CHARACTERIZATION OF OPTIMAL TRAJECTORIES

ON AN INFINITE HORIZON

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1. **Introduction**

We consider an infinite horizon problem of minimizing the expression
\[ \int_0^T L(x, \dot{x}) \, dt \] as \( T \) grows to infinity. In the case where \( L(\cdot, \cdot) \) is jointly convex the problem was studied by Brock and Haurie [1] and Leizarowitz [6], using the overtaking optimality notion. In [6] a certain differential inclusion \( \dot{x} \in G(x) \) was associated to the integrand \( L \) and optimality properties of the minimization problem were related to and studied in the light of dynamical properties of solutions of this differential inclusion. The convexity of \( L \) implied the simplification of \( x \to G(x) \) having a convex graph and then its dynamical properties really determined by some linear differential equation (see [8]).

In this work we consider a more general case where we only require \( v \to L(x, v) \) to be convex for every fixed \( x \). Again consideration of the overtaking optimality notion leads to examining a related differential inclusion, which reduces under a strict convexity assumption to an ordinary differential equation. Dynamical properties of this equation are related to optimality properties of the minimization problem: If all the \( \omega \)-limit sets are the same, independent of the initial condition, then there exists a weak overtaking optimal trajectory. If, moreover, the common \( \omega \)-limit set is a singleton, then there exists an overtaking optimal trajectory.

We want to emphasize that the interest in this work lies beyond the existence–uniqueness questions of optimal solutions. The optimal trajectories can, in certain situations, be computed as the solutions of a certain nth order o.d.e., and in general the solutions of this o.d.e. provide useful information about the behavior of the optimal trajectories. This o.d.e. is different in structure from the Euler–Lagrange equation for the integrand. For example, the latter has the order \( 2n \) while the one we consider is of the order \( n \). Moreover, the minimal cost growth can be explicitly computed in certain cases. For example, we'll show that in the scalar case it is given by \( \min_x L(x, 0) \).
An important tool in our analysis will be the consideration of Bellman's equation for the infinite horizon problem. We shall introduce the notion of "good solutions" for this equation, which depend on the state constraints imposed on \( x(\cdot) \). It will be shown that all the good solutions have the same constant in the right hand side of the Bellman equation (3.2), which is equal to the minimal time average integral. From this it will be clear how the geometry of the state constraint influences the minimal cost growth rate.

There are a number of applications and examples which demonstrate how our results can be used to compute optimal quantities (i.e., minimal cost growth rate and optimal trajectories). We consider the scalar case and compute explicit expressions for the minimal cost growth and for the differential equation which governs the behavior of the optimal trajectory. We also apply our results to study the optimization problem in \( \mathbb{R}^n \) in the case that \( L \) is in a separated form \( L(x, v) = g(v) + \ell(x) \).

The paper is organized as follows. In Section 2 we describe the framework and the assumptions. In Section 3 we discuss some properties of the Bellman equation for our problem. We also introduce the notion of a good solution to this equation.

In Section 4 we prove the existence of good solutions. It is then used in Section 5 to define a certain differential inclusion and the relation between its solutions and optimal trajectories is studied.

In Section 6 we study the scalar case and explicitly compute various optimality notions. In Section 7 we apply our results to compute expressions for the case where \( L(x, v) = g(v) + \ell(x) \) is in a separated form, \( x, v \in \mathbb{R}^n \).

2. The Framework

Let \( (x, v) \to L(x, v) \) be a continuous function defined on \( \mathbb{R}^n \times \mathbb{R}^n \) with values in \( (-\infty, +\infty] \). For every fixed \( x \) the function \( v \to L(x, v) \) is convex.

Definition 2.1. A function \( z(\cdot), z:[0, \infty) \to \mathbb{R}^n \) is called a trajectory of \( L(\cdot, \cdot) \) if it is absolutely continuous and if \( L(z(t), \dot{z}(t)) < \infty \) a.e. in \( [0, \infty) \).
To every trajectory \( z(\cdot) \) of \( L(\cdot, \cdot) \) we associate the cost function \( c(\cdot) \) defined by

\[
c(t) = \int_0^t L[z(s), z'(s)] \, ds.
\]

We shall use the following optimality notion, the **overtaking optimality criterion**. This concept was introduced and studied in the economic literature by Gale [4], Koopman [5], and von Weizsäcker [10], and in the control literature by Brock and Haurie [1].

