Eventual $C^\infty$-Regularity and Concavity for Flows in One-Dimensional Porous Media

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Abstract

We study the regularity and the asymptotic behaviour of the solutions of the initial value problem for the porous medium equation

$$u_t = (u^m)_{xx} \quad \text{in} \quad Q = \mathbb{R} \times (0, \infty),$$
$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R},$$

with \( m > 1 \) and \( u_0 \) a continuous, nonnegative function.

It is well known that, across a moving interface \( x = c(t) \) of the solution \( u(x,t) \), the derivatives \( v_t \) and \( v_x \) of the pressure \( v = (m/(m-1))u^{m-1} \) have jump discontinuities. We prove that each moving part of the interface is a \( C^\infty \)-curve and that \( v \) is \( C^\infty \) on each side of the moving interface (and up to it). We also prove that for solutions with compact support the pressure becomes a concave function of \( x \) after a finite time. This fact implies sharp convergence rates for the solution and the interfaces as \( t \to \infty \).
EVENTUAL $C^{\infty}$-REGULARITY AND CONCAVITY FOR FLOW IN ONE-DIMENSIONAL POROUS MEDIA

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with \( m > 1 \) and \( u_0 \) a continuous, nonnegative function.

It is well known that, across a moving interface \( x = \zeta(t) \) of the solution \( u(x,t) \), the derivatives \( v_t \) and \( v_x \) of the pressure \( v = (m/(m-1))u^{m-1} \) have jump discontinuities. We prove that each moving part of the interface is a \( C^\infty \)-curve and that \( v \) is \( C^\infty \) on each side of the moving interface (and up to it). We also prove that for solutions with compact support the pressure becomes a concave function of \( x \) after a finite time. This fact implies sharp convergence rates for the solution and the interfaces as \( t \to \infty \).
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Introduction.

Let $u = u(x,t)$ represent the density of an ideal gas flowing isentropically in an infinite one-dimensional porous medium. If $u$ is appropriately scaled its evolution is governed by the equation

$$u_t = (u^m)_{xx} \quad \text{in} \quad Q_T = \mathbb{R} \times (0,T) \quad \text{(0.1)}$$

together with the initial condition

$$u(\cdot,0) = u_0 \quad \text{in} \quad \mathbb{R}. \quad \text{(0.2)}$$

Here $m > 1$ and $T > 0$ are constants, while the initial distribution $u_0$ is a given nonnegative function. Because equation (0.1) is of parabolic type at the points where $u > 0$ but degenerates if $u = 0$, problem (0.1), (0.2) does not possess a classical solution if $u_0$ vanishes in some interval of $\mathbb{R}$. However the problem does possess a unique weak solution which is continuous everywhere in $Q_T$ under quite mild hypothesis on the initial data. Two of the main properties of the solutions of problem (0.1), (0.2) to which an extensive literature has been devoted are the regularity and the asymptotic behaviour, see [A3] for a survey of results. It is the aim of this paper to address the basic open problems for both properties.

One of the most important consequences of the degeneracy of equation (0.1) is the finite speed of propagation of disturbances from zero. For example, this implies that if $u_0$ is compactly supported then so is $u(\cdot,t)$ for every $t > 0$, and there appears an interface or free boundary which separates the sets $[u > 0]$ and $[u = 0]$. Across a moving part $\Gamma$ of this interface $u$ is not smooth, since both $(u^{m-1})_x$ and $(u^{m-1})_t$ are bounded functions with jump discontinuities on $\Gamma$. In the first part of this paper we prove that $\Gamma$ is a $C^\infty$-curve, and also
that $u^{m-1}$ can be continued up to $\tau$ from each side as a $C^\infty$-function. A full description of the results with reference to related works in given in Section 1, while the main estimates are proved in Sections 2 and 3.

The last section of the paper is devoted to the asymptotic behaviour of the solutions whose initial data $u_0$ have compact support. It is known that any such solution approaches, as $t \to \infty$, a concave profile given by an explicit solution $\bar{u}$ (the Barenblatt solution) which satisfies

$$(u^{m-1})_{xx} = -\frac{m-1}{m(m+1)t}.$$

Recently Bénilan and Vazquez [BV] have proved that whenever $u^{m-1}$ has compact support and is concave in its support, then as $t \to \infty$ we get

$$(u^{m-1})_{xx} = -\frac{m-1}{m(m+1)t}(1 + O(\frac{1}{t})).$$

inside the support of $u$. From (0.3) strong convergence results follow for the solution and its interface.

In section 4 we prove that any solution $u$ whose initial data has compact support is spatially concave in its support after a finite time so that (0.3) and its consequences hold without the restriction of initial concavity.

1. Regularity. Statement of results

Let us recall some of the main facts about problem (0.1), (0.2), and introduce convenient definitions and notations. The existence, uniqueness and well-posedness of weak solutions for an optimal class of (nonnegative) initial data has been discussed in the papers [AC], [BCP], [DK], among others, even for the $N$-dimensional analogue $u_t = \Delta(u^m)$ with $N > 1$. Without loss of generality we may assume that $u_0$ is a continuous nonnegative real function which grows as $|x| \to \infty$ at most like $O(|x|^\alpha)$ with $\alpha = 2/(m-1)$. In that case a unique continuous nonnegative function $u$ exists in a strip $Q_T$ with $0 < T < \infty$. 
which satisfies (0.1) in the sense of distributions in $Q_T$ and (0.2) in the sense of trace on $|R \times \{0\}$. The solution is global, i.e. $T = \infty$, precisely if $u_0(x) = o(|x|^\alpha)$ [BCP]. To avoid unnecessary complications we assume in the sequel that $T = \infty$. See, e.g., [V2] for details about the blow up if $T < \infty$. We set $Q = Q_\infty$.

Since (0.1) is strictly parabolic in the positivity set of a solution $u$, defined as

$$\mathcal{P}[u] = \{(x,t) \in Q_T : u(x,t) > 0\},$$

(1.1)

it follows that $u \in C^\infty(\mathcal{P})$, [OKC]. In case $u_0$ is, say, a continuous positive function, then $\mathcal{P} = Q$ and $u$ is a classical solution of (0.1). However, if $u_0$ vanishes on some open interval, then, because of the finite propagation property, the set $\mathcal{P}[u]$ is a proper subset of $Q$ and an interface appears which separates $\mathcal{P}[u]$ from the region where $u = 0$. We do not require that $u_0$ have compact support.

