HOMOCLINIC ORBITS FOR FLOWS IN $\mathbb{R}^3$

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HOMOCLINIC ORBITS FOR FLOWS IN $\mathbb{R}^3$.

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ABSTRACT. We propose a rough classification for volume contracting flows in $\mathbb{R}^3$ with chaotic behavior. In the simplest cases, one looks at the nature of a homoclinic loop for the flow. Most configurations have been studied at lenght in the literature ; here we examine briefly the "forgoten" case.

ORBITES HOMOCLINES DE FLOTS DE $\mathbb{R}^3$.

RÉSUMÉ. Nous proposons une classification grossière des flots de $\mathbb{R}^3$ dissipatifs présentant des comportements erratiques. Dans les cas les plus simples, on examine la nature d'une orbite homocline du flot. La plupart des configurations ont été abondamment étudiées dans la litterature ; nous examinons ici brièvement le cas "oublié".

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It happens that a great deal of informations can be learned about the origin and structure of chaotic behavior, as numerically observed with dissipative flows in $\mathbb{R}^3$, by looking at the characteristics of the trajectories (if any) which are bi-asymptotic (as time goes to $\pm\infty$) to a rough equilibrium (otherwise speaking, homoclinic orbits for flows in $\mathbb{R}^3$). Note that volume contraction in $\mathbb{R}^3$ prevents tori ; this enhances the importance of homoclinic orbits since one cannot find the type of chaos associated with the destruction of tori. On can of cource manage that the relevant equilibrium be at the origin O ;then the linear part of the differential equations reads (up to a possible time inversion) :

$$\begin{cases} \dot{X} = \lambda_1 X \\ \dot{Y} = -\lambda_2 Y \\ \dot{Z} = -\lambda_3 Z \end{cases} \qquad \begin{matrix} (\lambda_1, \lambda_2, \lambda_3 > 0 \ , \ \lambda_2 > \lambda_3) \\ (\lambda_2 = \lambda_3 \text{ is not robust }) \end{matrix} \qquad (1\text{-a})$$

or :

$$\begin{cases} \dot{X} = -\rho X - \omega Y \\ \dot{Y} = \omega X - \rho Y \\ \dot{Z} = \lambda Z \end{cases} \qquad (\lambda, \rho > 0) \qquad (1\text{-b})$$

Before going further with 3-dimensional flows, let us recall an old and well known result on the stability of saddle loops in $\mathbb{R}^2$. If the saddle is at O , and if the linear part of the equations is like the two first lines of (1-a) , one gets easily that attractivity (resp. repulsivity) with respect to orbits starting from neighbor points inside the loop , is insured by the condition $\lambda_1 < \lambda_2$ (resp. $\lambda_1 > \lambda_2$) . This result motivates the following terminology :

$\lambda_1 < \lambda_3$	: Real Stable	, for short	RS
$\lambda_1 > \lambda_3$	: Real Unstable	, " "	RU
$\lambda < \rho$	: Complex Stable	, " "	CS
$\lambda > \rho$	: Complex Unstable	, " "	CU

In the R cases, we compare only λ_3 to λ_1. This corresponds to the assumption that, if a homoclinic orbit exists, it enters O along the Z axis (according to the choice $\lambda_2 > \lambda_3$ we have made) ; the other case

is exceptional. The choice $\lambda'_2 > \lambda_3$ comes from Lorenz equations[3] .
Another important element in the analysis of a flow is the presence of a special symetry. One can find (together with homoclinic loops) :

G : general case (no symetry),

S_1 : invariance under the change of coordinates $(X, Y, Z) \rightarrow (-X, -Y, Z)$,

S_2 : " " " " $(X, Y, Z) \rightarrow (-X, Y, -Z)$,

S_3 : " " " " $(X, Y, Z) \rightarrow (-X, -Y, -Z)$.

It is then natural to classify [1] flows in $\mathbb{R}^3$ with a homoclinic orbit bi-asymptotic to 0 , according to table 1 . In this table, grey rectangles correspond to cases where no chaos is to be observed (see however remarks 1 and 2 below) , black rectangles to unrealisable situations, and the numbers to some references related to the corresponding situation .