**Definition 2.2.** The trajectory \( z^*(\cdot) \) is called an **overtaking optimal trajectory** of the integrand \( L(\cdot, \cdot) \) if the cost \( c^*(\cdot) \) corresponding to it has the property: Given a trajectory \( z(\cdot) \) with \( z(0) = z^*(0) \) and an \( \epsilon > 0 \), there is a time \( t_0 \) such that \( c^*(t) < c(t) + \epsilon \) for every \( t \geq t_0 \) where \( c(\cdot) \) is the cost corresponding to \( z(\cdot) \).

We shall consider also a weaker notion of overtaking optimality, defined as follows:

**Definition 2.3.** The trajectory \( z^*(\cdot) \) is called a **weak overtaking optimal trajectory** of the integrand \( L(\cdot, \cdot) \) if the cost \( c^*(\cdot) \) corresponding to it has the property: Given a trajectory \( z(\cdot) \) with \( z(0) = z^*(0) \) and an \( \epsilon > 0 \), there is an increasing sequence of times \( \{t_i\}_{i=1}^\infty \) such that \( c(t_i) < c(t) + \epsilon \) for every \( i \geq 1 \), where \( c(\cdot) \) is the cost corresponding to \( z(\cdot) \).

We denote by \( |\cdot| \) the Euclidean norm in \( \mathbb{R}^n \), and assume, throughout, the following.

**Assumption A.** The function \( (x, v) \mapsto L(x, v) \) is continuous, it is convex in \( v \) for every fixed \( x \), and has the following coercivity property

\[
L(x, v)/|v| \to \infty \quad \text{as} \quad |v| \to \infty
\]

for every \( x \in \mathbb{R}^n \).

Moreover, \( L \) satisfies the following Lipschitz condition: There is a constant \( c > 0 \) such that
\[ |L(x, v) - L(y, v)| \leq C|x - y|(1 + |v|) \]

for every \( x, y \in K \) and every \( v \in \mathbb{R}^n \). We also assume that there is a compact set \( K \subseteq \mathbb{R}^n \), the closure of an open and convex set in \( \mathbb{R}^n \), such that \( L(x, v) < \infty \) if and only if \( x \in K \). We call such a function \( L(\cdot, \cdot) \) an integrand.

The problem that we consider here is the existence and characterization of overtaking optimal trajectories for a given integrand. Certain infinite horizon control problems can be brought into the above framework by associating with them a suitable integrand. For details see, e.g., Brock and Haurie [1].

3. The Infinite Horizon Bellman Equation

The following is an heuristic discussion which comes to motivate the consideration of equation (3.2).

For the finite horizon problem of minimizing \( \int_0^T L(x, \dot{x}) \, dt \) the value function \( (x, t) \rightarrow \phi(x, t) \) satisfies the Bellman equation

\[
\begin{cases}
\frac{\partial \phi}{\partial t} + \min_v \left[ L(x, v) + v \cdot \nabla \phi(x, t) \right] = 0 \\
\phi(T, x) = 0
\end{cases}
\]  
(3.1)

(see, e.g., Fleming and Rishel [3], Theorem 4.1, p. 83). The function \( (x_0, t_0) \rightarrow \phi(x_0, t_0) \) is interpreted as the minimal value of \( \int_{t_0}^T L(x, \dot{x}) \, dt \) where \( x(\cdot) \) is subject to \( x(t_0) = x_0 \). In the infinite horizon case we expect the cost integral to grow in some minimal rate \( T \rightarrow \mu T \), but still expect the expressions

\[ \left[ \int_0^T L(x, \dot{x}) \, dt - \mu T \right] \]

to remain bounded as \( T \) varies in \([0, \infty)\), for the better trajectories \( x(\cdot) \). Thus replacing \( L(x, v) \) in (3.1) by \( [L(x, v) - \mu] \) would yield an equation for the excess cost, namely, the cost left after subtracting the linear part \( \mu T \) from the cost integral \( \int_0^T L(x, \dot{x}) \, dt \). Moreover, since the problem is
infinite horizon, $\phi$ is time-independent, the term $\partial \phi / \partial t$ should disappear, and we are led to consideration of the equation

$$\min_v \left[ L(x, v) + v \cdot \nabla p(x) \right] = \mu$$

for $x \in K$, $v \in \mathbb{R}^n$.

This is the Bellman equation for our infinite horizon problem. It is an equation both for the unknown function $p(\cdot)$ and for the scalar $\mu$ in the right hand side of (3.2). For a given $K \subset \mathbb{R}^n$, a pair $(p, \mu)$ can be given the control interpretation described above if it is a good solution, according to the following:

**Definition 3.1.** Let $z \mapsto p(z)$ be differentiable a.e. in $K$. The pair $(p, \mu)$ is a good solution of (3.2) if:

(i) Equality holds in (3.2) for almost every $x$ in $K$.