The basic properties of the interface are by now well understood. Without essential loss of generality we may consider the case where $u_0$ vanishes on $|R|$ and is a continuous positive function, at least, on an interval $(0,a)$ with $a > 0$. Then $\mathcal{P}[u]$ is bounded to the left in the $(x,t)$-plane by the left interface curve $x = \zeta(t)$, where

$$\zeta(t) = \inf \{x \in |R| : u(x,t) > 0\}.$$  

(1.2)

It is known that $\zeta(t)$ is a Lipschitz continuous nonincreasing function in $[0,\infty)$ with $\zeta(0) = 0$. Moreover, there exists a time $t^* \in [0,\infty]$, called the waiting time, such that $\zeta(t) = 0$ for $0 < t < t^*$ and $\zeta(t) < 0$ for $t > t^*$. $t^*$ can be zero or positive depending on the local behaviour of $u_0$ near 0, cf. [K], [ACK], [V2] for related results. Caffarelli and Friedman [CF] have shown that $\zeta \in C^1(t^*, T)$ and $\zeta'(t) < 0$ for every $t > t^*$, i.e., a moving interface never stops. In deriving these results they also established lateral $C^1$ regularity for the pressure variable.
\[ v = \frac{m}{m-1} u^{m-1}, \quad (1.3) \]

which satisfies the equation

\[ v_t = (m-1)vv_{xx} + v_x^2. \quad (1.4) \]

In fact, the functions \( v_x(x,t) \) and \( v_t(x,t) \) admit limits as \( (x,t) \in \mathcal{I} \) tends to a point \((\zeta(t_0), t_0)\) of the moving part of the interface. Moreover, the extended functions thus obtained are continuous in the set

\[ D = \{(x,t): t > t^*, \, \zeta(t) < x < 0\} \quad (1.5) \]

The relationship between these results is given by the following equation on the interface ([A2], [K], [CF])

\[ \zeta'(t) = -v_x(\zeta(t), t). \quad (1.6) \]

Finally it is proved in [CF] that \( v_t = v_x^2 \) on the moving interface. Observe that (1.6) means that \( v_x \) has a jump across the moving interface, since \( \lim_{x \to -\infty} v_x \) is zero as \((x,t) \to (\zeta(t_0), t_0)\) with \( x < \zeta(t) \).

After these results the main regularity question for the solutions of problems (0.1), (0.2) can be formulated as follows. How smooth is the interface and how smooth is the solution near the interface? Our answer to this question is the following

**THEOREM 1.** \( v \) is a \( C^\infty \) function in \( D \) and \( \zeta(t) \) is a \( C^\infty \) function in \((t^*, \infty)\).

In case \( t^* = 0 \) the interface is then \( C^\infty (\mathbb{R}^+) \). If \( t^* > 0 \) we have \( \zeta(t) = 0 \) for \( 0 < t < t^* \) so a lack of regularity of \( \zeta \) can only occur at \( t = t^* \). In fact it has been proved in [ACV] that for some initial data, e.g., if \( u_0(x) = o(|x|^{2/m-1}) \) as \( x \to 0 \), we have \( t^* > 0 \) and \( D^+ \zeta(t^*) < 0 \) so that \( \zeta \) has a corner at \( t = t^* \). Lipschitz continuity is therefore the optimal global
regularity for interfaces with a positive waiting time. However, there are solutions with a positive waiting time and a $C^1$ interface, cf. [ACK] and [LOT]. In this last work self-similar solutions are constructed with $t^* > 0$ and $\zeta(t) = -(t-t^*)^\alpha$ for $t > t^*$, where $\alpha$ is larger than 1. Therefore $\zeta \in C^\infty(\mathbb{R}^+)$ in those cases.

The main step in the proof of Theorem 1 consists in improving the known regularity for $v$ by establishing a bound for $v_{xx}$ near the points of the moving interface.

**LEMMA 1.1.** $v_{xx}$ is locally bounded in $D$.

Part of this estimate is well known. In fact, the lower bound

$$v_{xx} > -\frac{1}{(m+1)t} \quad \text{in} \quad \Omega'(Q_T),$$

established in [AB], is one of principal tools used in the theory of the porous medium equation. (The analogous estimate also holds in several space dimensions). Since $v$ is a $C^\infty$ function in $\Omega[u]$, it suffices to derive local upper bounds near points of the interface. At those points we construct (see Section 2) an explicit local barrier for $v_{xx}$ by taking advantage of the fact that the possible blow-up of $v_{xx}$ at the boundary is controlled by

$$v_{xx} = \frac{1}{(m-1)} (v_t - v_x^2) + 0$$

if $(x,t) \in D$ and $(x,t) + (\zeta(t_0), t_0)$ with $t_0 > t^*$. Again this bound cannot be extended to $t = t^*$. In fact, if $t^* > 0$ and $\gamma = D^+ \zeta(t^*) < 0$ then

$$\limsup_{(x,t) \to (0,t^*)} v_{xx}(x,t) = +\infty.$$  

Moreover $v_{xx}$ blows up as $(x,t) \to (0,t^*)$ along the line $x = \gamma(t-t^*)$ faster than $O(1/x)$, cf. [ACV].

Bounds for the higher order derivatives of $v$ with respect to $x$, i.e., $v^{(k)} = (a/\partial x)^k v$ for $k > 3$, are obtained by a similar process in Section 3. We
first derive the differential equation for \( v^{(k)} \) and obtain a preliminary estimate near the boundary by a rescaling argument together with interior estimates for nondegenerate equations. Finally we apply an iterative barrier argument to get a bound in \( L^\infty_{10c}(D) \). The barrier argument is different from, but related to the one used for \( k = 2 \). Since bounds for the mixed derivatives 
\((\partial / \partial t)^{\ell}(\partial / \partial x)^k v\) with \( \ell > 1 \) and \( k > 0 \) can be obtained from bounds for \( v^{(k)} \) with \( k > 0 \) by applying the differential operator \(( -\partial / \partial t)^{\ell-1}(\partial / \partial x)^k \) to the equation (1.4), it follows that \( v \) is \( C^\infty \) smooth in \( D \) (cf., [Al]). The \( C^\infty \) smoothness of \( \zeta \) is then easily obtained by repeatedly differentiating equation (1.6). Thus to prove Theorem 1 it suffices to derive bounds for \( v^{(k)} \).

The consideration of moving parts of a right hand interface, if there exists one, or of inner interfaces if the initial support contains holes, offers no difficulties due to the local character of our results. Observe in particular, that any moving inner interface must end after a finite time at a point where it meets another inner interface, and \( v_x \) cannot be continuous at that point, cf. [V1].

After we proved Theorem 1 we learned that the result had been obtained previously by Höllig and Kreiss [HK]. Their proof requires the additional restriction that the initial data have compact support and it is based on a technique of iterative weighted norms. A partial \( C^\infty \) result had been previously obtained by Höllig and Pilant [HP]. More recently Angenent [An] has announced a proof of the analytic regularity of the free boundaries in the case of a solution with compact support that exhibits a left hand and a right hand interface. The analyticity occurs for \( t \) larger than the maximum of both waiting times. His proof uses our Lemma 1.1 and is based on nonlinear semigroup theory.

