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THE LOW OF GLACIERS

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ABSTRACT

McTigue and others (Journal of Glaciology, 31 (1985), 120-126) initiated the enquiry whether normal stress effects can have practical significance in the flow of glaciers. Their discussion, however, were marred by having chosen the second-order fluid as the constitutive model for ice (which is inconsistent with Glen's flow law) and by problems in the quantification of material parameters in their model. In this note the issues discussed by McTigue and others are re-examined in light of two new models of polycrystalline ice, namely "the modified second-order fluid" and "the power-law fluid of grade 2", both of which include the second-order fluid and Nye's generalization of Glen's flow law as special instances. For ease of comparison, the same creep data of McTigue and others are used in quantifying the material parameters in our models. Both of our models predict far less pronounced normal stress effects in glaciers than those estimated by McTigue and others.
INTRODUCTION

McTigue and others (1985) discussed the possible significance of normal stress effects in the flow of glaciers. Their discussion was based on modeling polycrystalline ice as a second-order fluid. To give a quantitative illustration of normal stress effects, McTigue and others estimated the three material parameters in the second-order fluid from data of four triaxial creep tests on polycrystalline ice.\(^1\) (Those creep tests were performed at the same temperature and the same deviatoric stress, but at different confining pressures.) McTigue and others reported that their model gave "an excellent representation of both primary and secondary creep". Using the numerical estimates of material parameters obtained from fitting the data of the four creep tests, they concluded that normal stress effects could influence the development of crevasses and could lead to measurable depression of surface of glaciers on sufficiently steep slopes.

As noted by McTigue and others, the second-order fluid is in no way a replacement for, nor a generalization of, Glen's flow law (1952, 1955). Indeed, the second-order fluid "fails to represent the non-linear rate dependence of ice in shear"; as such, it cannot be considered as a candidate for the flow law of ice. Take, for instance, Nye's first model of glacier as an infinite slab of ice of uniform thickness flowing down a rough inclined plane (Nye, 1952, 1957). Consider steady laminar flow. For a slab of second-order fluid, the velocity field will be identical to that of an incompressible linearly-viscous fluid. Such an implication is
certainly undesirable.

Glen's flow law, as extended to cover multiaxial states of stress by Nye (1957), in effect models polycrystalline ice as an incompressible power-law fluid (cf. Bird and others, 1977, p. 208). It is the most widely used empirical flow law for polycrystalline ice under stress. While its successes are well documented (Hooke, 1981), it has its shortcomings, namely: it can only describe secondary creep, and it does not predict any normal stress effect in shearing flows of ice. Cf. McTigue and others (1985) for a more detailed criticism.

Motivated by a preliminary version of the paper of McTigue and others, Man and others (1985) initiated a research program in 1984, the purpose of which has been to seek a simple modification of Glen's flow law that does not share the aforementioned shortcomings. Based on results of pressuremeter creep tests performed by Shields and Kjartanson at The University of Manitoba (Kjartanson, 1986) and with the help of Sun in numerical work, Man has proposed two tentative modifications of Glen's flow law. The purpose of the present note is to re-examine the issues discussed by McTigue and others (1985) when the second-order fluid in their paper is replaced by the two new models. For ease of comparison, material parameters in the new models will be estimated by fitting data of the four triaxial creep tests of McTigue and others.

¹Here we follow common usage and call those creep tests "triaxial". McTigue and others used the adjective "biaxial".
In this note by Glen's flow law we shall mean either the formulation in terms of "octahedral strain rate" and octahedral shear stress (see Eq. (5) below) or the generalization due to Nye, i.e., the power-law fluid model, which is sometimes known as "the generalized flow law" in the literature (e.g., Paterson (1981), p. 30). It should be clear from the context which of the two versions we really mean when we refer to Glen's flow law below.
TWO MODIFICATIONS OF NYE'S GENERALIZATION OF GLEN'S FLOW LAW

In what follows we shall, with few exceptions, adopt the notation of McTigue and others (1985). Tensors will be denoted by Cartesian indicial notation. To conserve space, we shall not reproduce definitions of terms such as normal stress differences and Rivlin-Ericksen tensors, which are given in the paper of McTigue and others and in texts of non-Newtonian fluid mechanics (e.g., Schowalter, 1978).

