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Abstract
This paper describes two models for simulating flow in naturally fractured petroleum reservoirs, one for single phase flow of a fluid of constant compressibility, and the other for two-phase, incompressible, immiscible flow. Both models are based on the dual porosity concept. In each model the flow in an individual matrix block is simulated using the standard equations describing flow in unfractured media, and the matrix/fracture interaction is based on the imposition of proper boundary conditions on the surface of the block. The models are presented in an easily parallelizable form.

Introduction
Double porosity models of flow through a naturally fractured petroleum reservoir were first described by Barenblatt, Zheltov, and Kochina1 and Warren and Root2; their models were for single phase flow under the assumption of quasi-steady state flow in the matrix blocks. Kazemi3 and de Swaan O.4 considered the fully unsteady model. In this paper a somewhat more general single phase model will be considered, along with a model for two-phase, immiscible flow; Thomas et al5 have studied a different double porosity model of this problem.

The fractured reservoir \( \Omega \) will be idealized as a porous medium having a regular geometric pattern of fractures separating the medium into matrix blocks \( \Omega_i \). The diameter of each \( \Omega_i \) is supposed small in comparison to that of \( \Omega \). The fracture system and each matrix block will be considered to be distinct, coupled porous media. The flow in each matrix block will be treated in a standard manner, based on a proper form of Darcy's law and conservation of mass. Similarly, the flow in the fractures will be modelled through the same physics, except that a (distributed) source term is induced by the flow between the blocks and the fractures. No direct flow between blocks (i.e. without passing through the fractures) will be permitted; thus, each matrix block interacts, through proper boundary conditions, with the surrounding fractures, but with no other block. For convenience in the models formulated below, it will also be assumed that the blocks are not directly affected by external sources or sinks.

The Single Phase Model
In addition to the assumptions described above, assume that the single phase fluid is of constant compressibility; i.e.,

\[
R^{-1}dR = c dP, \quad \rho^{-1}dp = c dp, \quad (1)
\]

\( c \) a positive constant, in the entire system. (Capital letters generally denote fracture quantities, small letters the corresponding matrix quantities.) Gravitational terms will be linearized.

References at end of paper.
\[ R(x,t)^2 = [R_0(x) + |R(x,t) - R_0(x)|]^2 \approx R_0(x)[2R(x,t) - R_0(x)], \tag{2} \]

where \( R_0(x) \) is a fixed, conveniently chosen reference density, such as the initial density distribution. (This function can be updated from time to time in a simulation; this possibility will be ignored in the following discussion.)

The boundary condition relating the flow in a block to that in the surrounding fractures is based on the assumption, inherent in the concept of a double porosity model, that the blocks are quite small with respect to any practical spatial discretization parameter for the simulation of the flow in the fractures. Assume that

\[ p(x,t) = R(x_1,t), \quad x \in \partial \Omega_i, \quad 0 < t \leq T, \tag{3} \]

where \( x_1 \) is the centroid of \( \Omega_i \). Similarly, assume that

\[ p(x,0) = R(x_1,0) = R_0(x_1), \quad x \in \partial \Omega_i, \tag{4} \]

where \( R_0 \) is the initial (fracture) density.

The boundary condition is based on having three families of parallel surfaces to represent the fractures. If, for instance, the reservoir \( \Omega \) were a horizontal slab fractured by two families of parallel, vertical planes, a different boundary condition would be required to include properly the effect of gravity; only the case covered by (3) will be considered here.

The partial differential equation for the single phase flow in a block \( \Omega_i \) is given by

\[ \frac{\partial p}{\partial t} - \nabla \cdot \left( \frac{k}{\mu c} \nabla p \right) = 0, \quad x \in \Omega_i, \quad 0 < t \leq T; \tag{5} \]

note that, in line with the boundary condition (3), gravitational effects have been omitted on the block. Again, this omission would need to be addressed for other fracture geometries. The matrix block \( \Omega_i \) transmits through its surface a flow of fluid given by

\[ - \int_{\partial \Omega_i} \frac{k}{\mu c} \nabla p \cdot n \, d\sigma = - \int_{\Omega_i} \frac{\partial p}{\partial t} \, dx. \tag{6} \]

