DECOMPOSITIONS OF SEMIMARTINGALES ON DUALS OF COUNTABLY
HILBERT NUCLEAR SPACES

BY

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DECOMPOSITIONS OF SEMIMARTINGALES ON DUALS OF COUNTABLY
HILBERT NUCLEAR SPACES

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INTRODUCTION

The canonical decomposition of a real valued semimartingale is useful in proving functional central limit theorems for real valued semi-martingales as well as to study the absolute continuity of measures induced by them (see [3], [4] and [8]).

In the present work we find the canonical decomposition of a $\Phi'$-valued semimartingale, where $\Phi'$ is the dual of a countably Hilbert nuclear space (Theorem 3). We hope that this result can be used in the future to prove functional central limit theorems for $\Phi'$-valued semimartingales and to characterize $\Phi'$-valued stochastic processes with independent increments in terms of the predictable characteristics of this decomposition.

In addition we present (Theorem 1) a regularization theorem for $\Phi'$-valued semimartingales and some others decompositions for them (Theorem 2 and Lemmas 1 and 2).

The basic tools for the proofs of our results are the regularization Theorem of Ito [2] for continuous linear random functionals (see also Kallianpur [5]), a Baire category argument (Lemma I.2.3 in Xia [10]) and some results of Memin [6] on the topology of real valued semi-martingales. The idea of using Memin's results in the study of nuclear space valued processes is due to Ustunel [9].
1. DEFINITIONS AND NOTATION

Let \( \Phi \) be a Frechet space whose topology is given by an increasing sequence of Hilbertian norms \( \| \cdot \|_n \), \( n \geq 0 \). Let \( \Phi_n \) be the Hilbert space completion of \( \Phi \) with respect to \( \| \cdot \|_n \) and let \( \Phi'_n \) be the topological dual of \( \Phi_n \) with \( \| \cdot \|_n \) the dual norm on \( \Phi_n' \). The space \( \Phi \) is called nuclear if for each \( n \geq 0 \) there exists \( m > n \) such that the canonical injection \( i_n: \Phi_n \hookrightarrow \Phi_m \) is Hilbert-Schmidt, i.e. if \( \{ \phi_j \}_{j \geq 1} \) is a complete orthonormal system (CONS) in \( \Phi_m \) the following property holds:

\[
\sum_{j=1}^{\infty} \| \phi_j \|_n^2 < \infty.
\]

Let \( \Phi' \) be the topological dual space of \( \Phi \) with the strong topology.

It follows that

\[
\Phi = \bigcap_{n=1}^{\infty} \Phi_n \quad \text{and} \quad \Phi' = \bigcup_{n=1}^{\infty} \Phi'_n.
\]  

(1.1)

Throughout this work we assume that \( (\Omega, F, (F_t)_{t \geq 0}, P) \) is a stochastic basis, i.e. \( (\Omega, F, P) \) is a complete probability space and \( (F_t)_{t \geq 0} \) is an increasing right continuous family of sub \( \sigma \)-fields of \( F \). Let \( \mathcal{B} = \mathcal{B}(\Phi') \) be the \( \sigma \)-field generated by sets of the form

\[
\{ f \in \Phi' : f[\phi] < a \} \quad \phi \in \Phi \quad a \in \mathbb{R}.
\]

It can be shown that \( \mathcal{B} \) is the Borel \( \sigma \)-field of \( \Phi' \) generated by the open sets in the strong topology of \( \Phi' \).
A mapping $X_t(\omega): \mathbb{R}_+ \times \Omega \to \Phi'$ is a $\Phi'$-stochastic process if $X_t(\omega)[\phi]$ is a real valued process for each $\phi \in \Phi$, i.e.

$$\{ \omega: X_t(\omega) \in B \} \in F \quad \forall B \in \mathcal{B}.$$ 

A $\Phi'$-valued stochastic process $(X_t)_{t \geq 0}$ is a $\Phi'$-semimartingale if for each $\phi \in \Phi$ $X_t[\phi]$ is a real valued semimartingale, i.e.

$$X_t[\phi] = X^\phi_0 + M^\phi_t + A^\phi_t$$

where $X^\phi_0$ is a $F^\phi_0$-measurable random variable, $M^\phi$ is a real valued local martingale $M^\phi_0 = 0$ and $A^\phi$ is a real valued right continuous adapted process whose paths are of finite variation, $A^\phi_0 = 0$. A $\Phi'$-semimartingale is called special if for each $\phi \in \Phi$ there is a decomposition (1.2) for which $A^\phi$ is predictable.