2. The Upper Bound for \( v_{xx} \)

Let \( v = v(x, t) \) be the pressure corresponding to a solution \( u = u(x, t) \) of the porous medium equation (0.1). Then \( v = \frac{m}{m-1} u^{m-1} \) and it satisfies the equation (1.4):
\[ v_t = (m-1)v v_{xx} + v_x^2 \]

in the positivity set \( \overline{\Phi}[u] \). Let \( q = (x_0, t_0) \) be a point on the left interface, so that \( x_0 = \zeta(t_0), v(x, t_0) = 0 \) for all \( x < \zeta(t_0) \), and \( v(x, t_0) > 0 \) for all sufficiently small \( x > \zeta(t_0) \). We assume that the interface is moving at \( q \). Thus \( t_0 > t^* \), where \( t^* \) is the waiting time and

\[ \zeta'(t_0) = -v_x(x, t_0) \equiv -a < 0. \tag{2.1} \]

We shall use the notation

\[ R_{\delta, \eta} = R_{\delta, \eta}(t_0) \equiv \{(x, t) \in \mathbb{R}^2 : \zeta(t) < x < \zeta(t) + \delta, t_0 - \eta < t < t_0 + \eta\} \]

(see Figure 1).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

We prove the following precise version of Lemma 1.1.

**Lemma 2.1** Let \( q \) be a point on the left interface and assume that (2.1) holds. There exist positive constants \( C, \delta \) and \( \eta \) depending only on \( m \) and \( q \) such that

\[ v_{xx} < C \quad \text{in} \quad R_{\delta, \eta/2} \tag{2.2} \]

**Proof.** As we said above, [CF] proved that \( v_t, v_x \) and \( vv_{xx} \) are continuous in a closed neighborhood in \( \overline{\Phi}[u] \) of any point on a moving interface, and that
\[(\psi\psi_{xx})(x,t) \to 0 \text{ as } \mathcal{P}[u] \ni (x,t) \in (\zeta(t), \tau) \quad (2.3)\]

for any \( \tau > t^* \). Choose now an \( \epsilon > 0 \) such that

\[(a - 4m\epsilon)(a - \epsilon) > 4(m+1)\epsilon > 0. \quad (2.4)\]

Then there exists a \( \delta = \delta(\epsilon) > 0 \) and \( \eta = \eta_1(\epsilon) \in (0, t_0 - t^*) \) such that

\[R_{\delta, \eta} \subset \mathcal{P}[u], \quad \quad a - \epsilon < \psi_x < a + \epsilon, \quad (2.5)\]

and

\[\psi\psi_{xx} < \epsilon \quad (2.6)\]

in \( R_{\delta, \eta} \). In view of (2.5) we have

\[(a-\epsilon)(x-\zeta(t)) < \psi(x,t) < (a+\epsilon)(x-\zeta(t)) \text{ in } R_{\delta, \eta} \quad (2.7)\]

and

\[-(a+\epsilon) < \zeta'(t) < -(a-\epsilon) \text{ in } [t_1, t_2] \quad (2.8)\]

where \( t_1 = t_0 - \eta \) and \( t_2 = t_0 + \eta \). We set

\[\zeta^*(t) = \zeta_1 - b(t-t_1), \quad (2.9)\]

where \( \zeta_1 = \zeta(t_1) \) and \( b = a+2\epsilon \). Clearly \( \zeta(t) > \zeta^*(t) \) in \([t_1, t_2]\).

On \( \mathcal{P}[u] \), \( p = \psi_{xx} \) satisfies

\[\mathcal{L}(p) = p_t - (m-1)v\psi_{xx} - 2mv_x\psi_x - (m+1)p^2 = 0. \quad (2.10)\]

We shall construct a \textbf{barrier} for \( p \) in \( R_{\delta, \eta} \) of the form

\[\phi(x,t) = \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)}, \quad (2.11)\]

where \( \alpha \) and \( \beta \) are positive constants. To accomplish this we must show that \( \alpha \) and \( \beta \) can be chosen so that \( \mathcal{L}(\phi) > 0 \) in \( R_{\delta, \eta} \) and \( \phi > p \) on the parabolic boundary of \( R_{\delta, \eta} \). Indeed, we shall show that this can be done for arbitrarily small \( \alpha \) so that, in the limit as \( \alpha \to 0 \), \( \phi \) gives a finite bound for
p in $R_{\delta, \eta/2}$.

It is easy to verify that

$$
\mathcal{L}(\phi) = \frac{\alpha}{(x-\zeta)^2} \left\{ \zeta' - 2(m-1) \frac{v}{x-\zeta} + 2m v_x \right\} + \frac{\beta}{(x-\zeta^*)^2} \left\{ \zeta^* - 2(m-1) \frac{v}{x-\zeta^*} + 2m v_x \right\} - (m+1)\phi^2
$$

$$
> \frac{\alpha}{(x-\zeta)^2} \left\{ \zeta' - 2(m-1) \frac{v}{x-\zeta} + 2m v_x - 2(m+1)\alpha \right\} + \frac{\beta}{(x-\zeta^*)^2} \left\{ \zeta^* - 2(m-1) \frac{v}{x-\zeta^*} + 2m v_x - 2(m+1)\beta \right\}.
$$

From the estimates (2.5), (2.7), (2.8) and the definition (2.9) of $\zeta^*$ we conclude that

$$
\mathcal{L}(\phi) > \frac{\alpha}{(x-\zeta)^2} \left\{ a - (4m-1)e - 2(m+1)\alpha \right\} - \frac{\beta}{(x-\zeta^*)^2} \left\{ a - 4m e - 2(m+1)\beta \right\}.
$$

Set

$$
\beta = (a - 4m e) / 2(m+1)
$$

(2.12)

and note that (2.4) implies that $\beta > 0$. Then $\mathcal{L}(\phi) > 0$ in $R_{\delta, \eta}$, for all $\alpha \in (0, \alpha_0)$, where $\alpha_0 = (a - (4m-1)e) / 2(m+1)$.

Let us now compare $p$ and $\phi$ on the parabolic boundary of $R_{\eta, \delta}$. In view of (2.6) and (2.7) we have

$$
v_{xx} < \frac{e}{(a-e)(x-\zeta)} \text{ in } R_{\delta, \eta}
$$

so that, in particular,

$$
v_{xx}(\zeta(t) + \delta, t) < \frac{e}{(a-e)\delta} \text{ in } [t_1, t_2].
$$

By the mean value theorem and (2.8) it follows that for some $\tau \in (t_1, t_2)$

$$
\zeta(t) + \delta - \zeta^*(t) = \delta + (a+2e)(t-t_1) + \zeta'(\tau)(t-t_1) < \delta + 3e(t-t_1) < \delta + 6\epsilon \eta.
$$

Now set

$$
\eta = \min \{ \eta_1(\epsilon), \delta(\epsilon) / 6 \epsilon \}.
$$

Since $\epsilon$ satisfies (2.4) and $\beta$ is given by (2.12) it follows that
\( \phi(t+\delta,t) > \frac{\beta}{2\delta} > \frac{e}{(a-e)\delta} > v_{xx}(t+\delta,t) \) on \([t_1,t_2]\).

Moreover,

\[ \phi(x,t) > \frac{\beta}{x-\zeta} > \frac{e}{(a-e)(x-\zeta_1)} > v_{xx}(x,t) \] on \((\zeta_1, \zeta_1+\delta]\).