Remark 1: A system whith a sligth departure from a symetry would better be studied by first considering the symetric case, and then investigating the effects of a small symetry breaking (see e.g. [1,5-8,31]).

Remark 2: Homoclinic orbits for flows in $\mathbb{R}^3$ are destroyed by a general small perturbation. However, table 1 is also (even more) usefull for a more global study of one parameter families of flows containing a homoclinic situation (see most of the cited literature) , and even for families which come "close" to a homoclinic situation [1,26,31].
For instance, the methods used to study the cases CS[1,2,30,31] can be used to explain why chaos may occur when one does not consider the precise value of the parameter such that a CS-homoclinic orbit exists[1,28,31].

In this note, we shall concentrate on the case RU-S_3 ; RU-S_2 would lead to similar results, while RU-S_1 contains the Lorenz attractor[3]. The analysis we shall make for definitness in the unstable case will set us in a position to comment on the stable cases (S_3 or S_2) as well. In the same way, methods [4-7] which have been developped for studies in RU-S_1 have been easily adapted to RS-S_1 ; this allowed to discover and understand new types of non trivial dynamical behavior [1,16-18].

Indeed, the aproach we shall use here is also directly inspired from previous works on Lorenz attractor [4-7] : we construct geometrically a "model flow", in a situation "after" the existence of a homoclinic orbit (the meaning of "after" will become clear below). The analysis starts with a linear flow (1-a)-RU, in a cell C as represented in figure (1-a).

If (as it is often the case in applications) one is interested in a one parameter r family, it is natural to supose that for r small enough, one has only negative eigenvalues at O, and that the further evolution is (e.g.) as follows :

-for $r > r_s$, $\lambda_1 > 0$ and one gets a reasonable approximation by (1-a)RU in the cell C for $r > r_S > r_s$,

-for $r_s < r < r_h$, the two branches W_0^{u+} and W_0^{u-} of the unstable manifold of O, converge respectively to the two stable fixed points $M^{\pm}$ created by the bifurcation at $r = r_s$,

-for $r = r_h$, one has a pair of homoclinic orbits bi-asymptotic to O.

Now, we supose that for some $r > r_h$, the interesting part of the dynamics is contained in the figure height cell of figure (1-b) . In order to study the dynamics, we look at the first return map F on the uper face of C. In the spirit of [5-7] , it remains reasonable to supose that the rectangle R^+ of figure (1-a) is transformed by F as indicated in figure (2-a) ; in both figures, the lines drawn in R^+ and its image are intended to indicate that F preserves the (linear) strong stable foliation, otherwise speaking, $F(x,y)=(g_y(x),f(y))$ for (x,y) in R^+ (here g_y stands for a contraction whose rate depends on y). Let us emphasize that T^- should be (at least) deformed as represented in figure (1-b) by the non-linear part of the flow ; if no torsion is involved, all orbits will eventually diverge out of the figure height cell. Unfortunately, the torsion destroys the hope to get a F which preserves globally a strong stable foliation as in R-S_1 cases.

Part of the points of R^- give rise to orbits making at least two revolutions around M^- before the firs crossing whith $R^+ \cup R^-$. The remaining ones give contributions to $F(R^+ \cup R^-)$ like what is represented in figure (2-b). The orbits issued from points in R^- can indeed make an arbitrary number of loops around M^- ; collecting all contributions, one gets for $F(R^+ \cup R^-)$ the global aspect represented in figure (2-c).