**Proposition 1.** Every $\Phi'$-semimartingale $X_t$ has a $\Phi'$-valued version which is right continuous with left hand limits in the strong topology of $\Phi'$.

**Proof.** Since for each $\phi$ the real valued semimartingale $X_t[\phi]$ has a right continuous with left hand limits version (VII.23 Dellacherie and Meyer [1]) the proposition follows using Theorem 2 of Mitoma [7].
The space of all $\Phi'$-valued right continuous with left hand limits processes is denoted by $D(\mathbb{R}_+;\Phi')$. From now on we will assume $\Phi'$-semimartingales in $D(\mathbb{R}_+;\Phi')$.

A $\Phi'$-process $(M_t)_{t \geq 0}$ is called a $\Phi'$-local martingale if for each $\phi \in \Phi (M_t[\phi])_{t \geq 0}$ is a real valued local martingale. Every $\Phi'$-local martingale has a version in $D(\mathbb{R}_+;\Phi')$. Similarly a $\Phi'$-valued square integrable local martingale is defined. A $\Phi'$-process $(A_t)_{t \geq 0}$ is called a $\Phi'$-process of finite variation if for each $\phi \in \Phi (A_t[\phi])_{t \geq 0}$ is a real valued process of locally bounded variation. A $\Phi'$-valued purely discontinuous process is defined in a similar manner.
2. CHARACTERIZATION OF $\Phi'$-VALUED SEMIMARTINGALES

The aim of this section is to prove the following three theorems. The first one is a regularization theorem for $\Phi'$-valued semimartingales. The last two theorems give characterizations analogous to the finite dimensional results presented, for example, in Jacod and Mémin [3] and Kabanov, Lipcer and Shiryaev [4].

Theorem 1 (Regularization). Let $(X_t)_{t \geq 0}$ be a $\Phi'$-valued semimartingale. Then there exist a $\Phi'$-valued right continuous adapted process $(A_t)_{t \geq 0}$ whose paths are of finite variation, $A_0 = 0$ and a $\Phi'$-valued local martingale $(M_t)_{t \geq 0}$, $M_0 = 0$ such that for each $t \geq 0$

$$X_t = X_0 + M_t + A_t$$

(2.1)

where $X_0$ is a $\Phi'$-valued random variable $F_0$-measurable.

Theorem 2. Let $(X_t)_{t \geq 0}$ be a $\Phi'$-valued semimartingale. Then for each $t > 0$ $X_t$ admits the unique representation

$$X_t = X_0 + A_t + M^c_t + M^d_t + \sum_{0 < s \leq t} \Delta X_s 1_{\Delta X_s \in B^c}$$

(2.2)

where

1.1) $X_0$ is a $\Phi'$-valued random variable $F_0$-measurable.

1.2) $(A_t)_{t \geq 0}$ is a predictable $\Phi'$-right continuous processes of finite variation, $A_0 = 0$. 
1.3) \((M^C_t)_{t \geq 0}\) is a \(\phi'\)-valued continuous local martingale, \(M^C_0 = 0\).

1.4) \((M^d_t)_{t \geq 0}\) is a \(\phi\)-valued purely discontinuous local martingale.

1.5) \(B\) is a (strongly) compact set in \(\phi'\), such that \(0 \in B\).

**Theorem 3.** Let \((X^r_t)_{t \geq 0}\) be a \(\Phi'\)-valued semimartingale. Then for each \(t > 0\) \(X_t\) admits the unique decomposition

\[
X_t = X_0 + A_t + M^C_t + \int_0^t \int_B \psi d(u-v)(s,\psi) + \int_0^t \int_B \psi d\mu
\]

(2.3)

where \(X_0, A_t, M^C_t\) and \(B\) are as in Theorem 2 and

1.6) \(\mu(\omega; (0,t]; \Gamma) = \sum_{0 \leq s \leq t} 1\{\Delta X_s(\omega) \in \Gamma\} \Gamma \in B(\Phi'_0)
\)