Man has proposed the following two constitutive equations as tentative modifications of "the generalized flow law":

\[
(1) \quad T_{ij} + p\delta_{ij} = \mu^{ \Pi \text{m} / 2 } A_{ij}^{(1)} + \alpha_1 A_{ij}^{(2)} + \alpha_2 A_{ik} A_{kj}^{(1)},
\]

\[
(II) \quad T_{ij} + p\delta_{ij} = \Pi^{ \text{m} / 2 } (\mu A_{ij}^{(1)} + \alpha_1 A_{ij}^{(2)} + \alpha_2 A_{ik} A_{kj}^{(1)});
\]

here \( T_{ij} \) is the Cauchy stress tensor; \( p \) is the indeterminate pressure due to incompressibility; \( A_{ij}^{(1)} \) and \( A_{ij}^{(2)} \) are the first and the second Rivlin-Ericksen tensor, respectively; \( A_{11}^{(1)} + A_{22}^{(1)} + A_{33}^{(1)} \equiv 0 \) because of incompressibility; \( \Pi \equiv \frac{1}{2} A_{jk} A_{kj}^{(1)} \); \( \mu, \alpha_1, \alpha_2 \) and \( m \) are material parameters, which are constants for polycrystalline ice of given temperature and given structural features such as texture and fabric; the parameter \( m \) is related to the exponent \( n \) in Glen's flow law by \( n = 1/(m + 1) \) (thence \( n = 3 \) corresponds to \( m = -2/3 \)). When \( \alpha_1 = \alpha_2 = 0 \), both models (I) and (II) reduce to the power-law fluid, i.e., Glen's flow law. When \( m = 0 \), both models (I) and (II) become the second-order fluid. Both models (I) and (II) belong to a class of fluids introduced by
Rivlin and Ericksen (1955); they are special instances of (incompressible) Rivlin-Ericksen fluids of complexity 2 and are formally included in a class of fluids studied by Dunn (1982). They are also special instances of a class of fluids proposed by Morland and Spring (1981) for modeling the deformation of ice. Both models satisfy the principle of material frame-indifference (Truesdell and Noll, 1965, Sect. 19). Both models are meant for the description of primary and secondary creep, but not tertiary creep.

The specific choice of models (I) and (II) is mainly guided by empiricism and the requirement of simplicity. Model (II), however, can be viewed as the second in a hierarchy of more and more complex models, the first or simplest of which is the usual power-law fluid. We shall call it the power-law fluid of grade 2. In general by the power-law fluid of grade \( k \) we mean the constitutive equation that results when the terms between parentheses in Eq. (2) are replaced by terms appropriate to the fluid of grade \( k \) in the approximation scheme of Coleman and Noll (1960). The usual power-law fluid is then of grade 1. We call model (I) the modified second-order fluid.

Both models (I) and (II) gave good fits to data of single-stage and multistage pressuremeter creep tests (Man and others, 1985; Kjartanson, 1986), which were performed at \(-2^\circ\) C with cavity pressure ranging from 1.0 to 2.5 MPa. Between them, model (II) consistently gave better fits. Complete details of fitting are given in the Ph.D. thesis of Sun (1986).

For steady laminar flow in Nye's first model of glacier (1952, 1957), both models (I) and (II) deliver a velocity field identical to that which pertains to Glen's flow law. The stress distribution, however, is different for all of the three models.
DETERMINATION OF MATERIAL PARAMETERS

The material parameters \( m, \mu \) and \( \alpha_1 \) in both models (I) and (II) can be determined unambiguously from data of pressuremeter creep tests (Man and others, 1985; Kjartanson, 1986; Sun, 1986); the parameter \( \alpha_2 \) does not appear in the differential equation that governs pressuremeter creep tests. The purpose of the present note, however, is to re-examine the conclusions of McTigue and others (1985) in light of models (I) and (II). For comparison purposes we should determine the material parameters in models (I) and (II) by fitting data of the triaxial creep tests of McTigue and others.

For triaxial creep tests we shall adopt the same kinematical assumptions as McTigue and others, namely: during the test the cylindrical specimen undergoes a homogeneous, isochooric, uniaxial elongational flow so that its shape always remains cylindrical. We shall neglect body-forces and assume that the stress in the specimen be homogeneous during the test.