(ii) For every $x_0 \in K$ there is an absolutely continuous (a.c.) trajectory $x(\cdot)$ such that $x(0) = x_0$, $x(t) \in K$ for all $t \geqslant 0$ and

$$L(x(t), x'(t)) + x'(t) \cdot \nabla p(x(t)) = \mu$$

for a.e. $t$ in $[0, \infty)$. Such a trajectory will be termed a viable trajectory for $L$ in $K$.

We define the set-valued function $z \mapsto G(z)$ as

$$G(z) = \left\{ v \in \mathbb{R}^n : L(z, v) + v \cdot \nabla p(z) = \mu \right\}$$

whenever $\nabla p(z)$ is defined. Then the condition (ii) in Definition 3.1 can be phrased as:

For every $z_0 \in K$ there exists an absolutely continuous trajectory $z(\cdot)$ such that $z(0) = z_0$, $z(t) \in K$ for all $t \geqslant 0$, and

$$z'(t) \in G(z(t)) \quad \text{for a.e.} \quad t \geqslant 0$$

$$z(0) = z_0 \quad .$$

If $G(z)$ is a singleton, as in the case where $v \mapsto L(x, v)$ is strictly convex, then $z(\cdot)$ satisfies the o.d.e. (3.4).
If \((p, \mu)\) is a good solution of (3.2) and \(z_0 \in K\), let \(z(\cdot)\) be a viable trajectory. Then we have that a.e. in \([0, \infty)\)

\[
L(z(t), \dot{z}(t)) = \dot{z}(t) \cdot \nabla p(z(t)) + \mu
\]

which implies that

\[
\int_0^T L(z, \dot{z}) \, dt = \mu T + p(z_0) - p(z(T)) .
\tag{3.5}
\]

Since \(p(\cdot)\) is bounded on \(K\) it follows from (3.5) that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T L(z(t), \dot{z}(t)) \, dt = \mu .
\tag{3.6}
\]

On the other hand, for every a.c. trajectory \(x(\cdot)\) satisfying \(x(0) = x_0, x(t) \in K\), it can be shown that for almost every \(t \geq 0\) the following holds (see the proof of (4.11)):

\[
L(x(t), \dot{x}(t)) \geq \mu - \dot{x}(t) \nabla p(x(t))
\]

implying that

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T L(x, \dot{x}) \, dt \geq \mu .
\tag{3.7}
\]

We thus conclude from (3.6) and (3.7) the following:

**Theorem 3.2.** If \((p_i, \mu_i), \ i = 1, 2, \) are good solutions of (3.2), then \(\mu_1 = \mu_2\), namely there is uniqueness in the scalar part of the good solution. This unique value \(\mu\) is the minimal cost growth, or the minimal time average integral.

**Example 3.3.** Let \(K = [-1, 1] \subset \mathbb{R}\) and \(L(x, v) = \frac{1}{2} (v - f(x))^2\) where \(f(\cdot)\) is a continuous scalar function on \([-1, 1]\). The Bellman equation is then

\[
\min_v \left[ \frac{1}{2} (v - f(x))^2 + vp'(x) \right] = \mu
\tag{3.8}
\]

which is equivalent to
\[- \frac{1}{2} p'(x)^2 + p'(x)f(x) = \mu \]

or
\[p'(x) = -f(x) \pm \left[ f^2(x) - 2\mu \right]^{1/2}.

For every \(x\) the minimization in (3.9) is attained at \(v = f(x) - p'(x)\), thus we are looking for trajectories \(z(\cdot)\) which satisfy \(\dot{z}(t) = f(z) - p'(z)\) and \(|z(t)| \leq 1\) for all \(t \geq 0\). Then we must have
\[
(3.9) \quad 2\mu \leq \min_{|x| \leq 1} f^2(x).
\]

Assume that the minimum of \(f^2(\cdot)\) is attained at a unique point \(-1 < z_0 < 1\). If a strict inequality holds in (3.9) then, we claim, we cannot have continuous viable trajectory which will satisfy condition (ii) of Definition (3.1). Indeed, in order that \(z(\cdot)\) will be viable, we then must choose
\[
z(t) = \begin{cases} 
\sqrt{f^2(z) - 2\mu} & \text{if } z(t) > z_0 \\
-\sqrt{f^2(z) - 2\mu} & \text{if } z(t) < z_0
\end{cases}.
\]