Average this function over \( \Omega_i \) to give the source function

\[ Q(x,t) = \frac{1}{|\Omega_i|} \int_{\Omega_i} \frac{\partial p}{\partial t} \, dx, \quad x \in \Omega_i, \quad 0 < t \leq T. \tag{7} \]

The flow in the fractures can be described by the differential equation

\[ \frac{\partial \Phi}{\partial t} - \nabla \cdot \left( \frac{K}{\mu c} \nabla \Phi \right) = f_0 \cdot Q, \quad x \in \partial \Omega, \quad 0 < t \leq T. \tag{8} \]

where \( z=2(x) \) is the vertical coordinate and \( f_0 \) is the imposed external mass flow rate. For simplicity, the boundary condition on \( \partial \Omega \) will be taken to be "no flow:"

\[ \left[ \frac{K}{\mu c} \nabla R - 2R_0 g - \nabla z \right] \cdot n = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T. \tag{9} \]

Arbogast\(^6\) has studied the mathematical aspects of the model defined by (3) through (9) in a slightly modified form that has no effect whatsoever on the numerical model approximating it. Under the assumption that the data functions \( f_0 \) and \( R_0 \) and the reference density function \( R_0 \) are smooth, he has shown that the coupled differential system is well-posed, i.e., there exists a unique solution to the system and this solution depends continuously on the data. The continuous dependence is expressed explicitly for bounds in Sobolev spaces of \( R \) and \( p \) in terms of norms on \( f_0 \), \( R_0 \), and \( R_0 \). The effect of the matrix source function \( Q \) is determined to be one of stabilization; it induces a bound on the spatial block averages of \( \partial R/\partial t \) in the Sobolev space \( H^{1/2}(\Omega_i, T) \), which adds mildly to the usual bound for \( \partial R/\partial t \) in \( L^2(0,T;L^2(\Omega)) \) in terms of the same norms on \( f_0 \), \( R_0 \), and \( R_0 \) for a standard, nonfractured reservoir. In the case that each \( \Omega_i \) is a rectangular parallelepiped, Arbogast showed that the Warren-Root model is essentially equivalent to taking a term like the first term in the series representation of the source function \( Q \).

**A Finite Element Approximation of the Single Phase Model**

Define the bilinear forms \( B(u_i,u_2) \) on \( H^1(\Omega_i) \) and \( b_i(u_i,u_2) \) on \( H^1(\Omega_i) \) by

\[ B(u_i,u_2) = \left[ \frac{K}{\mu c} \nabla u_i, \nabla u_2 \right] = \int_{\Omega_i} \frac{K}{\mu c} \nabla u_i \cdot \nabla u_2 \, dx \tag{10} \]

and
be the approximation to \( p \) on \( \Omega_i \) at time \( t_{f+i} \).

Determine \( p_{1, n-1} \in H^1(\Omega_i) \) such that, for \( \lambda = 1, \ldots, N, \)

\[
\int_{\Omega_i} \left( \frac{\partial}{\partial t} p_{1, n-1} - \frac{\partial}{\partial t} p_{1, n-1} \right) \cdot \frac{\partial}{\partial z} \left( p_{1, n-1} - z \right) \ dx = 0, \quad z \in H^1(\Omega_i), \quad \lambda = 1, \ldots, N. \tag{17}
\]

and then compute

\[
Q_1^n = - \int_{\Omega_i} \left( \frac{\partial}{\partial t} p_{1, n-1} - \frac{\partial}{\partial t} p_{1, n-1} \right) \ dx. \tag{19}
\]

Next, let \( p_{2, \lambda} \in \{ \lambda/\Delta t, \lambda/\Delta t + z : z \in H^1(\Omega_i) \} \) for \( \lambda = 1, \ldots, N \) satisfy

\[
\int_{\Omega_i} \left( \frac{\partial}{\partial t} p_{2, \lambda} - \frac{\partial}{\partial t} p_{2, \lambda} \right) \cdot \frac{\partial}{\partial z} \left( p_{2, \lambda} - z \right) \ dx = 0, \quad z \in H^1(\Omega_i), \quad \lambda = 1, \ldots, N. \tag{20}
\]

and compute

\[
G_1^n = \left| \Omega_i \right|^{-1} \int_{\Omega_i} \left( \frac{\partial}{\partial t} p_{2, n} - \frac{\partial}{\partial t} p_{2, 0} \right) \ dx. \tag{22}
\]