Under the assumptions above it is straightforward to deduce the differential equations that govern triaxial creep tests for models (I) and (II), respectively. Since the derivation is similar to that presented by McTigue and others for the second-order fluid, here we are content to write down the end equations only. For model (I), the differential equation in question is

\[
3\alpha_1 \ddot{a} + 3\mu (3a^2)^{m/2} a + 3(\alpha_1 + \alpha_2)a^2 - \sigma = 0; \tag{3}
\]

here a superposed dot denotes differentiation with respect to time; \( a \equiv \ell/\ell \), where \( \ell(t) \) is the length of the cylindrical specimen at the instant \( t \); \( \sigma \) is the axial stress in excess of the confining pressure.
For model (II) or the power-law fluid of grade 2, the creep equation is

$$3a_1\dot{a} + 3\mu a + 3(a_1 + a_2)a^2 - (3a^2)^{-m/2}\sigma = 0. \quad (4)$$

When $m = 0$, both Eqs. (3) and (4) reduce to Eq. (13) of McTigue and others, i.e., the creep equation for the second-order fluid.

Let $\dot{\varepsilon} \equiv \frac{1}{3}A_{ij}^{(1)}D_{ij}$ be the "octahedral strain rate"; here $D_{ij} = \frac{1}{3}A_{ij}^{(1)}$ is the stretching tensor or "strain-rate" tensor. Let $\tau \equiv 3^{1/4}(T_{ij}T_{ij})^{1/2}$ be the octahedral shear stress; here $T_{ij}$ denotes the deviatoric stress tensor. Glen's flow law is often expressed in terms of $\dot{\varepsilon}$ and $\tau$ in the form

$$\dot{\varepsilon} = (\tau/B)^n, \quad (5)$$

where $B$ and $n$ are material constants (Hooke, 1981). For compression tests, the quantity $a$ in Eqs. (3) and (4) is related to $\dot{\varepsilon}$ by $a = -2^{1/4}\dot{\varepsilon}$, whereas $\sigma$ is related to $\tau$ by $\sigma = -(3/2)^{1/2}\tau$. When the absolute value of $a$ is at its minimum $|a|_{\text{min}}$, $\dot{a} = 0$. In order that Eqs. (3) and (4) should deliver Glen's flow law in secondary creep, the estimates below must be valid for models (I) and (II), respectively:

(I) $|a_1 + a_2| \ll 3^{m/2}\mu|a|_{\text{min}}^{m-1} \quad (6)$

(II) $|a_1 + a_2| \ll \mu/|a|_{\text{min}}. \quad (7)$
Models (I) and (II) are meant to be modifications of Nye's generalization of Glen's flow law. They should be consistent with laboratory data and field measurements which support Glen's flow law. Henceforth we shall assume that estimates (6) and (7) be valid for triaxial creep tests with octahedral shear stresses between 0.1 and 1 MPa (Hooke, 1981). After setting $\dot{a} = 0$ and ignoring the term $3(\alpha_1 + \alpha_2)a^2$, both Eqs. (3) and (4) reduce to the equation

$$3\mu(3a^2)^{m/2}a = \sigma. \quad (8)$$

Eq. (8) will agree with Eq. (5) if and only if

$$B = 2(6)^{m/2}\mu, \quad n = 1/(m + 1). \quad (9)$$

All four creep tests of McTigue and others (1985) were done at $\sigma = -0.47$ MPa and at essentially the same temperature ($-9.5 \, ^\circ C$ to $-9.8 \, ^\circ C$), but at different confining pressures. Within the framework of either of our models, which assume incompressibility of ice, the four tests would ideally produce the same creep curve should the initial conditions of the tests be identical. With $\sigma$ fixed, tests under different confining pressures will certainly tell us something about the assumption of incompressibility. Besides that, the four tests will have no more information content than repeating one single test four times.
Indeed we immediately face the problem of over-parametrization when we attempt to estimate the material parameters in Eqs. (3) and (4) by fitting the data of McTigue and others. The parameters are ill-determined because of insufficient data.\(^2\) Fortunately we can still proceed after making some reasonable assumptions.

In fitting the data of McTigue and others our first assumption was that the value of the parameter \( \alpha \) was close to \(-2/3\) (i.e., \( n \approx 3 \)). This assumption is in line with Hooke's conclusion in his review (1981) on Glen's flow law.