Since \(z(\cdot)\) can't stay in \([-1, 1]\) on either side of \(z_0\) for infinite time interval, it must jump from one side of \(z_0\) to the other, thus \(z(\cdot)\) can't be continuous. Thus in this example a good solution is given by
\[
\mu = \min_{-1 \leq x \leq 1} f^2(x)
\]
\[
p'(x) = \begin{cases} 
f(x) + \sqrt{f(x)^2 - 2\mu} & z_0 < x \leq 1 \\
 f(x) - \sqrt{f(x)^2 - 2\mu} & -1 \leq x < z_0
\end{cases}.
\]

By Theorem 3.2 \(\mu\) is the minimal cost growth. For every \(x_0\) the trajectory \(x(\cdot)\) which satisfies
\[
\dot{x}(t) = \sqrt{f^2(x) - 2\mu} \cdot \text{sgn}(z_0 - x_0), \quad x(0) = x_0
\]
is such that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} L(x(t), \dot{x}(t)) \, dt = \mu.
\]

In fact, this trajectory is even overtaking optimal, as will follow from Theorem 5.3. Clearly, and not surprisingly, \( z(t) \to z_0 \) as \( t \to \infty \), for every initial condition \( x_0 \).

4. **Existence of Good Solutions**

In the previous section we defined the notion of a good solution of the Bellman equation with respect to the constraint set \( K \). In this section we shall prove that this definition is not vacuous, and there indeed exist good solutions. The method of proof is by defining a finite horizon problem with the same cost as the infinite horizon one and applying known results for finite horizon control problems.

**Theorem 4.1.** Let \( L(\cdot, \cdot) \) be an integrand for the constraint set \( K \subset \mathbb{R}^n \). Then the Bellman equation (3.2) has a good solution \( (p(\cdot), \mu) \).

To prove this we shall need the following results.

**Proposition 4.2.** Under Assumption A there exist constants \( M \) and \( \mu \) such that

\[
(4.1) \quad \int_{0}^{T} \left[ L(z, \dot{z}) - \mu \right] \, dt \geq -M \quad \text{for all } T > 0 \text{ and all trajectories } z(\cdot),
\]

and there is a trajectory \( x(\cdot) \) so that

\[
(4.2) \quad \int_{0}^{T} \left| L(x, \dot{x}) - \mu \right| \, dt \leq M \quad \text{for all } T > 0.
\]

**Proof.** We shall prove first that (4.1) and (4.2) hold for every \( T \) integer and then the result will follow. To this end we define the function \( m : K \times K \to \mathbb{R}^1 \)

\[
(4.3) \quad m(x, y) = \inf \left\{ \int_{0}^{1} L(z, \dot{z}) \, dt : z(0) = x, \ z(1) = y \right\}
\]
where the infimum is over all the trajectories \( z(\cdot) \) satisfying \( z(0) = x, \ z(1) = y. \) The conditions in Assumption A guarantee that the inf in (4.3) is attained by some trajectory (see Fleming and Rishel \([3]\), Theorem 4.1, p. 68). Also, the fact that

\( \mathbb{K} \) is the closure of an open and convex set implies that \( m(x, y) \) is finite for every \( x, y \in \mathbb{K} \). Moreover, \( m(\cdot, \cdot) \) is continuous on \( \mathbb{K} \times \mathbb{K} \) (see Proposition 4.2).

Now consider, for every sequence \( \{x_i\}_{i=0}^\infty \subset \mathbb{K} \), the cost expressions

\[
C_N(\bar{x}) = \sum_{i=1}^{N-1} m(x_i, x_{i+1})
\]

where we denote \( \bar{x} = \{x_i\}_{i=0}^\infty \) and consider (4.4) as \( N \to \infty \). A study of such an infinite horizon discrete time control system is displayed in Leizarowitz \([7]\), and it follows from Theorem 3.1 there that there exist constants \( \mu \) and \( M_0 \) such that

\[
\sum_{i=0}^{N-1} m(x_i, x_{i+1}) - \mu N \geq -M_0 \quad \text{for all } N
\]

and every sequence \( \{x_i\}_{i=0}^\infty \), and there is a sequence \( \{x_i^*\}_{i=0}^\infty \) so that

\[
\left| \sum_{i=0}^{N-1} m(x_i^*, x_{i+1}^*) - \mu N \right| \leq M_0 \quad \text{for all } N \geq 0.
\]

Clearly this means that (4.1) and (4.2) hold if we consider just \( T \) natural.