Then, the matrix source function \( Q \) on \( \Omega_i \) at time \( t_f^n \) is approximated by

\[
Q(x, t_f^n) \approx Q_1^n - \frac{R_H^n(x_i) - R_H^{n-1}(x_i)}{\Delta t_f}. \tag{23}
\]

Now, approximate \( R \) at time \( t_f^n \) by \( R_H^n, \lambda \in H_i \) such that

\[
\int_{\Omega_i} \left( \frac{R_H^n - R_H^{n-1}}{\Delta t_f} \right) \ dx + \sum_{\lambda} G_1^n(1, \lambda) \frac{R_H^n(x_i) - R_H^{n-1}(x_i)}{\Delta t_f} + \left( R_H^n - \left( R_H^n, \lambda \right) \right) = \left( f_0(t_f^n, \lambda) \right) \tag{24}
\]

The stabilizing effect of the flow from the matrix
blocks into the fractures is readily apparent in (24), since $G_l > 0$.

The finite element algorithm can be summarized as follows. The constants $G_l$, which must be computed for blocks associated with quadrature points in the fracture calculation, can be evaluated in a pre-processor. Then, the initial condition $P^0$ must be approximated by $R_l^0 f(x_i)$, and the initial values $p^0_l$ are determined from it. Given $R_l^0 r^0_l$ and $p^0_l$, the general time step consists of three parts. First, the collection of functions $p^1_{l,n} x_l$ must be found by solving (17), (18) for each relevant $i$, and the corresponding values $Q_l x_l$ computed. Second, (24) must be solved for $R_l x_l^1$. Finally, $p^1_l$ must be evaluated:

$$p^1_l = p^1_{l,n} x_l + (R_l(x_i) - R_l r^0_l x_i) p^2_l x_l.$$  

so that $R_l$ and $p_l$, all $i$, have been updated and the time step completed.

The blocks can be treated simultaneously; i.e., in parallel. Then, only the single number $Q_l x_l^1$ must be transmitted from the solution block to the fracture calculation. Next, the calculation for the density $R_l x_l^1$ in the fractures takes place, after which the single number $R_l(x_i)$ must be returned to each block to permit the update (25) and the determination of $\mathbf{b}_{l,n} x_l$. The algorithm can be implemented very efficiently on a computer system having one quite fast node, such as a vector computer, tied to a collection of less expensive nodes, such as reasonable workstations. Note that the number of parameters associated with the fracture calculation (i.e. $\mathbf{b}_{l,n}$) should be expected to be much larger than the number associated with an individual block (i.e. $\mathbf{b}_{l}$). The blocks can be assigned to the workstations, a few to each station. (More sophisticated, a fraction to the fast node and the rest portioned out to the slower nodes.) The fracture calculation would be made by the fast node. Since so little information is passed between the fracture calculation and the block calculations, only a modest bus capacity is required.

The convergence of the approximate solution $R_l$ and $p_l$, all $i$, to the solution $R$ and $p$ follows easily from the argument of Arbogast for a Crank-Nicoolson version of the algorithm; Arbogast restricted his attention to $N=1$, but no significant change in his argument is needed to treat the case considered here.

A finite difference procedure can be constructed with the same concepts as used in the finite element case. The same parallel features will occur in the algorithm.

The Two-Phase, Immiscible Model

This model is intended to simulate an incompressible waterflood in the fractured reservoir $\Omega$, again employing a dual porosity model. The model will be formulated for a horizontal, linear flood here in order to meet the length constraint for the paper. Gravity and more space variables can be treated by essentially the same techniques. In the fractures, let $S=S_w, P=P_w, P_c=P_o-T_w, \Lambda = \kappa \epsilon / \mu_B (=d, d)$. And $\Lambda = \Lambda_w + \Lambda_o$. With $Q_w$ denoting the (usually negative) water source term resulting from imbibition into the matrix blocks and $Q_o$ the corresponding oil source term, the differential equations describing the flow in the fractures (external source terms are omitted here; again, they could be added with little complication) are the usual

$$\frac{\partial s}{\partial t} - \frac{\partial}{\partial x} \left[ \Lambda w \frac{\partial p}{\partial x} \right] = Q_w, \quad (26)$$

$$\frac{\partial s}{\partial t} - \frac{\partial}{\partial x} \left[ \Lambda_o \frac{\partial p}{\partial x} \right] = Q_o, \quad (27)$$

for $0 < x < L = |\Omega|$ and $t > 0$. If (26) and (27) are added, the pressure equation

$$\frac{\partial}{\partial x} \left[ \frac{\partial p}{\partial x} \right] = 0, \quad (28)$$

results, as incompressibility requires that $Q_w + Q_o = 0$.