Only those data (day 1 to day 17) pertaining to primary and secondary creep were used in the fitting. In each test the initial value of \( |\alpha| \) on day 1 was less than \( 3|\alpha|_{\text{min}} \). By Eq. (8) the value of \( |\alpha|_{\text{min}} \) which corresponds to \( \sigma = -0.47 \text{ MPa} \) will be only about 0.01 times that which corresponds to \( \tau = 1 \text{ MPa} \). Since estimates (6) and (7) should still be valid when \( \tau = 1 \text{ MPa} \), it follows that the term \( 3(\alpha_1 + \alpha_2)\alpha^2 \) in Eqs. (3)

---

\(^2\)McTigue and others, who adopted the second-order fluid model, also encountered the problem of over-parametrization. We noticed that the estimated values of material parameters reported by McTigue and others for Test 1 and Test 4 as well as the reported means of the estimates for Tests 2, 3 and 4 are inconsistent with their assumption that \( \xi^2 > 0 \) (see their Eq. (20) for the definition of \( \xi^2 \)), which they use when they develop their theoretical solution Eq. (22). There may be something wrong in their data-fitting.
and (4) can be ignored in the present data-fitting. That was exactly what we did; as a result, only three parameters, namely \( m \) \((= -2/3)\), \( \mu \) and \( \alpha_1 \), were left to be determined.

We can easily recast the data of Mctigue and others in terms of the stretch ratio \( \lambda(t) = \ell(t)/L \); here \( L \) is the original length of the specimen and \( \ell(t) \) is its length at the instant \( t \). By substituting \( a = \dot{\lambda}/\lambda \) and ignoring the term \( 3(\alpha_1 + \alpha_2)a^2 \) in Eqs. (3) and (4), we obtain for models (I) and (II) the following second-order differential equations:

\[
\text{(I)} \quad 3\alpha_1(\ddot{\lambda}/\lambda - (\dot{\lambda}/\lambda)^2) + 3\mu(3(\dot{\lambda}/\lambda)^2)^{m/2}(\dot{\lambda}/\lambda) - \sigma = 0, \tag{10}
\]

\[
\text{(II)} \quad 3\alpha_1(\ddot{\lambda}/\lambda - (\dot{\lambda}/\lambda)^2) + 3\mu\dot{\lambda}/\lambda - (3(\dot{\lambda}/\lambda)^2)^{-m/2}\sigma = 0. \tag{11}
\]

The data-fitting problem for model (I) is as follows: For each test we are given a set of data points \( (t_i, \lambda_i) \), where \( t_i = 1, 2, \ldots, 17 \) day, and \( \lambda_i \) is the measured value of \( \lambda \) at \( t_i \). From the data we can determine the rate of stretching \( \dot{\lambda}_i \) at \( t_i \) by polynomial interpolation. For a given set of parameters \( (m, \mu, \alpha_1) \), let \( \lambda(t; m, \mu, \alpha_1) \) be the solution to Eq. (10) with initial conditions \( \lambda(1; m, \mu, \alpha_1) = \lambda_1 \) and \( \dot{\lambda}(1; m, \mu, \alpha_1) = \dot{\lambda}_1 \). We want to determine values of the parameters \( m, \mu, \) and \( \alpha_1 \) which minimize the function

\[
F(m, \mu, \alpha_1) = \sum_{i}(\lambda(t_i; m, \mu, \alpha_1) - \lambda_i)^2. \tag{12}
\]
The data-fitting problem for model (II) is similar.

We solved the data-fitting problem for model (I) iteratively by using IMSL subroutine ZXSSQ (a finite difference analogue of the Levenberg-Marquardt method). We arrived at a set of initial guesses for \((m, \mu, \alpha_1)\) by the following procedure: First we chose a value of \(m\) close to \(-2/3\). Using this value of \(m\), we calculated the corresponding value of \(\mu\) from Eq. (8) and the measured value of \(\alpha_{min}\). By polynomial interpolation of the given data we estimated \(\dot{\lambda}\) and \(\ddot{\lambda}\) at \(t = 2\). From these estimates, \(\lambda_2\) and the first estimates of \(m\) and \(\mu\), we obtained a rough first estimate of \(\alpha_1\) from Eq. (10). To evaluate the function \(F\) at each iteration, the corresponding values \(\lambda(t_i; m, \mu, \alpha_1)\) are required. We obtain these values through numerical integration of Eq. (10) by using the fifth-order Runge-Kutta-Nyström method (cf. Lambert (1973), p. 122). Similarly, we solved the data-fitting problem for model (II). Complete details of the data-fitting with computer program are given in the doctoral thesis of Sun (1986).

