases} 1 & x \in S_i, \\ 0 & x \notin S_i \end{cases} \quad i = 1, \ldots, m
\]

Hankel function can be computed by summing a series. That is

\[
(6.1) \quad J_0(z) = \sum_{k=0}^{NO} (-1)^k (z/2)^{2k}/(k!)^2.
\]

\[
(6.2) \quad J_1(z) = \sum_{k=0}^{NO} (-1)^k (z/2)^{2k+1}/(k!(k+1)!),
\]

\[
(6.3) \quad N_0(z) = \frac{2}{\pi} \ln \left( \frac{z}{2} \right) + \nu J_0(g) - \frac{2}{\pi} \sum_{k=1}^{NO} (-1)^k/(k!)^2 (\frac{z}{2})^{2k} \sum_{m=1}^{k} \frac{1}{m^m}.
\]

\[
(6.4) \quad N_1(z) = \frac{2}{\pi} \ln \left( \frac{z}{2} \right) + \nu J_1(z) + \frac{2}{\pi z}
\]

\[
+ \frac{2}{\pi} \sum_{k=1}^{NO} (-1)^k (\frac{z}{2})^{2k-1} \left( \sum_{m=1}^{k} \frac{1}{m^m} - \frac{1}{2k} \right)/(k!(k+1)!),
\]

\[
(6.5) \quad H^{(2)}_j(z) = J_j(z) - i N_j(z),
\]

where \( N_0 \) is an integer number and \( J_j(z), N_j(z), H^{(2)}(z) \) are Bessel function, Neumann function, Hankel function of \( j \)-order, \( j = 0, 1 \), where the Euler's
constant $v$ is that
\[ v = 0.5772156649 \]

Further, the normal derivative of Hankel function is
\[ \frac{\partial}{\partial n} H_0^{(2)}(k_0 \rho) = -k_0 H_1^{(2)}(k_0 \rho) \frac{\partial}{\partial n}. \]

In the approximate problem (4.3), let $u_h = \sum_{i=1}^{m} u_i \phi_i$ and $\lambda_h = \sum_{i=1}^{m} \lambda_i \psi_i$ and $u_h |_{\Gamma_h} = \sum_{i=1}^{m} u_i \phi_i |_{\Gamma_h}$ and $v = \phi_j$ and $u = \psi_j$. Thus, we have
\[ \sum_{i=1}^{N} U_i A (\phi_i, \phi_j) + \sum_{i=1}^{M} \lambda_i <\phi_j, \psi_i> = (f, \phi_j) j = 1, \ldots, N. \]
\[ \sum_{i=1}^{N} \lambda_i 2G(\psi_i, \psi_j) + \sum_{i=1}^{M} U_i 2 \langle G_{n\phi_i}, \psi_j \rangle - \sum_{i=1}^{M} U_i <\phi_i, \psi_j> = 0. \]

Let
\[ F = ((f, \phi_i))_{N \times 1}, \quad U = (U_i)_{N \times 1}, \quad L = (\lambda_i)_{m \times 1} \]
\[ B = (2b(\psi_i, \psi_j))_{m \times m}, \quad C = (\langle \phi_i, \psi_j \rangle)_{m \times m}, \]
\[ D = (2 \langle G_{n\phi_i}, \psi_j \rangle - \langle \phi_i, \psi_i \rangle)_{m \times m}, \]
\[ A = (a(\phi_i, \phi_j))_{N \times N}. \]

From (6.7) and (6.8), we obtain algebraic equation system:
\[ AU + C^*L = F, \]
\[ D^*U + BL = 0, \]

where $C^* = (C, 0)^T$ is $N \times m$ matrix and $D^*$ is $m \times N$ matrix.

The equation system (6.9) can be solved by elimination method. Recalling the define of $A$, we know that $A$ is a matrix of symmetry, band type, positive-defined. From the first equation system of (6.9), we obtain
\[ U = A^{-1}F - A^{-1}C^*L. \]
From (6.10) and

\[ D^* U + BL = 0, \]

we obtain

(6.11) \[ (D^* A^{-1} C^* - B) L = D^* A^{-1} F. \]

Then \( L \) can be solved by (6.11) when \( D^* A^{-1} C^* \) and \( D^* A^{-1} F \) are computed. To this end, we compute \( D^* A^{-1} C^* \) and \( D^* A^{-1} F \) as follows.

Assume \( X_1, \ldots, X_n, X_{n+1} \) be \( n+1 \) \( N \)-dimension column vector, we solve the equation system:

(6.12) \[ A(X_1, \ldots, X_n, X_{n+1}) = (C^*, F) \]

Hence, we have

(6.13) \[ D^* A^{-1} C^* = D^*(X_1, \ldots, X_n), \]

(6.14) \[ D^* A^{-1} F = D^* X_{n+1}. \]

Next, computing the matrix

(6.22) \[ B L = D^* A^{-1} C^* - B, \]

We can obtain \( L \) by means of the subroutine \( GS1 \) of the reference [21] for (6.11). From (6.10), we obtain

(6.23) \[ U = X_{n+1} - (X_1, \ldots, X_n) L. \]

Below we shall describe the approximate solution and the analytic solution by FIG1, FIG2, FIG3 and FIG4.
FIG 1: REAL PART RU OF APPROXIMATE SOLUTION

TYPE 88 TO DRAW ANOTHER SURFACE
The coupling method presented in this paper is correction in theoretical and can be computed in practice. The advantages of this method are little work amount and optimal error estimatal and general application. The method can be generalized in the case of three-dimension.

REFERENCES