\(\Phi'_0 = \Phi' \setminus \{0\}\)

is the integer valued random measure of the jumps of \(X\) with compensated measure \(\nu\) that satisfies the conditions:

1.7) i) \(\nu(\omega; \{0\}; \Phi') = \nu(\omega; \mathbb{R}^+; \{0\}) = 0\),

ii) \(\nu(\omega; \{t\}; \Phi') \leq 1\) \(\forall t > 0\),

iii) \(\int_0^t \int_{\Phi'} \min(1, (\psi(\phi))^2) d\nu < \infty\) \(\forall \phi \in \Phi, t > 0\),

iv) \(\sum_{0 \leq s \leq t} |\int_B \psi(\phi) \nu(\omega, \{s\}, d\psi)| < \infty\) \(\forall \phi \in \Phi, t > 0\).

1.8) \(\left[\int_0^t \int_B \psi d(u-v)[\phi]\right] = \int_0^t \int_B \psi[\phi] d(u-v)\) \(\forall \phi \in \Phi, t > 0\),
\[ (\int_0^t \int \mathbb{B} \phi \, d\mu)[\phi] = \int_0^t \int \mathbb{B} \phi \, d\mu = \sum_{0 \leq s < t} \Delta X_s[\phi] \mathbb{I}_{\{\Delta X_s \in \mathbb{B}\}} \]

\[ \forall \phi \in \Phi, \ t > 0 \, . \]

For the proof of the above theorems we need the following lemmas that are important on their own. Furthermore, once these lemmas are shown the proofs of Theorems 2 and 3 are very similar to the real valued situation.

**Lemma 1.** Let \((Y_t)_{t \geq 0}\) be a \(\Phi'\)-valued semimartingale \((Y_0 = 0)\) with bounded jumps on the strong topology of \(\Phi'\). Then there exists a \(\Phi'\)-valued local martingale \(M_t\) and a predictable \(\Phi'\)-valued process \(A_t\) of finite variation with right continuous paths such that for each \(t > 0\)

\[ Y_t = M_t + A_t \tag{2.4} \]

i.e. \(Y_t[\phi] = M_t[\phi] + A_t[\phi] \quad \forall \phi \in \Phi\). Moreover this representation is unique up to indistinguishable \(\Phi'\)-valued processes.

**Proof.** Let \(\Delta Y = \{\Delta Y_t : t > 0\}\). Then by assumption \(\Delta Y\) is a strongly bounded set, i.e. for any bounded set \(\Lambda\) in \(\Phi\) and \(\varepsilon > 0\) \(\exists n > 0\) such that

\[ \Delta Y \subset n\{F \in \Phi' : \sup_{\phi \in \Lambda} |F[\phi]| < \varepsilon\} \, . \]

Then for any bounded set \(\Lambda\) in \(\Phi\) \(\exists K_\Lambda > 0\) such that

\[ \sup_{\phi \in \Lambda} |\Delta Y_t[\phi]| < K_\Lambda \quad \forall \ t > 0 \quad \tag{2.5} \]
and in particular for each \( \phi \in \Phi \) \( \exists K_\phi > 0 \) such that

\[
|\Delta Y_t[\phi]| < K_\phi \quad \forall \ t > 0.
\]

Hence for each \( \phi \in \Phi \) \( Y_t[\phi] \) is a special real valued semimartingale and therefore it admits a unique representation

\[
Y_t[\phi] = M_t^\phi + A_t^\phi
\]

(2.6)

where \( M_t^\phi \) is a real valued local martingale and \( A_t^\phi \) is a predictable real valued right continuous process of finite variation, \( M_0^\phi = A_0^\phi = 0 \).

