THE ASYMPTOTICS OF SOLUTIONS OF SINGULARLY PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS: DISTRIBUTED AND CONCENTRATED DELAYS ARE DIFFERENT

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ABSTRACT

This paper illustrates the differences between systems with distributed delays and systems having only concentrated delays in what concerns the asymptotic rates of solutions of singularly perturbed linear retarded functional differential equations. An example of a system with distributed delays shows that the introduction of a "slow" variable coupled with the "fast" variable may decrease the asymptotic rates of solutions observed when the perturbation parameter is close to zero. Such a situation cannot happen for ordinary differential equations, or even for differential-difference equations.

1. Introduction

This paper is concerned with the asymptotics of solutions of singularly perturbed systems of linear retarded functional differential equations (FDE) in relation to upper bounds on the real parts of the associated characteristic values.

The systems under study are written in the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 y(t) + A(x_t) + B(y_t) \\ \mu \dot{y}(t) &= C_0 x(t) + D_0 y(t) + C(x_t) + D(y_t) \end{aligned} \qquad (1_\mu)$$

where $t, \mu \in \mathbb{R}^+$, $x(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^n$ $(m \geqslant 0,\ n \geqslant 1)$, the delays lie in the interval $[-r, 0]$ for some fixed $0 < r \leqslant +\infty$, x_t, y_t are functions defined on $[-r, 0]$ by $x_t(\theta) = x(t+\theta)$, $y_t(\theta) = y(t+\theta)$, and A, B, C, D are linear operators defined on appropriate function spaces. More precisely,

$$A(\phi) = \int_{-r}^{0} a(\theta)\, \phi(\theta)\, d\theta + \sum_{k=1}^{h} A_k \phi(-w_k) \qquad (2)$$

and similarly for B, C, D, where a, b, c, d admit exponential bounds $|a(\theta)|, |b(\theta)|, |c(\theta)|, |d(\theta)| \leqslant k_0 e^{\gamma_0 \theta}$ for some $\gamma_0 > 0$, A_k, B_k, C_k, D_k are real matrices with all the eigenvalues of D_0 having negative real parts, and the concentrated delays w_k satisfy $0 < w_1 < w_2 < \ldots < w_h$ for some integer $h \geqslant 0$. It is common to refer to x and y in (1_μ) as the slow and fast variable, respectively.

In the particular case of differential-difference equations, the asymptotics of solutions at $t = +\infty$ was studied by Cooke [2] and Cooke and Meyer [3]. Through the introduction of the concept of σ_0-complete regularity, these authors formulated conditions for the existence of uniform exponential upper bounds on the fundamental matrix of the equation, and expressed the "best" exponential rates for these estimates in terms of the characteristic values of the degenerate system and the "smallest" value of σ for which the equation is σ_0-completely regular.

When all the eigenvalues of D_0 have negative real parts, it is easier to formulate the concept of σ_0-complete regularity, and the methods of Cooke and Meyer can be extended to apply to equations of the general type of (1_μ), which may involve distributed delays. This was done in [10]. The condition of all eigenvalues of D_0 having negative real parts is a convenient sufficient condition for solutions of all initial value problems for (1_μ), posed in adequately chosen phase spaces, to converge to solutions of the degenerate system (1_0), as $\mu \to 0^+$ (see [2-6, 8, 9, 11]). It is, therefore, a natural restriction on the systems studied.

When [10] was written, it was not known whether it would be possible to have a system of the form (1_μ) for which the upper limit M^* of the least upper bounds of the real parts of the characteristic values of (1_μ), as $\mu \to 0^+$, is strictly smaller than the corresponding value, N^*, for the reduced system which is obtained from (1_μ) by elimination of the slow variable:

$$\mu \dot{y}(t) = D_0 y(t) + D(y_t) \quad . \qquad (3_\mu)$$

It was known from the work of Cooke and Meyer that such a situation cannot occur for differential-difference equations, but this possibility was left open in [10], for systems which contain distributed delays. Some conditions, in terms of the coupling of the slow and fast variables, which are necessary for $M^* < N^*$ were also given in [10]. These results are briefly recalled in Section 3.

The purpose of this paper is to give an example of a system for which $M^* < N^*$. This shows that the coupling of a slow variable with a fast variable in an equation with distributed delays may decrease the asymptotic rates of solutions. This improvement of the stability properties of the solutions is not possible for differential-difference equations.