The main statement one can formulate about F and RU-S_{3-2} flows is that one can hope chaotic behavior but that a general existence theorem for chaos is hopeless since the global properties of the flow (like the torsion we invoked in our geometric construction) play a fundamental role in the dynamics. This is in contrast whith CU cases where a theorem by Sil'nikov [19-21] insures chaotic behavior (positive topological entropy on appropriate first return maps) when (or "close" to when) there exists a homoclinic orbit. On the other hand, all known results about chaos in RU-S_1 systems use some ad hoc hypotheses such as the existence of an invariant strong stable foliation [4, 9] or, at least, some hyperbolicity-like conditions [10,14] . These special properties are robust but not necessarily verified by an arbitrary flow in the RU-S_1 class. The worst peculiarity of S_{3-2} is that I cannot imagine a natural and simply formulated condition which would insure chaos like in S_1 cases ; an example in S_3 or S_2 of particular physical relevance could be of some help to find such a condition (the fact that one has not yet, to my knowledge, such an example is meaningless : [17] was discovered after and independently of the first investigations of RU-S_1

cases in [16]).

Having renounced to provide a rigorous analysis, let us make some qualitative comments on the first return map F we have constructed. It is easier to supose that the dissipation is very strong ; then the x projection of the dynamics under F is correctly described by a one dimensional map whose graph should look like what is represented in figure (2-d). This is enough to allow a safe prediction of well known phenomena such as cascades of period doubling bifurcations or intermittency (one hope also these phenomena in CU and some CS systems). The one dimensional map is also the more convenient setting for a discussion of the stable cases : one has mainly to replace the half cusp at the right of the graph by a parabolic-like curve in the $x_i > 0$ region, and to replace all sharp slopes in the $x_i < 0$ region which are due to an orbit of the flow coming close to 0, by pieces of curves with derivative going to zero. The occurence of chaos would then depend on details of the vector field. The only general statement on can make is that, like in all S cases, the proximity (both in phase space and in paramater space) of a homoclinic orbit prevents chaos[1,2,30,31] .

We insist on the fact that figure (1-b) corresponds to the simplest form of torsion ; with an explicit example in RU-S_{3-2} , F can be much more complicated than what we have goten with our geometrical construction. It is the case for the oscilator :

$$\dddot{X} + \nu.\ddot{X} + \frac{(\alpha - \exp(-X^2))}{\alpha}.\dot{X} + m.\frac{(a.X - b.X^9)}{(d + c.X^6)} = 0 \quad , (2)$$

which has been used to get the RU-S_3 "strage attractor" represented in figure 3 (this oscilator has been adapted from those used in [23,25,26 29] which cannot be in a RU class).

Our last remark is that heteroclinic loops may as well be usefull to elucidate the origin and structure of chaotic behavior [1,15] ; some new results will be reported elsewhere[31] .

Acknowledgements. The rough classification resumed by table 1 was first proposed in a seminar at the Ecole Polytechnique (1980), then in my thesis, in Les Houches (1981), and in many places in United States. This allowed me to refine many points since most cases in the table were presented each time. I would thus like to aknowledge all those who helped me by any mean , and particularly the warm hospitality of Courant Institute, Stevens Tech. , and the I.M.A. (Minneapolis).

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TABLE CAPTION

Table 1 : Résumé of the rough classification proposed in the paper. Shadded rectangles correspond to cases where no chaos is to be observed (modulo remarks 1 and 2 in the main text) , black rectangles to unrealizable situations, and the numbers to some references.

FIGURE CAPTIONS

Figure 1 : (a): the cell C where the model flow is supposed to be linear. The sets $R^{\pm}$, $T^{\pm}$ which are used in the analysis of the model flow have been represesnted ; $T^{\pm}$ is obtained by first intersection of the lateral faces of C whith the orbits issued from points in $R^{\pm}$. (b): the figure height cell which corresponds to the simplest re-injection process in C yielding non trivial dynamics.

Figure 2 : (a-c): the construction of $F(R^{+} \cup R^{-})$ by taking account of various contributions as explained in the main text. (d): the one dimensional map which represent reasonably the x-projection of the dynamics under F in case of strong dissipation. The numbers on the bottom line indicate the number of loops around M^{-}, starting from the upper face of C, before returning to it for the first time. The dots indicate typical points where the shape of the graph is necessarily modified in a stable case. The dotted line corresponds to the branching point of the "palm tree" in figure (2-c).