Let \( \tau = \{(x,t) \in \mathbb{R}^2 : x = \zeta(t), t_1 < t < t_2\} \). Clearly \( \tau \) is a compact subset of \( \mathbb{R}^2 \). Fix \( \alpha \in (0,\alpha_0) \). For each point \( s \in \tau \) there is an open ball \( B_s \) centered at \( s \) such that

\[
(vv_{xx})(x,t) < \alpha(a-e) \text{ in } B_s \cap \Phi[u].
\]

In view of (2.7) we have

\[
\phi(x,t) > \frac{\alpha}{x-\zeta} > v_{xx}(x,t) \text{ in } B_s \cap \Phi[u].
\]

Since a finite number of these balls suffices to cover \( \tau \) it follows that there exists a \( \gamma = \gamma(\alpha) \in (0,\delta) \) such that

\[
\phi(x,t) > p(x,t) \text{ in } R_{\gamma,\eta}.
\]

Thus for every \( \alpha \in (0,\alpha_0) \), \( \phi \) is a barrier for \( p \) in \( R_{\delta,\eta} \).

By the comparison principle for parabolic equations [LSU] we conclude that

\[
v_{xx}(x,t) < \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta(t)} \text{ in } R_{\delta,\eta},
\]

where \( \beta \) is given by (2.12) and \( \alpha \in (0,\alpha_0) \) is arbitrary. Now let \( \alpha \to 0 \) to obtain

\[
v_{xx}(x,t) < \frac{\beta}{x-\zeta^*(t)} < \frac{2\beta}{e\eta} \text{ in } R_{\delta,\eta/2}.
\]

REMARK: The above argument shows that \( \zeta, \delta \) and \( \eta \) may be chosen to vary continuously with the point \( q = (x_0,t_0) \), where \( x_0 = \zeta(t_0) \) and \( t_0 > t^* \).
3. Bounds for \( \left( \frac{\partial}{\partial x} \right)^j v \)

We now turn to the problem of estimating the derivatives of the form

\[
v(j) = \left( \frac{\partial}{\partial x} \right)^j v
\]

for \( j > 3 \), from which Theorem 1 easily follows as we explained in the Introduction. Here we shall use a barrier similar to the one employed in §2 in the estimation of \( v(2) \), but there are some crucial differences. In the estimate for \( v(2) \) we were able to let \( \alpha \to 0 \) because we knew apriori that

\[
v(2)(x,t) < o(1/d)
\]

as \( x + \zeta(t) \), where \( d = d(x,t) \equiv x - \zeta(t) \). For \( j > 3 \) the only apriori information we can get is the weaker estimate

\[
|v(j)(x,t)| < O(1/d)
\]

(Lemma 3.2). However to compensate for this, when \( j > 3 \) the equation for \( v(j) \) is linear so that there will be no restriction on the size of \( \beta \). We shall exploit this fact in an iterative barrier argument which enables us to derive a finite bound starting with an estimate which blows up on the interface. (Lemma 3.4).

For \( j > 3 \), the \( v(j) \) satisfy the equation

\[
\mathcal{L} v(j) = v_{tt}(j) - (m-1)v v_{xx}(j) - (2+j(m-1)) v_x v_{x}(j) - c_{mj} v_{xx} v(j) - \sum_{\ell=3}^{j^*} d_{mj}^\ell v(\ell)v(j+2-\ell).
\]

in \( \mathcal{G}(u) \), where \( j^* = [j/2] + 1 \), and the \( c_{mj} \) and \( d_{mj}^\ell \) are positive constants which depend only on their indices, but whose precise values are irrelevant.

Observe that the sum in \( \mathcal{L} j \) involves only derivatives of order \( < j \) and it is omitted for \( j = 3 \) since in that case \( j^* = 2 < 3 \). Our result is
PROPOSITION 3.1. Let \( q = (x_0, t_0) \) be a point on the interface for which (2.1) holds. For each integer \( j > 2 \) there exist constants \( C_j, \delta \) and \( n \) depending only on \( m, j \) and \( q \) such that

\[
| (\frac{\partial}{\partial x})^j v | < C_j \text{ in } R_{\delta, n/2}.
\]

(3.2)

The proof proceeds by induction on \( j \). Suppose that \( q = (x_0, t_0) \) is a point on the left interface for which (2.1) holds. Fix \( \epsilon \in (0, a) \) and take \( \delta_0 = \delta_0(\epsilon) > 0 \) and \( \eta_0 = \eta_0(\epsilon) \in (0, t_0 - t^*) \) such that \( R_0 = R_{\delta_0, \eta_0(\epsilon)} \subset \overline{P(u)} \) and (2.5) holds. Thus we also have (2.7) and (2.8) in \( R_0 \). Assume that there exist constants \( C_k \in \mathbb{R}^+ \) for \( k = 2, 3, \ldots, j-1 \) such that

\[
|v^{(k)}| < C_k \text{ on } R_0 \text{ for } k = 2, \ldots, j-1.
\]

(3.3)

Observe that, by Lemma 2.1 and the estimate (1.7), (3.3) holds for \( k = 2 \).

Rescaling and interior estimates allow us to obtain a first estimate near \( \Gamma \).

LEMMA 3.2. There exist constants \( K \in \mathbb{R}^+, \delta \in (0, \delta_0), \) and \( n \in (0, \eta_0) \), depending only on \( q, m \) and the \( C_k \) for \( k \in [2, j-1] \) with \( j > 3 \), such that

\[
|v^{(j)}(x, t)| < K/(x-\zeta(t)) \text{ in } R_{\delta, n}.
\]

Proof.

Set

\[
\delta = \min \{2\delta_0/3, 2s\eta_0\},
\]

\[
n = \eta_0 - \frac{\delta}{4s},
\]

...
and define

\[ R(\bar{x}, \bar{t}) \equiv \{(x, t) \in \mathbb{R}^2 : |x - \bar{x}| < \frac{\lambda}{2}, \bar{t} - \frac{\lambda}{4s} < t < \bar{t}\} \]

for \((\bar{x}, \bar{t}) \in R_{\delta, n}\), where \(s = a - \varepsilon\) and \(\lambda = \bar{x} - \zeta(\bar{t})\). Then \((\bar{x}, \bar{t}) \in R_{\delta, n}\) implies that \(R(\bar{x}, \bar{t}) \subseteq R_0\). Also observe that for each \((\bar{x}, \bar{t}) \in R_{\delta, n}\), \(R(\bar{x}, \bar{t})\) lies to the right of the line \(x = \zeta(\bar{t}) + s(\bar{t} - t)\).