Next, for each \( t > 0 \) define \( M_t(\cdot) : \Phi \to L_0^\phi(\Omega) \) and \( A_t(\cdot) : \Phi \to L_0^\phi(\Omega) \) by

\[
M_t(\phi) = M_t^\phi
\]

\[
A_t(\phi) = A_t^\phi
\]

Then since \( \forall t > 0 \ Y_t \in \Phi' \), by the uniqueness of the representation (2.6) the mappings \( M_t(\cdot) \) and \( A_t(\cdot) \) are linear on \( \Phi \). Then if we show that \( M_t(\cdot) \) and \( A_t(\cdot) \) are continuous in probability, by the regularization lemma of Ito \[2\] there exist \( M_t \in \Phi' \) and \( A_t \in \Phi' \) such that

\[
M_t[\phi] = M_t(\phi) \quad \text{a.s.} \quad \forall \ \phi \in \Phi
\]

and

\[
A_t[\phi] = A_t(\phi) \quad \text{a.s.} \quad \forall \ \phi \in \Phi.
\]
and the lemma will be shown. Hence we only have to show that $M_\ell (\cdot) : \Phi \to L_0(\Omega)$ and $A_\ell (\cdot) : \Phi \to L_0(\Omega)$ are continuous.

In order to do this we shall use some results from the topology of real valued semimartingales given in Memin [6]: Let $S$ be the linear space of real valued semimartingales and define on $S$ the metric

$$
\| Z \|_d = \sum_{m \geq 1} \frac{1}{2^m} \| Z \|_m
$$

where

$$
\| Z \|_m = \sup_{h \in \mathcal{G}} E(1\Lambda | h \cdot Z_m | ),
$$

$\mathcal{G}$ is the class of real valued predictable processes $h = h(t, \omega)$ of the form

$$
h = \sum_{i=1}^{n-1} a_i 1_{(t_i, t_{i+1}) \times \Omega},
$$

$0 < t_1 < t_2 < \ldots < t_n < \infty$, $a_i$ is a real valued random variable $F_{t_i}$ measurable, $|a_i| \leq 1$ for $i = 1, \ldots, n$ and

$$
h \cdot Z_m = \int_0^m h_s dZ_s = \sum_{i=1}^{n-1} a_i (Z_{t_{i+1} \Lambda m} - Z_{t_i \Lambda m}).
$$

Next if $Y$ is a $\Phi'$-valued semimartingale define $V : \Phi \to \mathbb{R}_+$ by

$$
V(\phi) = \| Y [\phi] \|_d, \quad \phi \in \Phi.
$$

Then the function $V$ satisfies the following properties:
a) \( V(\phi+\psi) \leq V(\phi) + V(\psi) \quad \forall \phi, \psi \in \Phi \).

b) \( V(\phi) = V(-\phi) \quad \forall \phi \in \Phi \).

c) \( V(\phi) < \infty \quad \forall \phi \in \Phi \).

d) \( V(\phi) \) is a lower semicontinuous function: for if \( \phi_n \to \phi \) by

using Fatou's lemma

\[
V(\phi) = \sum_{m=1}^{\infty} \frac{1}{2^m} \| Y[\phi] \|_m \leq \sum_{m=1}^{\infty} \frac{1}{2^m} \sup_{h \in \mathcal{H}} \left( \liminf_{n \to \infty} E(1\Lambda|h \cdot Y[\phi_n]_m) \right)
\]

\[
\leq \sum_{m=1}^{\infty} \frac{1}{2^m} \liminf_{n \to \infty} \sup_{h \in \mathcal{H}} E(1\Lambda|h \cdot Y[\phi_n]_m) \]

\[
\leq \liminf_{n \to \infty} \sum_{m=1}^{\infty} \frac{1}{2^m} \| Y[\phi] \|_m = \liminf_{n \to \infty} V(\phi_n).
\]

Then by Lemma I.2.3 in Xia [10] \( V(\phi) \) is a continuous function of \( \phi \)

and therefore if \( \phi_n \to \phi \) \( V(\phi_n - \phi) \to 0 \) i.e. \( Y[\phi_n] \) converges to \( Y[\phi] \) in

\((S, \| \cdot \|_d)\).