Now it follows from Assumption A that there is a constant \( \alpha \) such that

\[
\int_0^\delta L(x(t), \dot{x}(t)) \, dt > \alpha
\]

for every \( 0 < \delta \leq 1 \) and every trajectory \( x(\cdot) \) on \([0, \delta]\). This implies that (4.1) and (4.2) hold for every \( T \) with

\[
M = M_0 + |\alpha| + |\mu|.
\]
As a trajectory $x(\cdot)$ in (4.2) we take a trajectory such that

$$x(k) = x^*_k \quad \text{for all } k \geq 0$$

and in the interval $[k, k+1]$ it satisfies the equality

$$m(x^*_k, x^*_{k+1}) = \int_k^{k+1} L(x(t), \dot{x}(t)) \, dt.$$ 

This concludes the proof of the proposition. □

Once we have the validity of (4.1) and (4.2) we can go on to define the function $p : K \to \mathbb{R}^1$

(4.7) \quad p(y) = \inf_{x(\cdot) \to y} \lim_{T \to \infty} \int_0^T \left[ L(x(t), \dot{x}(t)) - \mu \right] \, dt.

By Proposition 4.1 it is well defined. We shall need the following two results from [9] in the sequel.

**Proposition 4.2.** The function $m(\cdot, \cdot)$ is Lipschitz continuous on $K \times K$.

**Proof.** See [9], Theorem 5.6.3.

**Proposition 4.3.** Let $\mathcal{F}$ be the set of all trajectories $z(\cdot)$ on $[0,1]$ such that

$$\int_0^1 L(z, \dot{z}) \, dt = m(x, y)$$

for some $x, y \in K$. Then there is a constant $B > 0$ such that

$$|\dot{z}(t)| \leq B$$

for every $z(\cdot) \in \mathcal{F}$ and a.e. $t \in [0,1]$.

**Proof.** See [9], Lemma 5.6.2.

**Proposition 4.4.** The function $p(\cdot)$ is Lipschitz continuous on $K$. 

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Proof: Let $m : \mathbb{R}^2 \to \mathbb{R}$ be as in (4.3). Then by Proposition 4.2 it is Lipschitz continuous in $\mathbb{R}^2$, namely there is a constant $k > 0$ such that

$$
\left| m(x_1, y_1) - m(x_2, y_2) \right| \leq k \left[ |x_1 - x_2| + |y_1 - y_2| \right]
$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$.

Now let $x \in \mathbb{R}^2$ and let $x(\cdot)$ be such that $x(0) = x$ and

$$
\liminf_{T \to \infty} \int_0^T \left[ L(x(t), \dot{x}(t)) - \mu \right] dt
$$

approximates $p(x)$ up to $\epsilon > 0$. For $x' \in \mathbb{R}^2$ we take a trajectory $z(\cdot)$ such that

$$
\int_0^1 L(z(t), \dot{z}(t)) dt = m(x', x(1))
$$

and $z(t) = x(t)$ for $t \geq 1$. Then it follows from (4.8) that

$$
p(x') \leq p(x) + k|x' - x| + \epsilon.
$$

This being true for every $\epsilon > 0$ and for every $x$ and $x'$ implies that

$$
\left| p(x') - p(x) \right| \leq k|x - x'|
$$

for all $x, x' \in \mathbb{R}^2$, proving the Lipschitz continuity of $p(\cdot)$ in $\mathbb{R}^2$. \hfill \Box

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Let $T > 0$ and consider the following minimization problem

$$
(4.9) \quad \min_{x(\cdot)} \left\{ \int_0^T L(x(t), \dot{x}(t)) dt + p(x(T)) - \mu T : x(0) = y \right\}
$$

It follows from the definition of $p(\cdot)$ in (4.7) that the minimal value of the minimization problem (4.9) is $p(y)$, and this is true for every $T > 0$. It follows now from standard results about the value function (see Fleming and Rishel [3], Theorem 4.1, p. 83) that at all points where the value function is differentiable it satisfies
the equality

\[ \frac{\partial p}{\partial t} + \min_v \left[ L(x, v) - \mu + \nabla p \cdot v \right] = 0 \]

By Proposition 4.4, \( p(\cdot) \) is Lipschitz continuous, hence by Rademacher's Theorem (see Federer [2], p. 216) it is differentiable almost everywhere in \( K \). Also, \( p(\cdot) \) does not depend on the time, thus the last equality implies that \( p(\cdot) \) satisfies (3.2) a.e. in \( K \). This proves the validity of (i) in Definition 3.1.