2. Notation

By a linear change of coordinates, system (1_μ) can be transformed into a system where the coupling between slow and fast variables is done only through delayed values of the variables. This fact was proved in [10] using a change of variables introduced for ordinary differential equations (ODE) by Chang [1], and can be stated in the following form:

Lemma 1. There exist $\mu_0 > 0$ and matrices $R = R(\mu)$, $S = S(\mu)$ depending continuously on μ for $0 \leqslant \mu \leqslant \mu_0$, satisfying $R(0) = D_0^{-1}C_0$ and $S(0) = -B_0D_0^{-1}$, such that the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I_m , & -\mu S \\ -R , & I_n + \mu RS \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

transforms system (1_μ) into a system of the form

$$\dot{v}(t) = \left(A_0 - B_0 R(\mu)\right) v(t) + \ldots$$

$$\mu\dot{w}(t) = \left(D_0 + \mu R(\mu) B_0\right) w(t) + \ldots$$

where the dots stand for the contribution of the delayed values of v and w.

This shows that, without loss of generality, we may assume $B_0 = C_0 = 0$ in (1_μ), provided we allow A_0, D_0, A, B, C, D to depend continuously on $\mu \geqslant 0$ in some neighborhood of the origin. To avoid overburdening the notation we omit the dependence of these elements on μ and use the same symbols as before.

In order to formulate the results given below, summarizing the relevant facts of [2, 3, 10], it is convenient to introduce the functions of complex variables Δ_μ and θ_μ defined by[1]

[1] For simplicity we use A, B, C, D also to denote the functions of complex variables defined by the same formal expressions used for the operators.

$$\Delta_\mu(\lambda) = \begin{bmatrix} \lambda I_m - A_0 , & 0 \\ 0 & , \mu\lambda I_n - D_0 \end{bmatrix} - \begin{bmatrix} A(e^{\lambda\cdot}) & , & B(e^{\lambda\cdot}) \\ C(e^{\lambda\cdot}) & , & D(e^{\lambda\cdot}) \end{bmatrix}$$

and

$$\theta_\mu(\lambda) = \mu\lambda I_n - D_0 - D(e^{\lambda\cdot}) \quad .$$

The former is equal to the characteristic function of system (1_μ) with the bottom blocks multiplied by μ, and the latter is equal to the product of μ and the characteristic function of the reduced equation (3_μ). The characteristic values of (1_μ) and of (3_μ) are therefore the zeros of $\det \Delta_\mu$ and $\det \theta_\mu$, respectively. The following quantities, related to the distribution of the characteristic values as $\mu \longrightarrow 0^+$, are considered:

$$M_\mu = \sup\{\operatorname{Re}\lambda : \det \Delta_\mu(\lambda) = 0\}$$

$$M^* = \limsup_{\mu \longrightarrow 0^+} M_\mu$$

$$N_\mu = \sup\{\operatorname{Re}\lambda : \det \theta_\mu(\lambda) = 0\}$$

$$N^* = \limsup_{\mu \longrightarrow 0^+} N_\mu \quad .$$

It was shown in [10] that N^* can be expressed in terms of the concept of σ_0-complete regularity which was introduced in [2, 3], as

$$N^* = \inf\{\sigma_0 \in \mathbb{R} : \text{equation } (1_\mu) \text{ is } \sigma_0\text{-completely regular}\} \quad .$$

3. Review of results

In order to recall the theory established in [2, 3, 10], the important results giving relationships between the quantities M_0, M^*, N_0, N^* are stated here. The proofs can be found in [10].

Theorem 2. The following relationships among the quantities M_0, M^*, N_0, N^* hold:

(i) $N_0 \leqslant N^*$,

(ii) either $N^* < M_0 = M^*$ or $M_0 \leqslant M^* \leqslant N^*$,

(iii) if $N_0 \neq N^*$ or $N_0 \leqslant M^*$, then $M^* = \max\{M_0, N^*\}$.

Theorem 3 (Cooke and Meyer). If (1_μ) is a differential-difference system (i.e., $a = b = c = d = 0$), then $N_0 \leqslant M^*$. Consequently $M^* = \max\{M_0, N^*\}$.