![Figure 2](image)

Set \(x = \lambda \xi + \bar{x}\) and \(t = \lambda \tau + \bar{t}\). The function

\[ v(j-1)(\xi, \tau) \equiv v(j-1)(\lambda \xi + \bar{x}, \lambda \tau + \bar{t}) = v(j-1)(x, t) \]

satisfies the equation

\[ v_{(j-1)} = \{(m-1)\lambda^j_v V_{(j-1)}(\xi) + (2 + (j-1)(m-1))v_x v(j-1)\}_{\xi} \]

\[ \quad + \lambda \sum_{\xi = 3}^{j-1} m, j-1 \lambda^j_v V_{(j+1-\xi)}(\xi) \]

in the region

\[ B = \{(\xi, \tau) \in \mathbb{R}^2 : |\xi| < \frac{1}{2}, -1/4s < \tau < 0\}, \]
and \(|V(j-1)| < C_{j-1}\) in \(B\) (see Figure 2). In view of (2.7) and (2.8)

\[
(a - \varepsilon) \frac{x - \zeta (L)}{\lambda} < \frac{v(x, t)}{\lambda} < (a + \varepsilon) \frac{x - \zeta (L)}{\lambda}
\]

and

\[
\zeta(l) < \zeta(t) < \zeta(l) + s(l - t) < \zeta(l) + \frac{\lambda}{4}.
\]

Therefore

\[
\frac{\lambda}{4} = \bar{x} - \frac{\lambda}{2} - \zeta(l) - \frac{\lambda}{4} < \bar{x} - \zeta(t) < \bar{x} + \frac{\lambda}{2} - \zeta(t) = \frac{3\lambda}{2}
\]

which implies

\[
\frac{a - \varepsilon}{4} < \frac{v}{\lambda} < \frac{3(a + \varepsilon)}{2},
\]

that is, equation (3.4) is uniformly parabolic in \(B\). Moreover, it follows from (2.5) and (3.3) that \(V(j-1)\) satisfies all of the hypothesis of Theorem 5.3.1 of [LSU]. Thus we conclude that there is a constant \(K = K(a, m, C_{1}, ..., C_{j-1}) > 0\) such that

\[
|\frac{\partial}{\partial \xi} V(j-1)(0, 0)| < K,
\]

that is,

\[
|v(j)(\bar{x}, l)| < K/\lambda.
\]

Since \((\bar{x}, l) \in R_{\delta, \eta}\) is arbitrary this proves the lemma.

We now turn to the barrier construction. If \(\gamma \in (0, \delta)\) we will use the notation

\[
R^{\gamma}_{\delta, \eta} = R^{\gamma}_{\delta, \eta}(t_0) = \{(x, t) \in \mathbb{R}^2: \zeta(t) + \gamma < x < \zeta(t) + \delta, t_0 - \eta < t < t_0 + \eta\}.
\]
LEMMA 3.3. Let $R_{\delta_1, \eta_1}$ be the region constructed in the proof of Lemma 3.1. For $j > 3$ and $(x,t) \in R_{\delta_1, \eta_1}^\gamma$, let

$$\phi_j(x,t) = \frac{\alpha}{x - \zeta(t) - \gamma/3} + \frac{\beta}{x - \zeta^*(t)} \quad (3.5)$$

where $\zeta^*$ is given by (2.9), and $\alpha$ and $\beta$ are positive constants. There exist $\delta \in (0, \delta_1)$ and $\eta \in (0, \eta_1)$ depending only on $a, m, C_1, \ldots, C_{j-1}$ such that

$$\ell_j(\phi_j) > 0 \text{ in } R_{\delta, \eta}^\gamma$$

for all $\gamma \in (0, \delta)$

Proof. Choose $\epsilon$ such that

$$0 < \epsilon < \frac{\alpha}{4 + 2J(m-1)} \quad (3.6)$$

There exist $\delta_2 \in (0, \delta_1)$ and $\eta \in (0, \eta_1)$ such that (2.5), (2.7) and (2.8) hold in $R_{\delta_2, \eta}^\gamma$. Fix $\gamma \in (0, \delta_2)$. For $(x,t) \in R_{\delta_2, \eta}^\gamma$,

$$\ell_j(\phi_j) = \frac{\alpha}{(x - \zeta - \gamma/3)^2} \left( \zeta' - \frac{2(m-1)v}{x - \zeta - \gamma/3} + (2 + j(m-1))v_x - c_{mj}(x - \zeta - \gamma/3)v_{xx} \right. \right.$$

$$\left. \left. - (x - \zeta - \gamma/3)^2 \sum_{\ell=3}^{*} d_{mj}^\ell v(\ell)v(j + 2 - \ell) \right) \right.$$
$$\mathcal{L}_j(\phi_j) > \frac{\alpha}{(x-\zeta-\gamma/3)^2} (a - (3 + 2j(m-1)) \varepsilon - \delta_2(c_{mj}C_2 + \delta_2 \sum_{k=3}^{j} d_{mj}^k C_k C_{j+2-k})$$

$$+ \frac{\beta}{(x-\zeta^*)^2} (a - (4 + 2j(m-1)) \varepsilon - \delta_2(c_{mj} C_2 + \delta_2 \sum_{k=3}^{j} d_{mj}^k C_k C_{j+2-k})$$.

Since $\varepsilon$ satisfies (3.6) we can choose $\delta = \delta_2(\varepsilon, m, C_2, \ldots, C_{j-1}) > 0$ so small that $\mathcal{L}_j(\phi_j) > 0$ in $R_\delta, \gamma$.

**LEMMA 3.4 (Barrier Transformation).** Let $\delta$ and $n$ be as in Lemma 3.3 with the additional restriction that

$$n < \delta/6 \varepsilon, \quad (3.7)$$

where $\varepsilon$ satisfies (3.6). Suppose that for some nonnegative constants $\alpha$ and $\beta$

$$v(j)(x,t) < \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \text{in} \quad R_\delta, \gamma. \quad (3.8)$$

Then $v(j)$ also satisfies

$$v(j)(x,t) < \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta + 2\alpha/3}{x-\zeta^*(t)} \quad \text{in} \quad R_\delta, \gamma. \quad (3.9)$$

**Proof.** By Lemma 3.3, for any $\gamma \in (0, \delta)$ the function

$$\phi_j(x,t) = \frac{2\alpha/3}{x-\zeta-\gamma/3} + \frac{\beta + 2\alpha/3}{x-\zeta^*}$$

satisfies $\mathcal{L}_j(\phi_j) > 0$ in $R_\delta, \gamma$. On the other hand, on the parabolic boundary of $R_\delta, \gamma$ we have $\phi_j > v(j)$. In fact, for $t = t_1$ and $\zeta_1 + \gamma < x < \zeta_1 + \delta$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_j(x,t_1) = \frac{2\alpha/3}{x-\zeta_1-\gamma/3} + \frac{\beta + 2\alpha/3}{x-\zeta_1} > \frac{4\alpha/3}{x-\zeta_1} + \frac{\beta}{x-\zeta_1} > v(j)(x,t_1),$$
while for $x = \zeta + \delta$ and $t_1 < t < t_2$ we get, in view of (3.7),

$$
\phi_j(\zeta+\delta,t) > \frac{2\alpha/3}{\delta-\gamma/3} + \frac{\beta}{\zeta+\delta-\zeta^*} + \frac{2\alpha/3}{\delta+\delta\varepsilon_n} \geq \frac{2\alpha/3}{\delta} + \frac{\beta}{\zeta+\delta-\zeta^*} + \frac{\alpha/3}{\delta} > v(j)(\zeta+\delta,t).
$$

Finally, for $x = \zeta + \gamma$, $t_1 < t < t_2$ we have

$$
\phi_j(\zeta+\gamma,t) = \frac{2\alpha/3}{\gamma-\gamma/3} + \frac{\beta+2\alpha/3}{\zeta+\gamma-\zeta^*} \geq \frac{\alpha}{\gamma} + \frac{\beta}{\zeta+\gamma-\zeta^*} > v(j)(\zeta + \gamma,t).
$$

By the comparison principle we get

$$
\phi_j > v(j) \text{ in } R_{\delta,n}^\gamma
$$

for any $\gamma \in (0,\delta)$, and (3.9) follows by letting $\gamma \downarrow 0$.