Now let \( y \in K \) and let \( y(\cdot) \) be a trajectory which minimizes

\[ (4.10) \quad \int_0^1 \left[ L(x(t), x(t)) - \mu \right] dt + p(x(1)) \]

subject to \( x(0) = y \). Then the value of (4.10) when \( y(\cdot) \) is substituted for \( x(\cdot) \) is \( p(y) \). Let \( Q_0 \subset K \) be the set where equality in (3.2) holds, and we know that the Lebesgue measure of \( K \setminus Q_0 \) is zero. Let

\[ A = \left\{ t \in [0, 1] : y(t) \in Q_0 \right\} \]

\[ B = \left\{ t \in [0, 1] : y(t) \in K \setminus Q_0 \right\} . \]

If \( t \in A \) then clearly by (3.2)

\[ (4.11) \quad L(y(t), \dot{y}(t)) + \dot{y}(t) \cdot \nabla p(y(t)) - \mu \geq 0 . \]

If, however, \( t \in B \), then the left hand side of (4.11) may be negative. We claim that the set of points in \( B \) for which the left hand side of (4.11) is negative is of a Lebesgue measure zero. Assuming that it is of a positive measure, then for some \( \alpha > 0 \) also the set

\[ (4.12) \quad C = \left\{ t \in B : L(y(t), \dot{y}(t)) + \dot{y}(t) \cdot \nabla p(y(t)) - \mu \leq -\alpha \right\} \]

is of a positive Lebesgue measure. Then \( C \) has a Lebesgue point \( 0 \leq t_0 < 1 \) and we consider the integral.
for $\delta > 0$ sufficiently small. Since it can by assumed, by Proposition 4.3, that $L(y(t), \dot{y}(t))$ is bounded from above, it follows from (4.12) and the fact that $t_0$ is a Lebesgue point of $C$, that the expression in (4.13) is less than $p(y(t_0')) - \frac{1}{2}a\delta$, for $\delta > 0$ sufficiently small. But we know that $p(y(t_0'))$ is the minimal possible value for the expression in (4.13), which is a contradiction. Thus we conclude that (4.11) holds for almost every $t \in [0, 1]$.

Let $D$ be the set

$$D = \left\{ t \in A : L(y(t), \dot{y}(t)) + \dot{y}(t) \cdot \nabla p(y(t)) = \mu > 0 \right\}$$

and we claim that it has a Lebesgue measure zero. Otherwise there would be a $\beta > 0$ so that

$$E = \left\{ t \in D : L(y(t), y(t)) + y(t) \cdot \nabla p(y(t)) - \mu \geq \beta \right\}$$

is of a positive Lebesgue measure, hence $E$ has a Lebesgue point $0 \leq \tau < 1$. Arguing as above, there is an interval $[\tau, \tau + \eta]$ for $\eta$ sufficiently small such that

$$(4.14) \quad \int_{\tau}^{\tau+\eta} \left[ L(y(t), \dot{y}(t)) - \mu \right] dt + \int p(y(\tau + \eta)) \geq \frac{1}{2} \eta\beta + p(y(\eta)) .$$

But this is a contradiction since $y(t)$ must be optimal on the interval $[\tau, \tau + \eta]$ and hence the value of the left hand side of (4.14) is $p(y(\tau))$. This proves that $D$ is of a Lebesgue measure zero. The conclusion of the discussion up to now is that

$$L(y(t), \dot{y}(t)) + \dot{y}(t) \nabla p(y(t)) - \mu = 0$$

for almost every $t$ in $[0, 1]$. We can now repeat the argument with $y(1)$ replacing $y(0) = y$, and inductively with $y(k)$ replacing $y(0) = y$ for every $k \geq 1$. This proves that $y(t)$ is a trajectory as required in part (ii) of Definition 3.2, and concludes the proof of Theorem 4.1. \qed
We have proved in the course of the proof of Theorem 4.1 that for every viable trajectory $y(\cdot)$ of $L(\cdot, \cdot)$ in $K$ the inequality (4.11) must hold a.e. in $[0, \infty)$. Since the derivative $\nabla p(\cdot)$ exists only a.e. in $K$, say on the set $Q_0 \subset K$, this means that $y(t) \in Q_0$ for almost every $t \geq 0$. We thus have the following consequence of the proof of Theorem 4.1:

**Corollary 4.5.** Let $x(\cdot)$ be a viable trajectory of $L(\cdot, \cdot)$ in $K$. Then $x(t) \in Q_0$ for almost every $t \geq 0$, and then

$$L(x(t), \dot{x}(t)) + \dot{x}(t) \nabla p(x(t)) \geq \mu$$

for almost every $t \geq 0$.

5. A Related Differential Inclusion

We observed during the proof of Theorem 4.1 that every viable trajectory of $L(\cdot, \cdot)$ in $K$ is such that inequality (4.11) must hold on $[0, \infty)$. Our claim is that viable trajectories for which equality in (4.11) holds for a.e. $t \geq 0$ are of a special importance from the optimality point of view. We recall the definition of the set valued function $z \mapsto G(z)$ in (3.3) and consider the differential inclusion

\[ (5.1) \quad \dot{x}(t) \in G(x(t)) \quad \text{for a.e. } t \geq 0 \]

\[ x(0) = x_0 \]

By Corollary 4.5 there really exists a trajectory $x(\cdot)$ satisfying (5.1) for every initial condition.