Initial values $S(x,0)$ must be specified, along with boundary conditions for $t > 0$. Assume injection of water at $x=0$ at a specified volumetric rate. Then, for $t > 0$,

$$-\Lambda w \frac{\partial p}{\partial x}(x,0, t) = f(t) \geq 0, \quad -\Lambda o \frac{\partial p}{\partial x}(x,0, t) = 0. \quad (29)$$

At $x=L$, the flow out of the domain splits proportionally to the mobilities of the phases:

$$-\Lambda w \frac{\partial p}{\partial x}(L, t) = \Lambda w f(t), \quad -\Lambda o \frac{\partial p}{\partial x}(L, t) = \Lambda o f(t). \quad (30)$$
These conditions amount to the assumption that viscous forces dominate capillary forces at the outflow face. The initial pressure is determined by (28) and the boundary conditions (29) and (30).

The flow in an individual matrix block is governed by the standard equations

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\lambda_p}{\lambda_w} \frac{\partial p}{\partial x} \right) = 0, \quad (31)
\]

\[
\frac{\partial}{\partial x} \left( \lambda_w \frac{\partial s}{\partial x} + \lambda_o p_c \frac{\partial s}{\partial x} \right) = 0, \quad (32)
\]

for \( x \in \Omega_i \) and \( t > 0 \). It will be assumed that each block is sufficiently small with respect to \( Q \) that imbibition dominates viscous forces on the block. Thus, the pressure change in the fractures across a block will be ignored. The first consequence of this assumption is that the initial condition in the block will be taken to be

\[
p(x,0) = P(x,0), \quad p_c(x,0) = P_c(x,0), \quad x \in \Omega_i; \quad (33)
\]

thus,

\[
p_c(s(x,0)) = P_c(S(x,0)), \quad x \in \Omega_i; \quad (34)
\]

(The function \( p_c \) can vary from block to block, but this possibility will not be considered here.) A related consequence is that the boundary conditions on the block are given by

\[
p(x,t) = P(x,t), \quad p_c(x,t) = P_c(x,t), \quad x \in \partial \Omega_i; \quad (35)
\]

so that

\[
p_c(s(x,t)) = P_c(S(x,t)), \quad x \in \partial \Omega_i; \quad (36)
\]

The matrix/fracture interaction in differential form is quite similar to that for the single phase problem. The term \( Q_w \) can be evaluated easily on \( \Omega_i \) as

\[
Q_w = -\frac{1}{|\Omega_i|} \int_{\Omega_i} \frac{\partial s}{\partial t} \, dx, \quad x \in \Omega_i; \quad (37)
\]

A Finite Difference Approximation for the Two-Phase, Immiscible Model

The finite difference procedure in an individual block should be appropriately designed to treat an imbibition-dominated flow. In the appendix, it is shown that, if

\[
\lambda_o(s_o) - \alpha(s_o - s_r), \quad \lambda_w(s_w) - \beta, \quad p_c(s_c) - \gamma.
\]

for \( s_o \) slightly larger than \( s_r \) and if the water saturation in the face at \( y=0 \) is set to \( 1-s_r \) and the capillary pressure external to the face set to zero, then the solution for \( s_o(y,t) \) has the asymptotic form

\[
s_o(y,t) \sim \left( \frac{\alpha}{\alpha Y} s_w(0,t) \right)^{1/(k+1)} \frac{1}{y^{1/(k+1)}}. \quad (39)
\]

In order that the differences of the values of \( s \) between neighboring mesh points be reasonably nearly equal, a mesh point distribution proportional to \( y^{k+1} \) near \( y=0 \) is indicated.