Completion of proof of Proposition 3.1. By Lemma 3.2, we have an estimate for $v(j)$ of the form (3.8) with $\alpha = K$ and $\beta = 0$. Iterating this estimate by the Barrier Transformation Lemma we obtain the sequence of estimates

$$
v(j)(x,t) < \frac{\alpha_n}{x-\zeta(t)} + \frac{\beta_n}{x-\zeta^*(t)}
$$

with $\alpha_n = (2/3)^nK$ and $\beta_n = \left\{ \left( \frac{2}{3} \right) + \ldots + \left( \frac{2}{3} \right)^n \right\}K$. Thus if we let $n \to \infty$ we obtain an upper bound for $v(j)$ of the form

$$
v(j)(x,t) \leq \frac{2K}{x-\zeta^*(t)} \text{ in } R_{\delta,n}
$$

(3.10)

As in the proof of Lemma 2.1, this implies that $v(j)$ is bounded above in $R_{\delta,n/2^*}$.

Since the equation (3.1) for $v(j)$ is linear, a similar lower bound can be obtained in the same way and the induction step is complete.
4. **Eventual concavity and asymptotic behaviour**

In this section we assume that $u_0$ has compact support, say the interval $I = (a,b)$. Without loss of generality we may assume that $u_0$ is continuous and $u_0(x) > 0$ for $x \in (a,b)$. According to the results of [VI] there are in that case two expanding interfaces, $\zeta_-(t)$ and $\zeta_+(t)$, such that

$$\mathcal{P}[u] = \{(x,t) \in Q; \zeta_-(t) < x < \zeta_+(t)\},$$

and as $t \to \infty$ we have the expansion

$$\zeta_{\pm}(t) = \mp r(t) + x_0 + o(1), \quad (4.1)$$

where $r(t) = c_m(M^{m-1} t)^{m+1}$, $c_m$ is constant that depends only on $m$,

$$M = \int u_0(x) dx, \quad (4.2)$$

and

$$x_0 = M^{-1} \int x u_0(x) dx. \quad (4.3)$$

The **total mass** $M$ and **center of mass** $x_0$ are two invariants of the evolution. Moreover, $x = \mp r(t) + x_0$ is precisely the interface of a particular solution of (0.1), called the Barenblatt solution, whose pressure is given by

$$\bar{v}(x,t) = \frac{(r^2(t) - |x - x_0|^2)^+}{2(m+1)t}. \quad (4.4)$$

We remark that the Barenblatt solution takes intial values $u_0(x) = M \delta(x)$ with $M$ given by (4.2) and $\delta$ being the Dirac measure concentrated at $x_0$. The asymptotic closeness of $v$ and $\bar{v}$ is not restricted to formula (4.1). In fact we also have as $t \to \infty$,

$$\zeta_{\pm}'(t) = \mp r'(t) + o(1/t) \quad \text{and} \quad |\zeta_{\pm}'(t)| < r'(t), \quad (4.5)$$

$$v(x,t) - \bar{v}(x,t) = o(t \frac{m}{m+1}) \text{ uniformly in } x, \quad (4.6)$$
and

$$v_x(x,t) + x - x_0 \frac{1}{(m+1)t} = 0 \left( \frac{1}{t} \right) \text{ uniformly in } (\zeta_-(t), \zeta_+(t)). \quad (4.7)$$

The convergence rates in the above formulas are not optimal. Optimal rates are obtained in [V1] under the assumption of spatial symmetry. In [BV] the case of a solution whose initial pressure $v_0$ is concave in its support (i.e., $v'' < -c$ in $\mathcal{D}'(a,b)$) is considered and it is shown that $v(\cdot,t)$ is a concave function in $(\zeta_-(t), \zeta_+(t))$ for every $t > 0$ and also that

$$v_{xx} = -\frac{1}{(m+1)t} + 0 \left( \frac{1}{t^2} \right) \quad (4.8)$$

uniformly in $(\zeta_-(t), \zeta_+(t))$. From (4.8) sharp estimates for $v$ and $v_x$ can be derived by integration and, using formula (1.6), for $\zeta_+, \zeta_+^i$ and $\zeta_+^ii$ (see [BV]). In particular,

$$\zeta_+^ii = \pm r''(t)(1+O(\frac{1}{t^2})) \quad (4.9)$$

Here we show that the assumption of initial concavity can be removed in the above results. We prove

THEOREM 2. For every solution $u$ of (0.1), (0.2) which is compactly supported in space there exists a finite time $T$ after which the pressure $v(x,t)$ is a concave function with respect to the space variable in $\mathcal{C}[u]$, the functions $|\zeta_+(t)|$ are concave in $(T,\infty)$, and formulas (4.8), (4.9) hold as $t \to \infty$.

We divide the proof of Theorem 2 into a series of Lemmas. In the first two Lemmas we show that (4.8) holds locally in $\mathcal{C}[u]$ as $t \to \infty$. The remaining Lemmas show that this estimate can be made global. The key step is to show that the estimates in Sections 2 and 3 hold uniformly as $t \to \infty$ (Lemma 4.4) and for this we need a uniform rate of convergence of $vv_{xx}$ to zero at the interface (Lemma 4.3).
Without loss of generality we set $M = 1$ and $x_0 = 0$. We also introduce the sets

$$
\mathcal{P}_\alpha = \{(x,t) \in \Omega: |x| < \alpha r(t)\}
$$

for $\alpha \in (0,1)$. Since $|\xi_\pm(t)| > r(t)$ (cf. [V1: Theorem A]) we have

$$
\mathcal{P}_\alpha \subset \mathcal{P}[u].
$$

**Lemma 4.1.** For every $\alpha \in (0,1)$ there exist positive constants $C_\alpha$ and $t_\alpha$ such that

$$
|v_{XXX}(x,t)| < C_\alpha t^{-\frac{m+2}{m+1}} \quad (4.10)
$$

if $(x,t) \in \mathcal{P}_\alpha$ with $t > t_\alpha$.