Let $x(\cdot)$ be a viable solution of $L$ in $K$ and assume that

\[ (5.2) \quad \int_0^\infty \left[ L(x(t), \dot{x}(t)) + \dot{x}(t) \nabla p(x(t)) - \mu \right] dt < \infty. \]

Let $[a_j, b_j]$ be a sequence of time intervals, $b_j - a_j = \ell$ for some fixed positive $\ell$, suppose that $a_j \to \infty$ as $j \to \infty$, and let $x_j(\cdot)$ be defined on $[0, \ell]$ by

$$x_j(t) = x(a_j + t).$$
Proposition 5.1. Let $x(\cdot)$ be a viable trajectory of $L(\cdot, \cdot)$ in $K$ and let $[a_j, b_j]$ and $x_j(\cdot)$ be as above. Suppose further that (5.2) holds. Then there is a function $y : [0, \ell] \rightarrow \mathbb{R}^n$ such that $y(t) \in K$ for all $0 \leq t \leq \ell$, $y(t) \in G(y(t))$ for a.e. $t \in [0, \ell]$, and there is a sequence $\{j_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\lim_{k \rightarrow \infty} \max_{0 \leq t \leq \ell} |x_{j_k}(t) - y(t)| = 0 .$$

Proof: Since Assumption A holds, the conditions for existence of optimal trajectories on finite intervals are satisfied. Thus there is a subsequence of the natural numbers, $\{j_k\}_{k=1}^{\infty}$, and a function $y(\cdot)$ for which (5.3) holds and also

$$\int_0^\ell L(y(t), \dot{y}(t)) dt \leq \liminf_{k \rightarrow \infty} \int_0^\ell L(x_{j_k}(t), \dot{x}_{j_k}(t)) dt .$$

But since (5.2) is true for $x(\cdot)$ we must have

$$\int_0^\ell \left[ L(x_{j_k}, \dot{x}_{j_k}) - \mu + \dot{x}_{j_k} \cdot \nabla p(x_{j_k}) \right] dt \rightarrow 0$$

as $k \rightarrow \infty$. But then it follows from (5.4) that

$$\int_0^\ell \left[ L(y, \dot{y}) + \dot{y}(t) \nabla p(y(t)) - \mu \right] dt = 0 .$$

Since the integrand in the last integral is nonnegative we conclude that $y(\cdot)$ satisfies the differential inclusion (5.1) a.e. in $[0, \ell]$, concluding the proof of the Proposition. \(\square\)

Let us consider dynamical properties of the relation (5.1). For a solution $x(\cdot)$ of (5.1) we denote, as usual, by $\omega(x_0)$ the set of all points $z \in K$ such that $x(t_j) \rightarrow z$ for some sequence of times $t_j \rightarrow \infty$.

Theorem 5.2. Suppose that there is a set $S \subset K$ such that

$$\omega(x_0) = S \quad \text{for all } x_0 \in K .$$

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Then there is a weak overtaking optimal trajectory for the infinite horizon minimization problem for \( L \) in \( K \), for every initial condition \( x_0 \in K \). It is given by any viable solution of the differential inclusion (5.1).

**Proof:** Let \( x(\cdot) \) be a viable trajectory for \( L \) in \( K \), \( x(0) = x_0 \). Then we have from (4.11) that

\[
(5.5) \quad \int_0^T \left[ L(x, \dot{x}) - \mu \right] dt \geq p(x_0) - p(\bar{x}(T)).
\]

If

\[
\lim \sup_{T \to \infty} \int_0^T \left[ L(x, \dot{x}) - \mu \right] dt < \infty
\]

then it follows from Proposition 5.1 that there exists a viable solution \( y(\cdot) \) of (5.1) defined on \([0, \infty)\) so that for every interval \([0, \ell]\) there is a sequence of intervals \( I_j = [a_j, b_j] \) with \( a_j \to \infty \) such that

\[
(5.6) \quad \max_{0 \leq t \leq \ell} |x(a_j + t) - y(t)| \to 0 \quad \text{as} \quad j \to \infty.
\]

This \( y(\cdot) \) is constructed using Proposition 5.1 and a diagonal argument.