The problem of over-parametrization persists in the data-fitting even after the parameter \(\alpha_2\) is dropped from the picture. For both models and for each test there are many combinations of \(m (\approx -2/3)\), \(\mu\) and \(\alpha_1\) which give essentially the same value to the sum of squared residuals \(F\). For a given test and for a fixed value of \(m\), however, the estimates of \(\mu\) and \(\alpha_1\) are sharp for both models. Moreover, when \(m\) ranges over values close to \(-2/3\), the scatter in the numerical estimates of \(\mu\) and
\( \alpha_1 \) is rather narrow. For instance, for Test 1, as \( m \) ranges from -0.65 to -0.68, we found that for model (I) \( \mu \) ranges from \( 2.82 \times 10^3 \) to \( 2.31 \times 10^3 \) kPa d\(^{1+m} \), and \( \alpha_1 \) increases from \( 1.26 \times 10^5 \) to \( 1.40 \times 10^5 \) kPa d\(^2 \); for model (II) \( \mu \) decreases from \( 2.84 \times 10^3 \) to \( 2.32 \times 10^3 \) kPa d\(^{1+m} \), and \( \alpha_1 \) decreases from \( 2.75 \times 10^3 \) to \( 2.03 \times 10^3 \) kPa d\(^{2+m} \).

In this note we shall be content to obtain for models (I) and (II) an order-of-magnitude estimate of the normal stress effects discussed by McTigue and others. Moreover, we shall be interested only when the normal stress effects have sufficiently large magnitude to be practically significant. Since the data of McTigue and others all pertain to the same \( \sigma \) and are not sufficient to give a sharp estimate of \( m \), in what follows we shall take \( m = -2/3 \) (i.e., \( n = 3 \)) and estimate \( \mu \) and \( \alpha_1 \) on that basis. While this specific choice of \( m \) is somewhat arbitrary, it is the best choice available and is good enough for our present purpose: our conclusions below are insensitive to variations of \( m \) near -2/3. For instance, as the reader can easily do the calculations himself, taking \( m = -0.65 \) or \( m = -0.68 \) and the corresponding values of \( \mu \) and \( \alpha_1 \) (estimated from Test 1) given above will not affect the qualitative conclusions to be drawn below when we take \( m = -2/3 \). Indeed, a sampling of other choices of \( m \) from the range -0.65 to -0.71 all leads to the same qualitative conclusions; cf. Sun (1986).

The least squares estimates of \( \mu \) and \( \alpha_1 \) when \( m \) is fixed at -2/3 are listed in Tables 1 and 2. The scatter in the estimated values of the parameters is much narrower than that reported by McTigue and others. Under
the estimated values of the material parameters the fits are excellent for all four tests and for both models. The percentage errors in the fitted values of \( \lambda \) as compared with the measured values are mostly under \( \pm 0.02\% \); the highest percentage errors are about \( \pm 0.04\% \).

Having obtained the value of \( \mu \) for models (I) and (II), respectively, we can now look at estimates (6) and (7) in detail. Let us consider estimate (6) first. Within the range of octahedral shear stresses under which Glen's flow law is presumed to be valid (i.e., \( 0.1 \text{ MPa} \leq \tau \leq 1 \text{ MPa} \)), the maximum value of \( |\alpha|_{\min} \) corresponds to \( \tau = 1 \text{ MPa} \). Substituting \( \sigma = -3/2 \text{ MPa} \) and \( \mu = 2.41 \times 10^3 \text{ kPa d}^{1/3} \), we calculate from Eq. (8) that \( |\alpha|_{\min} = 0.0758 \text{ d}^{-1} \), for which the right-hand side of estimate (6) is equal to \( 1.23 \times 10^5 \text{ kPa d}^2 \). Since \( \alpha_1 \) is estimated to be equal to \( 1.61 \times 10^5 \text{ kPa d}^2 \), estimate (6) dictates that for model (I) \( |\alpha_1 + \alpha_2| \) should be at least an order of magnitude smaller than \( \alpha_1 \). Similarly, for model (II), corresponding to \( \tau = 1 \text{ MPa} \) and \( \mu = 2.43 \times 10^3 \text{ kPa d}^{1/3} \), the right-hand side of estimate (7) is equal to \( 3.29 \times 10^4 \text{ kPa d}^{4/3} \). Thence we infer that for model (II) \( |\alpha_1 + \alpha_2| \) should be at most of the same order of magnitude as \( \alpha_1 \).
NORMAL STRESS DIFFERENCES IN SHEARING FLOWS

Both models (I) and (II) can exhibit non-zero and unequal normal stress differences in shearing flows. (Cf. Schowalter (1978, p. 71) for the definition of normal stress differences.) For models (I) and (II), respectively, the normal stress differences are given by

\begin{align}
\text{(I)} & \quad N_1 = -2\alpha_1 \kappa^2, \quad N_2 = (2\alpha_1 + \alpha_2)\kappa^2, \quad (13) \\
\text{(II)} & \quad N_1 = -2\alpha_1 \kappa^{2+m}, \quad N_2 = (2\alpha_1 + \alpha_2)\kappa^{2+m}; \quad (14)
\end{align}

Here \( \kappa \) is the shear rate. The formulae for model (I) will be identical to those for the second-order fluid (McTigue and others, 1985) if we replace \( \alpha_1 \) and \( \alpha_2 \) in Eq. (13) by the parameters \( \mu_2 \) and \( \mu_3 \) of McTigue and others. It is thus possible to compare directly the magnitudes of \( N_1 \) and \( N_2 \) according to our model (I) with those reported by McTigue and others for the second-order fluid model.