It follows from Theorem 2-(ii) that $M^* \leqslant \max\{M_0, N^*\}$ and that the strict inequality $M^* < \max\{M_0, N^*\}$ can only happen provided $M^* < N^*$. This situation never occurs for ODEs or even for differential-difference equations, as a consequence of Theorem 3.

It is possible to give other sufficient conditions for the equality $M^* = \max\{M_0, N^*\}$, in terms of the coupling between the fast and the slow variables.

Proposition 4.

(i) If $B \equiv 0$ or $C \equiv 0$, then $M_\mu \geqslant N_\mu$ and $M^* \geqslant N^*$,

(ii) if $D \equiv 0$, then $M^* = M_0 \geqslant N^* = -\infty$.

In both cases $M^* = \max\{M_0, N^*\}$.

This result also illustrates another point. The study of Halanay in [5, 6] and the study of Klimushev in [8, 9] were restricted to the situations where $C = D \equiv 0$ and $B = D \equiv 0$, respectively. In both cases we have $M^* = M_0$, a situation that Cooke and Meyer call regular degeneration, which amounts to the continuity of the least upper bound of the real parts of the characteristic values of (1_μ) at the point

$\mu = 0$. In [2] Cooke discovered the interesting fact that differential-delay equations do not always have regular degeneration. This was done through an example of a neutral equation, but this example can be readily modified to give a retarded equation for which $M^* = N^* > N_0$ (see [10]).

4. An example with $M^* < N^*$

The results reviewed in the preceding section imply that we can only hope to find an example with $M^* < N^*$ in systems which contain distributed delays, satisfy $N_0 = N^*$ and $B \not\equiv 0$, $C \not\equiv 0$, $D \not\equiv 0$.

In order to facilitate the analysis, we look for simple systems with both variables x, y being scalar. We start by finding a scalar reduced system (3_μ) with distributed delays and such that $N_0 = N^* = 0$, and then proceed to construct the full system with the desired properties.

For systems of the form

$$\mu\dot{y}(t) = -y(t) + \int_{-1}^{0} d(\theta)\, y(t+\theta)d\theta$$

we have

$$\theta_\mu(\lambda) = \mu\lambda + 1 - \int_{-1}^{0} d(\theta)e^{\lambda\theta}d\theta \quad .$$

If we choose the function d so that $\int_{-1}^{0} d = 1$, then $\lambda = 0$ is a zero of θ_0, and, therefore, $N_0 \geqslant 0$. The zeros of θ_μ with positive real parts must satisfy

$$|\mu\lambda + 1| = \left|\int_{-1}^{0} d(\theta)e^{\lambda\theta}d\theta\right| < \int_{-1}^{0} |d(\theta)|\, d\theta \quad .$$

Consequently, in order to avoid zeros with positive real part, it is sufficient to take $d(\theta) > 0$ for all $-1 \leqslant \theta \leqslant 0$, satisfying $\int_{-1}^{0} d = 1$, since then all zeros with positive real parts would have to lie in the open disc centered at the point $-1/\mu$ and having radius $1/\mu$, a region that does not contain points with positive real parts. Taking into account Theorem 2-(i), we have $N_0 \leqslant N^*$, and, consequently, the preceding argument shows that $N_0 = N^* = 0$ whenever the function d is positive and $\int_{-1}^{0} d(\theta) = 1$.

For a system (1_μ) with B_0, C_0, A vanishing and such that both x and y are scalar variables, we have

$$\det \Delta_\mu(\lambda) = \det \begin{bmatrix} \lambda - A_0, & -B(e^{\lambda \cdot}) \\ -C(e^{\lambda \cdot}), & \theta_\mu(\lambda) \end{bmatrix} = (\lambda - A_0)\theta_\mu(\lambda) - B(e^{\lambda \cdot})C(e^{\lambda \cdot}) \quad .$$