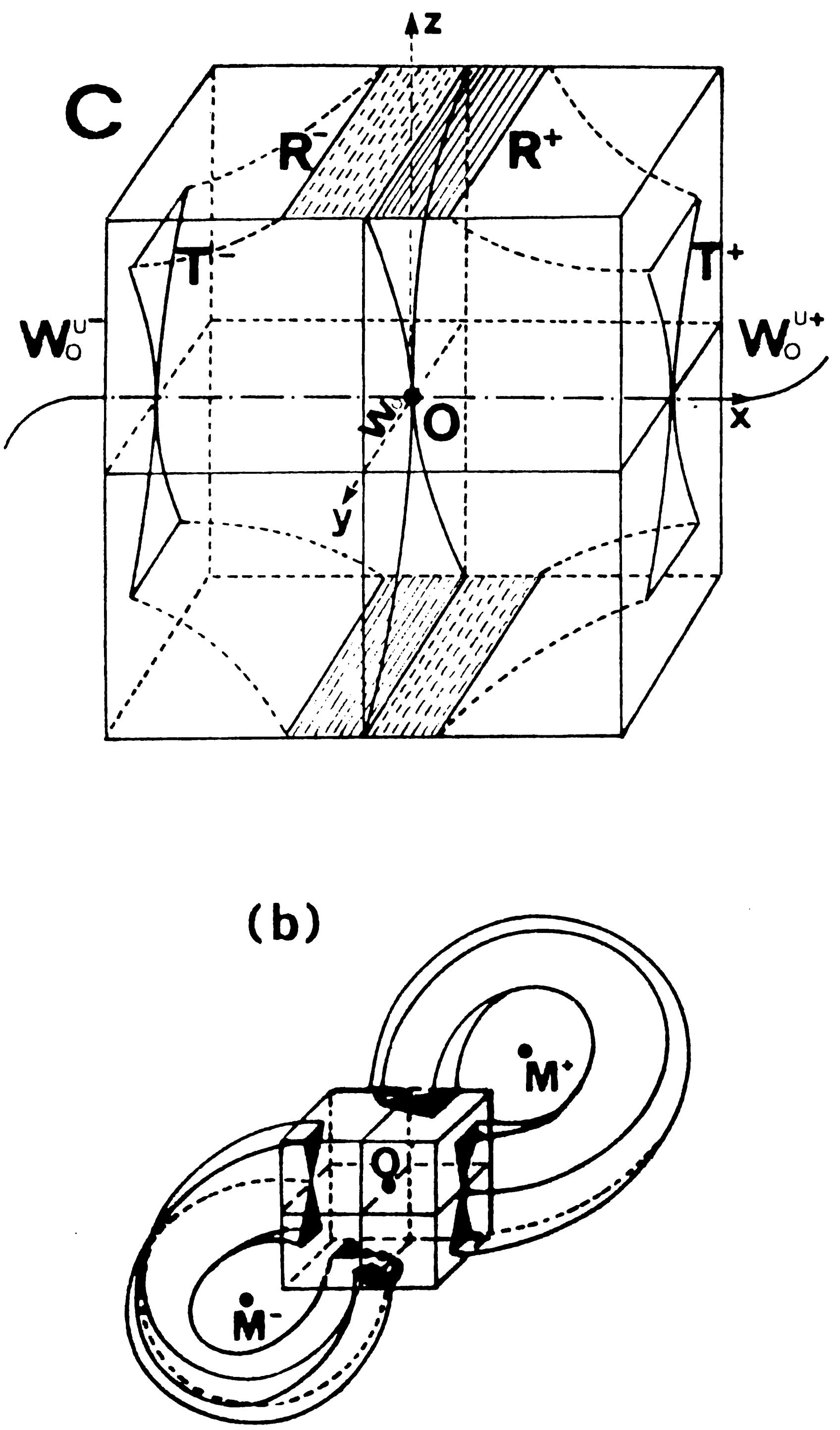
Figure 3 : The "strange attractor" numerically obtained with equation (2) of the main text. One has taken a=1, b=0.00002, c=0.005, d=0.3, ν=0.2, and α=0.3. This yields : $\lambda_1 \simeq 1.687$, $\lambda_2 \simeq 1.145$, $\lambda_3 \simeq 0.741$.