For model (I), by using the mean value of \( \alpha_1 \) in Table 1 and by changing units, we obtain

\[ N_1 = -2.40 \times 10^{18} \kappa^2 \text{ Pa}, \quad (15) \]

where the shear rate \( \kappa \) is in units of \( \text{s}^{-1} \). Since all that we know about the parameter \( \alpha_2 \) is the estimate (6), we can at best give an order-
of-magnitude estimate of $N_2$. Since $|\alpha_1 + \alpha_2|$ is estimated to be at least an order of magnitude smaller than $\alpha_1$, $|N_2|$ will have the same order of magnitude as $|N_1|$.

For the second-order fluid model, McTigue and others found that $N_1 = 2.1 \times 10^{19} \kappa^2 \text{ Pa}$ and $N_2 = 3.4 \times 10^{21} \kappa^2 \text{ Pa}$ (cf. their Eqs. (23a) and (23b)). Their value of $N_1$ is one order of magnitude bigger than the first normal stress difference in our model (I); moreover, it has a different sign. Their value of $N_2$ is three orders of magnitude bigger than that in our model (I).

The tremendous discrepancy in the estimates of normal stress differences, we believe, is due in part to the problem of over-parametrization in the data-fitting. As noted by McTigue and others, "good fits ... to the creep data can be found for broad ranges of [the] parameters [$\mu_2$ and $\mu_3$]". We overcame the problem of over-parametrization by fixing the value of $m$ at $-2/3$ (i.e., $n = 3$) and by ignoring in Eqs. (3) and (4) the term $3(\alpha_1 + \alpha_2)a^2$, which should be negligible for consistency with Glen's flow law. Working with the second-order fluid model is like setting $m = 0$ in our models at the outset, but there is no longer Glen's flow law to lean on, for the second-order fluid model is inconsistent with Glen's flow law. There does not seem to be any convincing way to evade the problem of over-parametrization when one attempts to fit the data of McTigue and others to the second-order fluid model.

Over-parametrization was not a problem in the pressuremeter study of
Man and the Manitoba team on polycrystalline ice at -2 °C (Man and others, 1985; Kjartanson, 1986; Sun, 1986). For both models (I) and (II) the parameter $\alpha_2$ simply does not appear in the respective differential equation that governs pressuremeter creep tests. With data of single-stage and mult-stage creep tests in which the cavity pressure ranged from 1.0 to 2.5 MPa and the early time responses were recorded in detail, the parameters $m$, $\mu$ and $\alpha_1$ could be determined unambiguously for both models. For the specimens of ice they studied, they found that $m$ was indeed close to $-2/3$ and $\alpha_1$ was positive for both models. For model (I) the value of $\alpha_1$ was estimated to be $9.17 \times 10^{14}$ Pa s$^2$, for which $N_1 = -1.83 \times 10^{15}$ Pa.
FREE-SURFACE DEPRESSION (OR HEAVE) IN OPEN-CHANNEL FLOW

McTigue and others made use of an approximation scheme (due to Wineman and Pipkin (1966), and Tanner (1970); cf. also Schowalter (1978), pp. 255-257) to calculate the free-surface depression or heave of a second-order fluid when it flows steadily down an inclined open semicircular channel. Using the values of material parameters which they estimated from their creep tests, they applied their surface-depression formula to the flow of glaciers.

The free-surface depression or heave studied by McTigue and others is due to a non-vanishing second normal-stress difference $N_2$. Both of our models can also exhibit the same phenomenon. Indeed, by following the same approximation procedure, it is straightforward to work out the corresponding formulae for our models. In our derivation the usual power-law fluid assumes a role parallel to that of the Newtonian fluid in the derivation of McTigue and others.