Next given \( \phi \in \Phi \) let \( \phi_n \to \phi \). Then there exists a bounded set

\( \Lambda \subset \Phi \) such that \( \phi \in \Lambda \) and \( \phi_n \in \Lambda \) \( n \geq 1 \). Moreover since \( \Delta Y \) is a strongly

bounded set, there exists \( K_{\Lambda} > 0 \) s.t. (2.5) is satisfied. Then by

Remark IV.3 and Theorem IV.4 in Memin [6] (see also VIII 76.3(b)

in [1]) for \( \eta \in \Lambda \) the applications

\[
Y[\eta] \to M^n
\]

\[
Y[\eta] \to A^n
\]
are $\| \cdot \|_d$-continuous. Then if $\phi_n \to \phi$ on $\Phi$

$$\| M^\phi - M^{\phi_n} \|_d \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\| A^\phi - A^{\phi_n} \|_d \to 0 \quad \text{as} \quad n \to \infty .$$

Finally from (2.7) for each $T > 0$ there exists $\theta_T > 0$ such that for $0 \leq t \leq T$

$$E(1_A | M^\phi_t - M^{\phi_n}_t |) \leq \theta_T \| M^\phi - M^{\phi_n} \|_d$$

and

$$E(1_A | A^\phi_t - A^{\phi_n}_t |) \leq \theta_T \| A^\phi - A^{\phi_n} \|_d .$$

Hence the mappings $M^\phi_t (\cdot) : \Phi \to L^0 (\Omega)$ and $A^\phi_t (\cdot) : \Phi \to L^0 (\Omega)$ are continuous.

The proof of the lemma is complete.

Q.E.D.

**Lemma 2.** Let $(M^c)_t \geq 0$ be a $\Phi'$-valued local martingale $M^c_0 = 0$ with bounded jumps on the strong topology of $\Phi'$. Then there exist a $\Phi'$-valued continuous local martingale $(M^c)_t \geq 0$ and a $\Phi'$-valued purely discontinuous local martingale $(M^d)_t \geq 0$ such that for each $t \geq 0$

$$M_t = M^c_t + M^d_t$$

i.e.

$$M^\phi_t = M^c_t [\phi] + M^d_t [\phi] \quad \forall \phi \in \Phi .$$

This decomposition is unique up to indistinguishable $\Phi'$-valued processes.
Proof. Since for each $\phi \in \Phi$ $M_t[\phi]$ is a local martingale, by Theorem VII.43 in [1] the following decomposition holds uniquely

$$M_t[\phi] = M^c_{t,\phi} + M^d_{t,\phi} \quad (2.10)$$

where $M^c_{t,\phi}$ is a real valued continuous local martingale and $M^d_{t,\phi}$ is a purely discontinuous local martingale. Thus for each $t > 0$ the mappings

$$M^c_{t}(\cdot) = M^c_{t,\cdot},$$
$$M^d_{t}(\cdot) = M^d_{t,\cdot},$$

from $\phi$ to $L^0(\Omega)$ are well defined and linear on $\Phi$. Then it is enough to show that they are $L^0(\Omega)$ continuous since by the regularization lemma in [2] they will have a $\Phi'$-valued version. To prove this we observe that similar to Theorem IV.4 and Remark IV.3 in Memin [6], one shows that in the set of real valued local martingales with jumps bounded by a constant $k$, the applications

$$Z \rightarrow Z^c \quad Z \rightarrow Z^d$$

are $\|\cdot\|_d$-continuous (using the notation of the proof of Lemma 1).

Thus if $\phi_n \rightarrow \phi$ there exists a bounded set $\Lambda$ in $\Phi$ such that $\phi \in \Lambda$ and $\phi_n \in \Lambda$ for $n \geq 1$. Moreover there exists $K_\Lambda > 0$ such that (2.5) holds. Then since $(M_t)_{t \geq 0}$ is a $\Phi'$-valued process we have that
\[ \| M^c(\phi) - M^c(\phi_n) \|_d \xrightarrow{n \to \infty} 0 \]

and

\[ \| M^d(\phi) - M^d(\phi_n) \|_d \xrightarrow{n \to \infty} 0 . \]

Since from (2.7) for each \( T > 0 \) \( \exists \Theta_T > 0 \) s.t. for \( 0 \leq t \leq T \)

\[ E(1_A |M^c_t(\phi) - M^c_t(\phi_n)|) \leq \Theta_T \| M^c(\phi) - M^c(\phi_n) \|_d \]

and

\[ E(1_A |M^d_t(\phi) - M^d_t(\phi_n)|) \leq \Theta_T \| M^d(\phi) - M^d(\phi_n) \|_d \]

then the mappings \( M^c_t(\phi): \Phi \to L_0(\Omega) \) and \( M^d_t(\phi): \Phi \to L_0(\Omega) \) are continuous.