**Proof.** For every $\lambda > 0$ the function

$$
\lambda(x,t) = \lambda(u(x, \lambda^{m+1}t))
$$

is again a solution of (0.1) with $M = 1$ and $x_0 = 0$. If $\xi_\pm(t)$ and $\nu$ denote respectively the interfaces and pressure for $u$, then the corresponding quantities for $\lambda$ are

$$
\xi_\pm^\lambda(t) = \frac{1}{\lambda} \xi_\pm(\lambda^{m+1}t) \quad (4.11)
$$

and

$$
\nu^\lambda(x,t) = \lambda^{-1}v(\lambda x, \lambda^{m+1}t),
$$

where $\nu^\lambda$ satisfies

$$
\nu^\lambda_t = (m-1)\nu_{XX}^\lambda + (\nu^\lambda_x)^2 \quad \text{in} \ \mathcal{P}[u^\lambda].
$$

From (4.6) and (4.7) we have
\[ v^\lambda(x,t) = \frac{(r^2(t) - x^2)_+}{2(m+1)t} + o\left(\frac{1}{\lambda}\right) \text{ in } RX[\frac{1}{2},2] \quad (4.12) \]

and

\[ v^\lambda(x,t) = -\frac{x}{(m+1)t} + o\left(\frac{1}{\lambda}\right) \text{ in } RX[\lambda u^\lambda, \lambda u^\lambda] \cap \{t \in [\frac{1}{2},2]\} \quad (4.13) \]

uniformly as \( \lambda + \infty \). Set \( \beta = (1+\alpha)/2 \). Then in view of (4.1), (4.5) and (4.11)

\[ |s^\lambda_t(t)| > \frac{1}{\lambda} r(\lambda^{m+1}t) = r(t) > \beta r(t) > \alpha \gamma(t) \text{ on } [\frac{1}{2},2]. \]

It follows from (4.12) and (4.13) that there exists a constant \( \lambda_\alpha \in R^+ \) such that for \( \lambda > \lambda_\alpha \) we have \( v^\lambda \) strictly positive and bounded, and \( |v^\lambda_x| \) bounded in \( R_\beta \) for \( t \in [\frac{1}{2},2] \). We now apply the interior regularity argument used in the proof of Lemma 1 of [A1] to conclude that for \( \lambda > \lambda_\alpha \) all of the derivatives of \( v^\lambda \) are bounded independent of \( \lambda \) in \( R_\alpha \) for \( t \in [\frac{3}{4},2] \). In particular, there exists a constant \( C_\alpha \in R^+ \) depending only on \( \alpha \) (and \( m \)) such that

\[ |v^\lambda_{xxx}(y,1)| < C_\alpha \text{ if } |y| < \alpha r(1) \text{ and } \lambda > \lambda_\alpha. \]

Since \( v^\lambda_{xxx}(y,1) = \lambda^{m+2}v_{xxx}(\lambda y, \lambda^{m+1}) \) and \( \frac{1}{t^{m+1}} r(t) = r(t) \), (4.10) follows.

We can now estimate \( v_{xx} \) in \( R_\alpha \).

**Lemma 4.2** As \( t \to \infty \),

\[ v_{xx}(x,t) = -\frac{1}{(m+1)t} + o(t^{-\gamma}), \quad (\gamma = 1 + \frac{1}{2(m+1)}), \quad (4.14) \]

uniformly in \( x \) for \( (x,t) \in R_\alpha \) with \( \alpha \in (0,1) \)

**Proof.** The function

\[ h(x,t) = v_{xx}(x,t) + \frac{1}{(m+1)t} \]
is of class $C^\infty$ in $\mathcal{P}[u]$ for $t > 0$ and by the estimate (1.7), is non-negative. It therefore suffices to estimate $h$ from above in $\mathcal{P}_\alpha$.

For $t$ large we have

$$
\int_{\xi_-}^{\xi_+} h(x,t) dx = v_x(\xi_+(t),t) - v_x(\xi_-(t),t) + \frac{\xi_+(t) - \xi_-(t)}{(m+1)t} =
$$

$$
= - \xi'_-(t) + \xi'_+(t) + o\left(\frac{1}{t}\right) + \frac{\xi_+(t) - \xi_-(t)}{(m+1)t} = o\left(\frac{1}{t}\right),
$$

(4.15)

by virtue of (4.7). Therefore $h(\cdot, t) \in L^1(\xi_-(t), \xi_+(t))$ with $\|h(\cdot, t)\|_1 = o(1/t)$ as $t \to \infty$. On the other hand, by Lemma 4.1, we have $h_x = v_{xxx} = 0(t^{-\mu})$ as $t \to \infty$, where $\mu = (m+2)/(m+1)$. We claim that $h(\cdot, t) \in L^\infty_{\text{loc}}(\xi_-(t), \xi_+(t))$ and $\sup_x h(x, t) = o(t^{-\gamma})$ in sets of form $|x| < \alpha r(t)$. In fact, assume that there exists a constant $c_1 > 0$ and a sequence $(x_n, t_n) \in \mathcal{P}_\alpha$ with $t_n \to \infty$, such that $h(x_n, t_n) > c_1 t_n^{-\gamma}$. Then, by the estimate on $h_x$, there exists $c_2$ such that

$$
h(x, t_n) > c_1 t_n^{-\gamma} - c_2 t_n^{-\mu} |x - x_n|.
$$

This means that $h(x, t_n) > c_1 t_n^{-\gamma}/2$ in the interval $I_n$ centered at $x_n$ with length

$$
g_n = \frac{c_1}{c_2} t_n^{-\gamma} = \frac{c_1}{c_2} t_n \frac{1}{2(m+1)} = o(r(t_n)).
$$

Hence

$$
\int_{I_n} h(x, t_n) dx > \left(\frac{c_1}{2} t_n^{-\gamma}\right) \left(\frac{c_1}{c_2} t_n \frac{1}{2(m+1)}\right) = \frac{c_1^2}{2c_2} t_n^{-1},
$$

which contradicts (4.15).

The next step is the proof of Theorem 2 is to rederive the convergence of $vv_{xx}$ to zero at the interface [CF] so as to obtain a rate of convergence which is uniformly valid for all large $t$. For convenience of calculation we state our
results instead for \( v^\lambda(x,t) \) at \( t = 1 \) for large \( \lambda \) and only consider the right hand interface \( \zeta^\lambda(t) = \zeta^\lambda_1(t) \).

**Lemma 4.3.** There exists a constant \( C > 0 \) independent of \( \lambda \) such that

\[
v_t^\lambda(x,1) - (v_x^\lambda(x,1))^2 < C(\zeta^\lambda(1) - x)^{1/2} + O(\lambda^{-1/2}) \tag{4.16}
\]

for all \( x < \zeta^\lambda(1) \) such that \( \zeta^\lambda(1) - x \) is small independent of \( \lambda \).