Let $\rho$ be the density of ice, $g$ be the acceleration due to gravity, $h$ be the central (maximum) rise or fall, $\beta$ be the channel slope, and $R$ be the channel radius (see Fig. 3 of McTigue and others). The formulae in question are found for models (I) and (II), respectively, to be as follows:

(I) \hspace{1cm} h = -\left(2\alpha_1 + \alpha_2\right) - \frac{(3 + m)}{2\rho g \cos \beta} \frac{R \rho g \sin \beta}{2\mu} \frac{2}{(1+m)} ;

(16)
\[(II) \quad h = -4(2\alpha_1 + \alpha_2) \frac{(3 + 2m)}{(2 + m)\rho g \cos \beta} \frac{(Rg \sin \beta)^{(2+m)}/(1+m)}{4\mu}. \quad (17)\]

When \( m = 0 \), both Eqs. (16) and (17) reduce to the formula for the second-order fluid, i.e., Eq. (26) of McTigue and others. For the usual power-law fluid, which is the special instance of our models with \( \alpha_1 = \alpha_2 = 0 \), both Eqs. (16) and (17) give \( h = 0 \).

Earlier we have argued that for model (I) \( |\alpha_1 + \alpha_2| \) should be at least an order of magnitude smaller than \( \alpha_1 \); for model (II) \( |\alpha_1 + \alpha_2| \) should at most have the same order of magnitude as \( \alpha_1 \). For a given \( R \) and \( \beta \), we can easily obtain an order-of-magnitude estimate of \( |h| \) for models (I) and (II). Let us consider model (I). First we set \( \alpha_1 + \alpha_2 = 0 \). The right-hand side of Eq. (16) can then be evaluated. Let us denote the resulting value by \( h_o \). Since \( |\alpha_1 + \alpha_2| \) is at least an order of magnitude smaller than \( \alpha_1 \), \( |h_o| \) and \( |h| \) should have the same order of magnitude. Similarly, for a given \( R \) and \( \beta \), we infer that for model (II) \( |h| \) can at most have the same order of magnitude as \( |h_o| \).

Let us evaluate \( |h_o| \) for some sample channel slopes and channel radii. We take \( \rho = 9 \times 10^2 \text{ kg/m}^3 \), \( g = 9.8 \text{ m/s}^2 \), \( m = -2/3 \), and give \( \mu \) and \( \alpha_1 \) the mean values given in Tables 1 and 2. For \( \beta = 10^\circ \) and \( R = 500 \text{ m} \), we obtain the following values for models (I) and (II), respectively: (I) \( |h_o| = 0.35 \text{ m} \); (II) \( |h_o| = 0.07 \text{ m} \). For the same \( \beta \) and \( R \), McTigue and others estimated \( h = -42 \text{ m} \). For \( \beta = 5^\circ \) and \( R = 250 \text{ m} \), McTigue and others obtained \( h = -2.6 \text{ m} \); for models (I) and (II) we get
(I) \( |h_0| = 0.08 \text{ mm}, \) and (II) \( |h_0| = 0.26 \text{ mm}. \)

For a given \( R \) and \( \beta \), let \( h^* \) be the value of \( h \) estimated by McTigue and others. In general \( |h^*| \) is related to \( |h_0| \) of our models by what follows:

\[
\text{(I) } |h_0|/|h^*| = 1.4 \times 10^{-10} (R \sin \beta)^4; \\
\text{(II) } |h_0|/|h^*| = 2.1 \times 10^{-7} (R \sin \beta)^2; 
\]

Here \( R \) is in units of metres. For the usual ranges of \( R \) and \( \beta \) of glaciers on earth, the free-surface depression or heave according to both of our models is much smaller in magnitude than the prediction of McTigue and others. Indeed, according to both of our models and with the values of material parameters given in Tables 1 and 2, the free-surface depression or heave induced by the second normal-stress difference will usually be negligible for glaciers.