Let the local coordinates for \( \Omega_i \) be denoted by \( y \), \( 0 \leq y \leq y_{i,j} \), and let \( (y_{i,j}: j=0,\ldots,j) \) be a partition of \( \Omega_i \), with \( h_{i,j} = y_{i,j} - y_{i,j-1} \). Discretize (31) and (32) in the following manner. For \( 1 \leq j \leq j-1 \) and \( j=1,\ldots,N \), let

\[
\frac{\phi_{i,j+1/2}}{\Delta t_m} - \frac{2}{h_{i,j}} \left( \lambda_w n_{i,j+1} - \frac{p_{n_{i,j+1}}}{h_{i,j+1}} - \frac{p_{n_{i,j}}} {h_{i,j}} \right) = 0, \quad (40)
\]

\[
\lambda w n_{i,j+1} - \frac{p_{n_{i,j+1}}}{h_{i,j+1}} - \frac{p_{n_{i,j} - p_{n_{i,j-1}}}}{h_{i,j}} = 0, \quad (41)
\]

\[
\lambda w n_{i,j+1} - \frac{p_{n_{i,j+1}}}{h_{i,j+1}} - \frac{p_{n_{i,j} - p_{n_{i,j-1}}}}{h_{i,j}} = 0. \quad (41)
\]

The boundary conditions (35) and (36) can be applied as follows:

\[
p_{n_{i,j}} = p_{n_{i,j+1}} = \frac{p_{n_{i,j}} + p_{n_{i,j+1}}} {2}, \quad (42)
\]

\[
s_{n_{i,j}} = s_{n_{i,j+1}} = s_{n_{i,j+1}}^{(1)} / p_{n_{i,j+1}} = s_{n_{i,j+1}}^{(1)} / p_{n_{i,j+1}}, \quad (43)
\]

\[
s_{n_{i,j}} = s_{n_{i,j+1}} = s_{n_{i,j+1}}^{(1)} / p_{n_{i,j+1}} = s_{n_{i,j+1}}^{(1)} / p_{n_{i,j+1}}, \quad (44)
\]
for \( l = 1, \ldots, N \). The initial condition for the saturation in the block \( Q_l \) is given by

\[
s_n^{0,ij} = s_n^{0,ij} = s_n^{0,ij} = 1 < j < J_l. \tag{45}
\]

Note that in general, (43) implies that \( s_n^{0,1,0} = s_n^{0,1,0} \) and \( s_n^{0,1,0} = s_n^{0,1,0} \); (43) restores the capillary equilibrium that is slightly lost by the linearization in (44).

The coefficients \( \lambda \) and \( \lambda_0 \) should be evaluated differently. Since \( \lambda = \lambda_w + \lambda_0 \) is a smooth function of \( s \) and is bounded away from zero, it suffices to take

\[
\lambda^{n+1}_{l,j+1/2} = \lambda((1/2)(s_n^{n+1,ij} + s_n^{n+1,ij+1})). \tag{46}
\]

However, since \( \lambda_0 \) vanishes at \( s = s_n^{0,0} \), it is better to use a harmonic average of its values. So, let

\[
(\lambda_0 \lambda_{PC}^{n+1}_{l,j+1/2}) = \frac{\Delta y_l}{\int_{y_l,j}^{y_l,j-1} (\lambda_0 \lambda_{PC}^{n-1}(y)) dy}. \tag{47}
\]

Evaluate the integral approximately by interpreting \( s_n^{n+1}(y) \) as the linear interpolant of \( s_n^{n+1,1-j} \) and \( s_n^{n+1,1-j+1} \) and then using a two-point Gauss quadrature rule. The evaluation of \( \lambda_w^{n+1}_{l,j+1/2} \) can be made analogously to that of \( \lambda^{n+1}_{l,j+1/2} \). Note that the coefficients on \( Q_l \) are not being held fixed over the fracture time step \( (t^m_{l-1}, t^m_l) \).