In order to facilitate the analysis, it is convenient that this function, for $\mu = 0$, be the characteristic function of a simple differential delay equation, preferably with just one delay. This is indeed the case if we choose

$$d(\theta) = \frac{\alpha e^{\alpha\theta}}{1 - e^{-\alpha}} , \qquad \alpha > 0$$

$$B(y_t) = b\,y(t - 1/2)$$

$$C(x_t) = c\,x(t - 1/2)$$

$$A_0 = -\alpha \, .$$

Then

$$\det \Delta_\mu(\lambda) = (\lambda + \alpha)\left[\mu\lambda + 1 - \frac{\alpha(1 - e^{-(\lambda+\alpha)})}{(1 - e^{-\alpha})(\lambda + \alpha)}\right] - bce^{-\lambda}$$

$$= \mu\lambda^2 + \alpha\mu\lambda + \lambda + \frac{\alpha}{1 - e^{\alpha}} + \left(\frac{\alpha}{e^{\alpha} - 1} - bc\right) e^{-\lambda}$$

and

$$\det \Delta_0(\lambda) = \lambda + \frac{\alpha}{1 - e^{\alpha}} + \left(\frac{\alpha}{e^{\alpha} - 1} - bc\right) e^{-\lambda} \quad .$$

This last function is the characteristic equation of

$$\dot{z}(t) = -a\,z(t) - kz(t-1) \tag{4}$$

with $a = \alpha/(1 - e^{\alpha})$ and $k = \alpha/(e^{\alpha} - 1) - bc$. The region of asymptotic stability of (4) in the ak-plane is well known (cf. [7]) and includes all points with $a > -1$ and $k = 1$. It is easy to see that the function $f(\alpha) = \alpha/(1 - e^{\alpha})$ satisfies $f(\alpha) > -1$ for $\alpha > 0$. Consequently, we can choose $\alpha > 0$ and $bc = \alpha/(e^{\alpha} - 1) - 1$. In particular, we can take $\alpha = b = 1$ and $c = (2 - e)/(e - 1)$, to obtain the system

$$\dot{x}(t) = -x(t) + y(t - 1/2)$$

$$\mu\dot{y}(t) = -y(t) + \frac{2 - e}{e - 1} x(t - 1/2) + \frac{1}{1 - e^{-1}} \int_{-1}^{0} e^{\theta} y(t+\theta)d\theta \ . \qquad (5_\mu)$$

The preceding arguments guarantee, for system (5_μ), that $N_0 = N^* = 0$, $M_0 < 0$. We just need to verify that $M^* < 0$ also.

We know, from Theorem 2-(ii), that $M^* \leqslant \max\{M_0, N^*\}$. Consequently, for system (5_μ), $M^* \leqslant 0$. Assume that $M^* = 0$. Then there exist sequences $\{\lambda_j\}$, $\{\mu_j\}$, with $M_0 < \text{Re}\lambda_j$, $\mu_j > 0$ $\mu_j \longrightarrow 0$, $\text{Re}\lambda_j \longrightarrow 0$ as $j \longrightarrow \infty$, such that

$$\det \Delta_{\mu_j}(\lambda_j) = 0 \ .$$

This means that

$$(\mu_j\lambda_j)\lambda_j + \mu_j\lambda_j + \lambda_j + \frac{1}{1 - e} + e^{-\lambda_j} = 0 \ . \qquad (6)$$

If the λ_j were bounded, then there would exist a convergent subsequence $\lambda_j \longrightarrow \lambda_0 = iw_0$, $w_0 \in \mathbb{R}$, and letting $j \longrightarrow \infty$ in (6) we would get

$$iw_0 + \frac{1}{1 - e} + e^{-iw_0} = 0$$

or

$$\cos w_0 = 1/(e - 1) \quad \text{and} \quad \sin w_0 = w_0 \ .$$

Since these two equalities never hold together, this shows that $\{\lambda_j\}$ is unbounded. So, for some subsequence, $\lambda_j = x_j + iy_j$, $x_j \longrightarrow 0$, $|y_j| \longrightarrow \infty$ as $j \longrightarrow \infty$.

If the $\mu_j\lambda_j$ were unbounded, then the term $(\mu_j\lambda_j)\lambda_j$ would dominate all other terms in (6) as $j \longrightarrow \infty$, implying that the left hand side of (6) would be unbounded. This is a contradiction, since it must equal zero. Consequently, $\{\mu_j\lambda_j\}$ is bounded and, therefore, there exists a subsequence such that $\mu_j\lambda_j \longrightarrow iz$ as $j \longrightarrow \infty$ for some $z \in \mathbb{R}$. Dividing equation (6) by iy_j and letting $j \longrightarrow \infty$ we get

$$iz + 1 = 0.$$

This is impossible because $z \in \mathbb{R}$. Consequently $M^* < 0$.

We have therefore proved the following

Proposition 5. For system (5_μ), we have $M^* < N^*$.

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