For both models (I) and (II), the velocity field in the flow under consideration is, to the order of approximation adopted in the derivation of Eqs. (16) and (17), identical to that which pertains to Glen's flow law. For both models the stress distribution is different from that of the usual power-law fluid. With the values of material parameters adopted above, however, the difference in stress distribution is found to be negligible for the usual ranges of \( R \) and \( \beta \) for glaciers. Thus we cannot accept the argument of McTigue and others as regards the significance of normal stress effects in the formation of crevasses.
The values of $|h_o|$ are even smaller if we use values of $m$, $\mu$ and $\alpha_1$ estimated by Man and the Manitoba team from pressuremeter creep tests on polycrystalline ice at $-2^\circ C$. The estimates were as follows: $m = -0.711$, $\mu = 2.97 \times 10^7 \text{ Pa s}^{1+m}$; for model (I), $\alpha_1 = 9.17 \times 10^{14} \text{ Pa s}^{2}$; for model (II), $\alpha_1 = 2.46 \times 10^{10} \text{ Pa s}^{2+m}$. For $\beta = 10^\circ$ and $R = 500 \text{ m}$, it follows that for model (I) $|h_o| = 9.8 \text{ mm}$ and for model (II) $|h_o| = 2.3 \text{ mm}$. For $\beta = 5^\circ$ and $R = 250 \text{ m}$, $|h_o| = 0.7 \times 10^{-3} \text{ mm}$ for model (I), and $|h_o| = 4.8 \times 10^{-3} \text{ mm}$ for model (II). These values of $|h_o|$ are about thirty to a hundred times smaller than those calculated above. Granted that the specimens in the pressuremeter tests would have structural features different from the specimens of McTigue and others, there still remains the possibility that the significant difference in the estimated values of $|h_o|$ could be attributed to the difference in temperature between the pressuremeter tests ($-2^\circ C$) and the triaxial tests ($-9.5$ to $-9.8^\circ C$). This suggests that lowering of temperature might greatly enhance normal stress effects in the creep of ice.
CONCLUSION AND DISCUSSION

McTigue and others (1985) initiated the enquiry whether normal stress effects can have practical significance in the flow of glaciers, to which their answer was leaning towards the affirmative. Their analysis, however, can be reproached on the following grounds: (i) Their analysis is based on modeling ice as a second-order fluid, which is inconsistent with Glen's flow law. (ii) Their conclusion naturally depends very much on the quantification of material parameters in their model. They did not indicate how they handled the problem of over-parametrization in their data-fitting. Moreover, there seems to be apparent error in their fitting (cf. Footnote 2 above).

In this note we have re-examined the issues discussed by McTigue and others in light of two new models, namely "the modified second-order fluid" and "the power-law fluid of grade 2", both of which can be taken as simple modifications of Glen's flow law. For ease of comparison with the work of McTigue and others, we have used the same creep data in quantifying the material parameters in our models. As indicated above there is a problem of over-parametrization in the data-fitting, and we have shown how we can circumvent that problem to obtain order-of-magnitude estimates for the normal stress effects discussed by McTigue and others. For both of our models, the normal stress effects are found to be far less pronounced than the estimates of McTigue and others. Indeed, should glacial ice have creep properties similar to the specimens of McTigue and others, our models indicate that except for extremely thick ice and steep slopes normal stress effects of the type discussed by McTigue and others would not play a significant role in
glacier flow.

The preceding conclusion remains valid, should we use values of material parameters estimated by Man and the Manitoba team from data of pressuremeter creep tests on polycrystalline ice at \(-2^\circ\text{C}\) (Man and others, 1985; Kjartanson, 1986; Sun, 1986); cf. Footnote 3 above. The problem of over-parametrization did not appear in their pressuremeter work. Pressuremeter creep tests, however, can only deliver estimates for the parameters \(m\), \(\mu\) and \(\alpha_1\). As indicated above, thanks to Glen's flow law, we do not need to know the value of \(\alpha_2\) to obtain an order-of-magnitude estimate of the normal stress effects discussed in this note. On the other hand, procurement of experimental data that suffice for the estimation of all the material parameters will not only tighten our argument; it will also provide more stringent tests for the models. For a given ice specimen, it should be possible to obtain the values of \(m\), \(\mu\), \(\alpha_1\) and \(\alpha_2\) for models (I) and (II) by performing both pressuremeter and triaxial creep tests which record early time response in detail.

A correct assessment of the practical significance of normal stress effects in glacier flows can be made only after a sufficiently accurate flow law is established and values of material parameters in the flow law are ascertained for various ice-forms and temperatures. At present it is premature on both counts to give a sweeping conclusion. For instance, as pointed out in Footnote 3, lowering of temperature might greatly enhance normal stress effects in the creep of ice.
ACKNOWLEDGMENT

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Table 1. Least squares estimates of $\mu$ and $\alpha_1$ for model (I) when $m$ is fixed at $-2/3$.

<table>
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<th>Test</th>
<th>$\mu$ ($10^3$ kPa d$^{1/3}$)</th>
<th>$\alpha_1$ ($10^5$ kPa d$^2$)</th>
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Table 2. Least squares estimates of $\mu$ and $\alpha_1$ for model (II) when $m$ is fixed at $-2/3$.

<table>
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<th>Test</th>
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