	G	S_1	S_2	S_3
RU	1,2	1,3 - 15	1	1
RS	1,2	1,16 - 18	1	1
CU	1,19-28,31			1,29,31
CS	1,2,31			1,30,31

Table 1

Figure 1

(a)



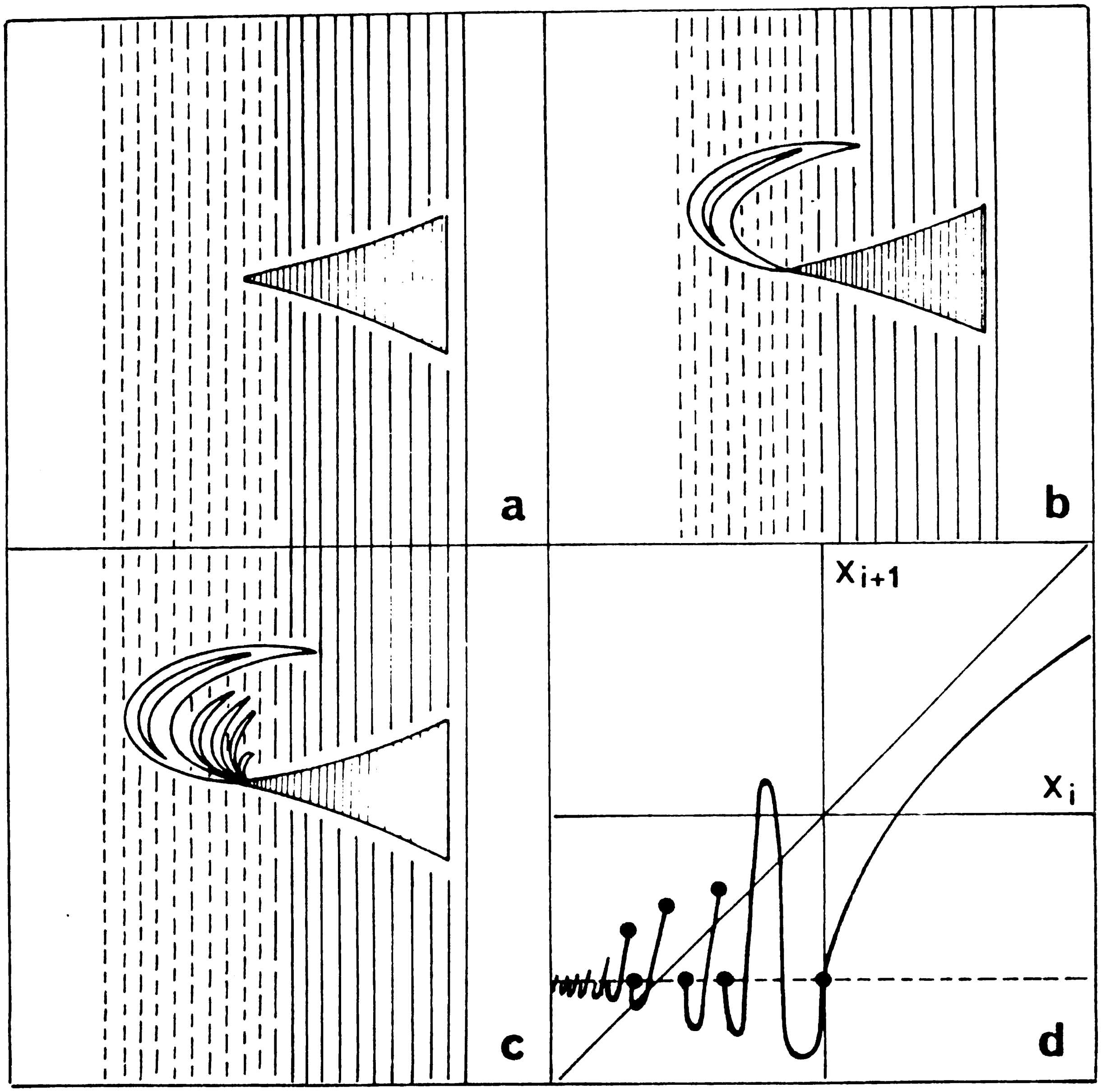


Figure 2

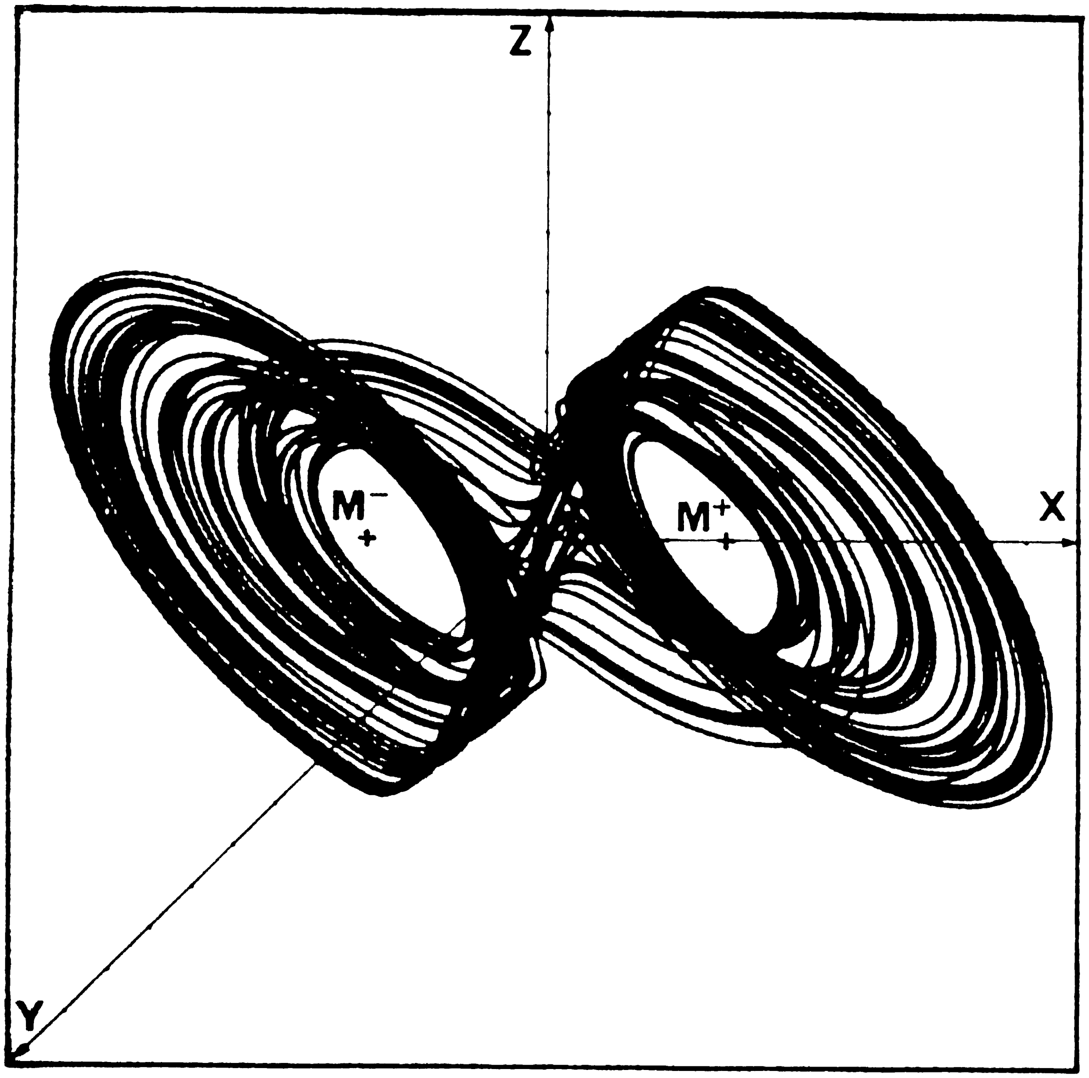


Figure 3

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