The calculation for \( s_n^{n+1}_{l,j} \) can be decoupled from that for \( s_n^{n+1} \) in a fashion corresponding to the splitting (17), (18) and (20), (21) for the solution in the block \( Q_l \) in the single phase problem. Let \( s_n^{l,1} = s_n^{l,1} s_2 n^{l,1} \), where \( s_1 \) reflects the effect of the conditions existing in the block at time \( t^m_{l-1} \) and \( s_2 \) reflects the effect of changing the saturation in the surrounding fractures; note that changing the pressure in the fractures has no effect on the saturation in the block, as no flow results from a uniform change in the pressure. Then, for \( j = 1, \ldots, J_l - 1 \) and \( l = 1, \ldots, N \),

\[
\phi_{l,j}^{n+1}_{l,j} - \frac{s_1 n^{l,1}_{l,j} - s_1 n^{l,1}_{l,j+1}}{\Delta t_m} \]

\[
\frac{2}{\Delta t_m} \left[ \lambda w^{n+1}_{l,j+1/2} \frac{p_1 n^{l,1}_{l,j+1} - p_1 n^{l,1}_{l,j}}{n_{l,j+1}} - \lambda w^{n+1}_{l,j-1/2} \frac{p_1 n^{l,1}_{l,j} - p_1 n^{l,1}_{l,j-1}}{n_{l,j}} \right] = 0, \tag{48}
\]

\[
\lambda w^{n+1}_{l,j+1/2} \frac{p_1 n^{l,1}_{l,j+1} - p_1 n^{l,1}_{l,j}}{n_{l,j+1}} - \lambda w^{n+1}_{l,j-1/2} \frac{p_1 n^{l,1}_{l,j} - p_1 n^{l,1}_{l,j-1}}{n_{l,j}} = 0, \tag{49}
\]

with

\[
p_1 n^{l,1,0} = p_1 n^{l,1,0} = p_1 n^{l,1,0} \text{ (or zero)}, \tag{50}
\]

\[
s_1 n^{l,1,0} = s_1 n^{l,1,0} = s_1 n^{l,1,0}, \tag{51}
\]

\[
s_1 n^{l,1,j} = s_1 n^{l,1,j} \quad 1 < j < J_l, \tag{52}
\]

and

\[
p_2 n^{l,1} = \frac{P_c(s_n^{n+1})}{P_c(s_n^{n+1})} (s_n^{n} - s_n^{n+1}) q_n n^{l,1}, \tag{53}
\]

\[
s_2 n^{l,1} = \frac{P_c(s_n^{n+1})}{P_c(s_n^{n+1})} (s_n^{n} - s_n^{n+1}) r_n n^{l,1}, \tag{54}
\]

where

\[
\frac{r_n n^{l,1} - r_n n^{l-1}}{\Delta t_m} \]

\[
\frac{2}{\Delta t_m} \left[ \lambda w^{n+1}_{l,j+1/2} \frac{q_n n^{l,1}_{l,j+1} - q_n n^{l,1}_{l,j}}{n_{l,j+1}} - \lambda w^{n+1}_{l,j-1/2} \frac{q_n n^{l,1}_{l,j} - q_n n^{l,1}_{l,j-1}}{n_{l,j}} \right] = 0, \tag{55}
\]

\[
\lambda w^{n+1}_{l,j+1/2} \frac{q_n n^{l,1}_{l,j+1} - q_n n^{l,1}_{l,j}}{n_{l,j+1}} - \lambda w^{n+1}_{l,j-1/2} \frac{q_n n^{l,1}_{l,j} - q_n n^{l,1}_{l,j-1}}{n_{l,j}} = 0, \tag{49}
\]

\[
(\lambda_0 \lambda_{PC}^{n+1}_{l,j+1/2}) = \frac{s_1 n^{l,1}_{l,j} - s_1 n^{l,1}_{l,j+1}}{n_{l,j+1}}. \tag{47}
\]
with
\( q_{n+1,0} = q_{n,1} = 0 \),
\( r_{n+1,0} = r_{n,1} = \Lambda N^{-1} \),
\( r_{0,1} = 0, \quad 1 < j < J_1 \).
\( (57) \quad (58) \quad (59) \)

Note that \( q \) and \( r \) must be recalculated each fracture time step, since the coefficients change.