Q.E.D.

**Lemma 3.** Let \( (X_t)_{t \geq 0} \) be a \( \Phi' \)-valued semimartingale. Then the random measure \( \mu \) of the jumps of \( X \) given in (1.6) of Theorem 3 is well defined and admits a compensated measure \( \nu \), i.e. \( \nu = \nu(\omega, dt, d\psi) \) is a predictable measure such that for an arbitrary non-negative predictable function \( f = f(\omega, t, \psi) \)

\[ E \int_0^\infty \int_\Phi' f d\mu = E \int_0^\infty \int_\Phi' f d\nu \]  

(2.11)
Moreover, for each $\phi \in \Phi$, the stochastic integrals:

$$\int_0^t \int_B \psi[\phi] d(\mu - \nu)$$  \hspace{1cm} (2.12)

and

$$\int_0^t \int_{B^c} \psi[\phi] d\nu$$  \hspace{1cm} (2.13)

are well defined for each $\phi \in \Phi$ and $t > 0$.

**Proof.** Since we are assuming that $X \in D([0, \infty); \Phi')$ then the random measure

$$\mu(\omega; (0, t], \Gamma) = \sum_{0 < s \leq t} 1_{\{\Delta X_s \in \Gamma\}} < \infty \text{ a.s. } \Gamma \in B(\Phi'_o) \quad t > 0$$

is an integer random measure. Next by Theorem 2.1.7 in Ito [2] $(\Phi', B(\Phi'))$ is a standard measurable space and so is $(\Phi'_o, B(\Phi'_o))$. Then by VIII (66.b) in Dellacherie and Meyer [1] there exists a compensator $\nu$ of $\mu$ such that (2.11) holds and

$$\nu(\omega; \{0\}; \Phi') = \nu(\omega; R^+; \{0\}) = 0$$  \hspace{1cm} (2.14)

$$\nu(\omega; \{t\}; \Phi') \leq 1 \quad \forall t > 0$$  \hspace{1cm} (2.15)

To show that the integral (2.12) exists we must show (see Theorem 2.1) in Kabanov, Lipcer and Shiryayev [4]) that for each $\phi \in \Phi$
\[ \int_0^t \int_{\Phi'} |\psi[\phi]|^2 \Lambda |\psi[\phi]| \nu(ds, d\psi) < \infty \quad \text{a.s.} \quad (2.16) \]

and that the integral (2.12) is a purely discontinuous local martingale. This is done in exactly the same way as in the real valued (see page 646 in [4]). The existence of the integral (2.13) is obvious.

Q.E.D.

Proofs of Theorems 1 and 2.

Let

\[ Z_t = \sum_{0 < s \leq t} \Delta X_s 1_{\{\Delta X_s \in B^c}\} \quad (2.17) \]

Then \( Z_t \) is a \( \Phi' \)-valued semimartingale and moreover it is a \( \Phi' \)-valued process of locally bounded variation. Next since \( B \) is a compact set in \( \Phi' \) (then closed and bounded) the \( \Phi' \)-valued semimartingale

\[ Y_t = X_t - Z_t \quad (2.18) \]

has bounded jumps on the strong topology of \( \Phi' \). Then Theorem 1 follows from Lemma 1. Theorem 2 follows using Lemmas 1 and 2.

Q.E.D.
Proof of Theorem 3.-

Once we have obtained Theorem 2 and Lemma 3 the proof of Theorem 1 is very similar to the real valued case (see [4]): From Theorem 2 it is enough to show that for each \( t > 0 \)

\[
M^d_t = \int_0^t \int_B \psi d(\mu - \nu)
\]

i.e.

\[
M^d_t[\phi] = \int_0^t \int_B \psi[\phi] d(\mu - \nu) \quad \forall \phi \in \Phi.
\]

But since both are purely discontinuous local martingales, it is enough to show that \( \forall \phi \in \Phi \) and \( t > 0 \)

\[
\Delta M^d_t[\phi] = \int_B \psi[\phi] d\mu (\{t\}; d\psi) - \int_B \psi[\phi] d\nu (\{t\}; d\psi) \quad (2.19)
\]