**Proof.** To simplify the notation we temporarily drop the superscript \( \lambda \) from \( v^\lambda \) and \( \zeta^\lambda \). Set

\[
k = c_m/(m+1).
\]

From (4.1) and (4.5) we have

\[
\zeta(1) = c_m + o(1),
\]

\[
\zeta'(1) = k + O(1/\lambda) \text{ and } |\zeta'(t)| < kt \frac{m}{m+1} \text{ in } \mathbb{R}^+ , \tag{4.17}
\]

so that (cf. [V3: Theorem 1])

\[
|v_x(x,t)| < kt \frac{m}{m+1} \text{ in Q.} \tag{4.18}
\]

We shall follow the argument of [CF: Lemma 4.4]. Let \( \alpha \) and \( \varepsilon \) be small positive constants and take \( \lambda \) so large that \( \zeta(1) > c_m/2 \). Set \( x_0 = \zeta(1) - \varepsilon \) and \( t_0 = 1 + \alpha \varepsilon \). Then by the Taylor's theorem

\[
v(x_0,t_0) = v(x_0,1) + v_t(x_0,1)\alpha \varepsilon + \frac{1}{2} \alpha^2 \varepsilon^2 v_{tt}(x_0,t_0 + \theta \varepsilon)
\]

for some \( \theta \in (0,1) \). By [CF: Lemma 4.2]

\[
v_{tt}(x,t) \leq \frac{c_1}{d(x,t)} \text{ in } N_6,
\]

in Q.
where $N_\delta$ is a neighbourhood of $(\zeta(1),1)$ in $\mathfrak{g}^\lambda = \{(x,t) \in Q: u^\lambda(x,t) > 0\}$, $d(x,t)$ is the distance from $(x,t) \in N_\delta$ to the interface $x = \zeta(t)$ and $C_1$ is a positive constant. It can easily be checked that $C_1$ and $\delta$ do not depend on $\lambda$. Therefore, if $\epsilon$ is small we get
\[ v(x_0,t_0) > v(x_0,1) + \alpha \epsilon v_t(x_0,1) - \frac{1}{2} \alpha^2 \epsilon C_2, \quad (4.19) \]

where $C_2$ does not depend on $\lambda$. Moreover, since
\[ \nabla_x(\zeta(1),1) = \zeta'(1) = k + o(1/\lambda) \]
and $\nabla_{xx}(x,1) > -1/(m+1)$ we have
\[ v(x_0,1) > (k + o(1/\lambda)) \epsilon - \frac{\epsilon^2}{2(m+1)} \]

Substituting this into (4.19) we get
\[ v(x_0,t_0) > (k + o(1/\lambda)) \epsilon - \frac{\epsilon^2}{2(m+1)} - \frac{1}{2} \alpha^2 \epsilon C_2 + \alpha \epsilon v_t(x_0,1). \quad (4.20) \]

On the other hand (4.18) implies that
\[ v(x_0,t_0) < k t \frac{m}{m+1} (\zeta(t_0) - x_0), \]
and from (4.17) it follows that $\zeta(t_0) - \zeta(1) < k(t_0 - 1) = \alpha \epsilon k$, hence
\[ v(x_0,t_0) < (\alpha \epsilon^2 + \epsilon k) t_0 \frac{m}{m+1}. \quad (4.21) \]

Combining (4.20) and (4.21) gives an estimate for $v_t$:
\[ v_t(x_0,1) < (k^2 + \frac{k}{\alpha})(1 + \alpha \epsilon) - \frac{m}{m+1} - \frac{1}{\alpha} (k + o(\frac{1}{\lambda})) \]
\[ + \frac{\epsilon}{2(m+1) \alpha} + \frac{1}{2} \alpha C_2 \]
\[ < k^2 + \frac{1}{\alpha} \{ o(\frac{1}{\lambda}) + \frac{\epsilon}{Z(m+1)} \} + \frac{1}{Z} \alpha c_2 \]

Now let \( \alpha = o(\lambda^{-1/2}) + \epsilon^{1/2} \) to obtain

\[ v_t(x_0,1) < k^2 + o(\lambda^{-1/2}) + c_3 \epsilon^{1/2} \quad (4.22) \]

where \( c_3 \) does not depend on \( \lambda \). On the other hand, it follows from (4.7) that for \( t > 1 \)

\[ v_x(x,t) + \frac{x}{(m+1)t} = o(\frac{1}{\lambda}) \quad (4.23) \]

hence

\[ v_x(x_0,1) = k - \frac{\epsilon}{m+1} + o(\frac{1}{\lambda}) \quad (4.24) \]

Since \( \epsilon = \zeta(1) - x_0 \), (4.16) follows from (4.22) and (4.24).

\# REMARK. A lower bound for \( v_t - v_x^2 \) is easier to obtain. In fact, by (1.7),

\[ v_t - v_x^2 = (m-1)\nu v_{xx} > -(m-1)\nu \frac{(m+1)\epsilon}{c_1} \]

Therefore as \( x \to \zeta^\lambda(1) \), \( v_\lambda^\lambda(x,1) - (v^\lambda(x,1))^2 > 0(\zeta^\lambda(1) - x) \), uniformly in \( \lambda \).

**LEMMA 4.4.** There exist bounds \( C_k \) for the derivatives \( (a/\partial k)^k v_\lambda(x,t) \) in a neighbourhood of \( (\zeta^\lambda(1), 1) \) in \( \Omega[v_\lambda] \) which are independent of \( \lambda \).

Proof. We have only to reexamine the proofs in Sections 2 and 3, and observe that all of the estimates can be made uniform in \( \lambda \), for \( \lambda \) sufficiently large. Thus to obtain the bound (2.2) for \( v_\lambda^\lambda_{xx} \) we must show that given \( \epsilon \in (0, a/4m) \) satisfying (2.4) we can choose \( \delta \) and \( \eta \) independent of \( \lambda \)
for large $\lambda$ so that (2.5) and (2.6) hold in $\mathbb{R}_{\delta, \eta}$. The fact that $\delta$ and $\eta$ can be chosen in this manner is a consequence of the estimates derived in the proof of Lemma 4.3, namely, (4.18) and (4.23) for $v_{x}^{\lambda}$ and (4.16) for

$$(m-1)v_{xx}^{\lambda} = v_{t}^{\lambda} - (v_{x}^{\lambda})^2.$$ 

Similarly, the estimates for higher derivatives in Section 3 are also independent of $\lambda$.

We may now extend the validity of the estimates of Lemmas 4.1 and 4.2 uniformly up to the boundary.

**Lemma 4.5.** For every $t_0 > \max(t^\star_1, t^\star_2)$ there exists a constant $C_0 \in \mathbb{R}^+$ depending only on $t_0$ such that

$$|v_{xxx}(x,t)| < C_0t^{-\frac{m+2}{m+1}} \text{ in } \mathcal{P}[u] \cap \{t > t_0\}.$$ 

**Proof.** It is clear from Lemmas 4.1 and 4.4 that there exists a $\lambda_0 \in \mathbb{R}^+$ such that $\{v_{xxx}^{\lambda}(x,1)\}$ is bounded in $(\zeta_{-}^1(1), \zeta_{+}^1(1))$ independent of $\lambda$ for $\lambda > \lambda_0$. The assertion is an easy consequence of this observation.

With this result it is easy to see that the argument in the proof of Lemma 4.2 can now be made global. Thus we obtain

**Lemma 4.6.** As $t \to \infty$, (4.14) holds uniformly for $x \in \mathcal{P}[u]$. In particular, there exists a time $T > 0$ such that $v(\cdot, T)$ is a strictly concave function in its support.

More precisely for every $\varepsilon > 0$ we may obtain from (4.14) a time $T_{\varepsilon}$ such that
$$v_{xx}(x,t) \leq -\frac{1 - \varepsilon}{(m+1)t}$$

if $(x,t) \in \mathcal{P}(u)$ and $t > T_\varepsilon$. The sharp asymptotic formulas (4.8) and (4.9) are now a consequence of Theorems 1 and 2 of [BV]. This completes the proof of Theorem 2.

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