Next, evaluate the matrix/fracture interaction:
\[ Q_{n+1} = -\frac{1}{2\Delta t} \sum_{j=1}^{J_1} \phi_{i,j}(s_{n+1,i,j-1} - s_{n+1,i,j}) - s_{i,1} \quad 1 < j < J_1 \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\[ G^n = \frac{1}{2\Delta t} \sum_{j=1}^{J_1} \phi_{i,j}(r_{n+1,i,j-1} - r_{n+1,i,j}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\( (60) \quad (61) \quad (62) \)

The finite difference equations in the fractures can be constructed as follows. Partition (here uniformly, for simplicity of notation) \( Q^n : H_j = H = L/M, x_i = H, i = 0, \ldots, M \). For \( i = 1, \ldots, M-1 \), let
\[ \left[ \phi_{i,j} + G^n \right] \frac{P_C(S_{n+1} - S_{n-1})}{P_C(S_{n+0})} \Delta t_f \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\[ \left( \phi_{i,j} + G^n \right) \frac{P_C(S_{n+1} - S_{n-1})}{P_C(S_{n+0})} \Delta t_f \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\[ = Q^n \quad 1 < j < J_1 \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\( (63) \quad (64) \quad (65) \)

The boundary conditions (29) and (30) can be incorporated in a discretization of (26) and (27) for \( i=0 \):
\[ \left[ \phi_{0,j} + G^n \right] \frac{P_C(S_{n+1} - S_{n-1})}{P_C(S_{n+0})} \Delta t_f \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\[ = 2H^{-1}f(t^n) + Q^n \quad 1 < j < J_1 \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \quad \Lambda n^{-1,2}(P_{n+1} - P_{n-1}) \]
\( (66) \quad (67) \quad (68) \quad (69) \quad (70) \)

The sequence of calculations above completes a time step in the fractures. The remarks on parallelization for the single phase, fractured problem apply in like fashion to this problem.

**Nomenclature**

Symbols separated by a semicolon refer to fracture and matrix quantities, respectively. Generally, capitals denote fracture quantities, while small letters denote the corresponding matrix quantities.
\( B(u_1,u_2) = ((K/\mu c)\nabla u_1,\nabla u_2)) \)

\( b_i(u_1,u_2) = ((K/\mu c)\nabla u_1,\nabla u_2)) \)

\( C \) positive constant

\( c \) compressibility

\( \partial \sigma \) surface differential

\( f \) volumetric waterflood rate

\( f_e \) external source/sink

\( G_i \) scaled effect on matrix source function due to changes on \( \Omega_i \)

\( g \) gravitational constant

\( H_i; h_i, h_{i,j} \) spatial discretization parameter

\( J_i \) number of intervals partitioning \( L_i \)

\( K; \theta \) permeability

\( k_{iB}, k_{iF} \) relative permeabilities

\( L_i; L \) length of linear reservoir; linear block

\( M = L/h_i \) number of intervals partitioning \( L \)

\( N = \Delta t/\Delta t_m \) number of block time steps per fracture time step

\( P; p \) pressure \( [= P_w; p_w \text{ in immiscible flow}] \)

\( P_e; \theta \) pressure

\( P_c; \mu \) capillary pressure, oil minus water

\( p_{1_m} \) matrix pressure due to conditions on the block at time \( t_{f,m} \)

\( p_{2,m} \) matrix pressure due to effect of saturation change on \( \Omega_i \)

\( Q; \theta \) matrix source function

\( Q_i L_m \) matrix source due to conditions on the block at time \( t_{f,m} \)

\( q \) scaled pressure due to effect of saturation change on \( \Omega_i \)

\( \theta; \theta \) order of approximation of \( \mathbb{A}_i; \mathbb{B}_i, h_i \)

\( \Lambda; \lambda \) \( = \lambda_i + \lambda_{i,0} \lambda_i + \lambda_{i,0} \)

\( \Lambda_{i,0}; \lambda_{i,0} \) asymptotic constant

\( \mu, \mu_{i,0} \) viscosity

\( \nu; \nu \) outer unit normal to \( \Omega_i \) to \( \partial \Omega_i \)

\( \rho \) density in the blocks

\( \rho_i \) finite element approximant of \( \rho \) on \( \Omega_i \)

\( \rho_{1,m} \) matrix density due to conditions on the block at time \( t_{f,m} \)

\( s; s \) scaled saturation due to effect of saturation change on \( \Omega_i \)