But from Theorem 2

\[
\Delta X_t[\phi] = \Delta A_t[\phi] + \Delta M^d_t[\phi] + \Delta X^c_t[\phi] 1_{\{\Delta X_t \in B^c\}}
\]

and

\[
\int_B \psi[\phi] d\mu (\{t\}; d\psi) = \Delta X_t[\phi] 1_{\{\Delta X_t \in B\}}.
\]
Then it is enough to show that $\psi_0 \in \Phi$ and $t > 0$

$$\Delta A_t[\phi] = \int_B \psi[\phi] d\nu(\{t\}, d\psi). \quad (2.20)$$

By Theorem 86 of Chapter IV in [1], it is enough to show that for each finite predictable time $\tau$

$$\Delta A_\tau[\phi] = \int_B \psi[\phi] d\nu(\{\tau\}, d\psi).$$

But the last expression holds since both processes are predictable, $\nu$ is the compensator measure of $\mu$ and

$$\int_B \psi[\phi] d\nu(\{\tau\}, d\psi) = E\left( \int_B \psi[\phi] \mu(\{\tau\}, d\psi) \mid F_\tau^- \right)$$

$$= E(\Delta X_\tau[\phi] \mathbf{1}_{\{\Delta X_\tau \in B\}} \mid F_\tau^-) = E(\Delta A_\tau[\phi] + \Delta M^d_\tau[\phi] \mid F_\tau^-)$$

$$= \Delta A_\tau[\phi] + E(\Delta M^d_\tau[\phi] \mid F_\tau^-) = \Delta A_\tau[\phi].$$

Hence (2.3) is established.

Q.E.D.

**Observations.** It is important to observe that the measure $\nu$ is the predictable compensator of the measure of the jumps and therefore it does not depend on the compact set $B$. Also the continuous martingale part of a semimartingale is unique and therefore does not depend on $B$. 
The only thing in the representation (2.3) that depends on \( B \) is the predictable process \( A_t \) which can be obviously modified if we change the compact set \( B \).

The representation (2.3) is a strong canonical decomposition of the \( \phi' \)-valued semimartingale \( X_t[\phi] \). Then for each \( \phi \in \Phi \) and \( t > 0 \) we have that

\[
X_t[\phi] = X_0[\phi] + A_t[\phi] + M^c_t[\phi] + \int_0^t \int_B \psi[\phi]d(\mu - \nu) + \int_0^t \psi[\phi]du . \tag{2.21}
\]

On the other hand the real valued semimartingale \( X_t[\phi] \) admits the (weak) canonical decomposition (see (43) in [4])

\[
X_t[\phi] = X_0[\phi] + A^\phi_t + M^{\phi,c}_t + \int_0^t \int_{|x| \leq 1} x d(\mu^\phi - \nu^\phi) + \int_0^t \int_{|x| > 1} x du^\phi . \tag{2.22}
\]

where

\[
\mu^\phi(\omega; (0, t]; \Gamma) = \sum_{0 < s \leq t} 1\{\Delta X_s[\phi](\omega) \in \Gamma\} \Gamma \in \mathcal{B}(\mathbb{R}^0), \mathbb{R}^0 = \mathbb{R} \setminus \{0\},
\]

\( \nu^\phi \) is the compensating measure of \( \mu^\phi \), \( A^\phi_t \) is a predictable real valued process with paths of finite variation and \( M^{\phi,c}_t \) is a continuous local martingale. Since the continuous part of a real valued semimartingale is unique then \( M^c_t[\phi] = M^{\phi,c}_t \psi \phi \in \Phi \) and clearly \( X^\phi_0 = X_0[\phi] \psi \phi \in \Phi \). Then from (2.21) and (2.22) we obtain the following equality for each \( \phi \in \Phi \) and \( t > 0 \).
\[
A_t[\phi] + \int_0^t \int_B \psi[\phi]d(\mu - \nu) + \int_0^t \int_{B^c} \psi[\phi]d\mu = \\
A^\phi_t + \int_0^t \int_{|x| \leq 1} x d(\mu_\phi - \nu_\phi) + \int_0^t \int_{|x| > 1} x d\mu_\phi. 
\] (2.23)

We hope that the decomposition (2.21) and equality (2.23) can be useful to prove functional central limit theorems for \( \Phi' \)-valued semimartingales. We hope to investigate such problem in the future.

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