\( s_{o,0} \) residual oil saturation

\( s_{o,0} \) saturation due to conditions on the block at time \( t_{f,m} \)

\( s_2 \) saturation due to effect of saturation change on \( \Omega_i \)

\( T \) maximal time of interest

\( t \) time

\( t_{f,m} \) \( = n\Delta t, n \mu \) time level

\( u \) dummy function

\( w \) test function

\( x \) space coordinate

\( x_i \) centroid of \( i \text{th} \) block, also \( i \text{th} \) fracture mesh location

\( y \) local space coordinate on a matrix block

\( y_{i,j} \) \( i \text{th} \) block's \( j \text{th} \) mesh location

\( z \) horizontal coordinate

\( z \) test function

\( \mathbb{A}_i \) fracture's Galerkin approximation space

\( \mathbb{B}_i; h_i \) \( i \text{th} \) block's Galerkin approximation space

\( \mathbb{A}_{i,0}; h_{i,0} \) \( i \text{th} \) block with imposed boundary condition

\( \alpha \) asymptotic constant

\( \beta \) asymptotic constant

\( \gamma \) asymptotic constant

\( \Delta t_{f,m} \) time discretization parameter

\( \delta \) asymptotic constant

\( \partial \Omega; \partial \Omega_i \) boundary of \( \Omega \) to \( \partial \Omega_i \)

\( \Theta; \Theta \) relative permeabilities

\( \Lambda_i; \lambda_i \) \( = \lambda_{i,0} \lambda_i + \lambda_{i,0} \lambda_i \)

\( \Lambda_i; \lambda_{i,0} \) asymptotic constant

\( \mu, \mu_i \) viscosity

\( \nu; \nu \) outer unit normal to \( \partial \Omega_i \) to \( \partial \Omega_i \)

\( \rho \) density in the blocks

\( \rho_i \) finite element approximant of \( \rho \) on \( \Omega_i \)

\( \rho_{1,m} \) matrix density due to conditions on the block at time \( t_{f,m} \)

\( \rho_{2,m} \) scaled matrix density due to effect of density change on \( \Omega_i \)

\( \phi; \phi \) porosity

\( \psi; \psi \) \( = \phi(x_i); \phi(u_{i,j}) \)

\( \Omega; \Omega_i \) reservoir; \( i \text{th} \) matrix block

\( \int_{\Omega} u_{i,j} \ dx \)

\( \int_{\Omega} u_{i,j} \ dx \)

\( |\Omega_i| \) volume (or length) of \( \Omega_i \) of \( \Omega_i \)
subscripts
f fracture
l i th matrix block, also i th fracture mesh location or interval
i+1/2 an evaluation at midpoint of fracture mesh
i,j j th matrix mesh location or interval on i th block
j+1/2 an evaluation at midpoint of block mesh
m matrix
n last fracture time \( t_f^{-1} \) [see superscript l]
o oil
w water
\( \theta \) o or w

superscripts
l block time level from last fracture time level [see subscript n]
\( n \) fracture time level

References

Appendix
Asymptotic Behavior for the Imbibition Problem
We consider the behavior of the saturation near an imbibition face at \( y = 0 \) when water saturation at that face is set at \( s_w^{-1} - s_r \). Normalize so that \( s_r = 0 \). Assume that for \( s_0 \) small

\[
\lambda_0 (s_0) - \alpha s_0^{\delta}, \delta > 0, \quad (71)
\]

\[
\lambda_w (s_0) - \beta, \quad (72)
\]

\[
p_C (s_0) - y. \quad (73)
\]

Then, the total flow \( q_w q_0 = 0 \) can be expressed as

\[
(\beta + \alpha s_0^{\delta}) \frac{\partial q_w}{\partial y} + \alpha s_0^{\delta} \frac{\partial s_0}{\partial y} = 0. \quad (74)
\]

So,

\[
- \beta \frac{\partial q_w}{\partial y} = q_w^{\alpha s_0^{\delta}} \frac{\partial s_0}{\partial y} \sim \alpha s_0^{\delta} \frac{\partial s_0}{\partial y} \quad (75)
\]

and

\[
s_0 (y, t) \sim \left[ \alpha \frac{s_0}{\partial q_w (0, t) y} \right]^{1/(\delta + 1)}. \quad (76)
\]
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