It follows from our assumptions that for every \( z \in S \) and every \( \epsilon > 0 \) there is some \( t_0 \geq 0 \) so that \(|y(t_0) - z| < \epsilon\). This, with (5.6), implies that for every \( z \in S \) there is a sequence of times \( \tau_k \to \infty \) so that

\[
(5.7) \quad x(\tau_k) \to z \quad \text{as} \quad k \to \infty.
\]

The function \( p(\cdot) \) is continuous on \( K \) and attains its minimal value on the compact set \( S \), say at \( z_0 \in S \), namely

\[
(5.8) \quad p(z) \geq p(z_0) \quad \text{for every} \quad z \in S.
\]

Let \( x(\cdot) \) be any viable trajectory of \( L(\cdot, \cdot) \) in \( K \), \( x(0) = x_0 \), and let \( y(\cdot) \) be a viable solution of (5.1), \( y(0) = x_0 \). If
\[
\lim_{T \to \infty} \int_0^T \left[ L(x(t), \dot{x}(t)) - \mu \right] dt = \infty
\]
then we have, from
\[
\int_0^T \left[ L(y(t), \dot{y}(t)) - \mu \right] dt = p(x_0) - p(y(T))
\]
that
\[
\int_0^T L(x, \dot{x}) dt - \int_0^T L(y, \dot{y}) dt \to \infty \quad \text{as } T \to \infty .
\]
(5.9)

If, on the other hand,
\[
\limsup_{T \to \infty} \int_0^T \left[ L(x, \dot{x}) - \mu \right] dt < \infty ,
\]
then we have, by (5.7), a sequence of times \( \tau_k \to \infty \) such that
\[
(5.10) \quad x(\tau_k) \to z_0 \quad \text{as } k \to \infty .
\]
Therefore we have, from (5.5)
\[
(5.11) \quad \int_0^{\tau_k} \left[ L(x, \dot{x}) - \mu \right] dt \geq p(x_0) - p(x(\tau_k))
\]
while \( y(\cdot) \) satisfies, by (5.1),
\[
(5.12) \quad \int_0^{\tau_k} \left[ L(y, \dot{y}) - \mu \right] dt = p(x_0) - p(y(\tau_k)) .
\]

As \( k \to \infty \), \( y(\tau_k) \) converges to \( S \), therefore for a given \( \epsilon > 0 \) we have from
(5.8) and (5.10) that
\[ p(\dot{y}(\tau_k)) \geq p(\dot{x}(\tau_k)) - \varepsilon \quad \text{for all large } k. \]

This, however, implies by using (5.11) and (5.12), that

\[ \varepsilon + \int_0^{\tau_k} L(x, \dot{x}) \, dt \geq \int_0^{\tau_k} L(y, \dot{y}) \, dt \]

for all sufficiently large \( k \). This, combined with (5.9), concludes the proof of the weak overtaking optimality of \( y(\cdot) \).

\[ \square \]

In order to conclude that a solution of (5.1) is in fact overtaking optimal, we need to impose a stronger assumption on its asymptotic dynamics.

**Theorem 5.3.** Suppose that there is a point \( s \in K \) so that

\[ \lim_{t \to \infty} x(t) = s \]

for every viable solution \( x(\cdot) \) of (5.1). Moreover, the limit in (5.13) has a uniform rate in the sense that given an \( \varepsilon > 0 \) then there is a \( t_0 \) so that \( |x(t) - s| < \varepsilon \) for all \( t > t_0 \) and all viable solutions \( x(\cdot) \) of (5.1). Then there is an overtaking optimal trajectory for the infinite horizon minimization problem for \( L \) in \( K \). Such a trajectory is given by a viable solution of (5.1) with initial value \( x_0 \).

**Proof:** Let \( x(\cdot), x(0) = x_0 \), be a viable trajectory for \( L \) in \( K \). Let also \( y(\cdot) \) be a viable solution of (5.1) satisfying \( y(0) = x_0 \). Comparing the corresponding costs \( c(\cdot) \) and \( c^*(\cdot) \) of \( x(\cdot) \) and \( y(\cdot) \) respectively in light of the overtaking criterion, it is enough to consider the case

\[ \limsup_{T \to \infty} \int_0^T \left[ L(x, \dot{x}) - \mu \right] \, dt < \infty. \]

Then it follows from Proposition (5.1) and the uniform rate of convergence of solutions of (5.1) to \( s \) that also

\[ x(t) \to s \quad \text{as } t \to \infty. \]
Since $x(\cdot)$ is viable on $[0, \infty)$ we have

$$\int_0^T \left[ L(x, \dot{x}) - \mu \right] dt \geq p(x_0) - p(x(T))$$

(5.14)

while for $y(\cdot)$ the following holds:

$$\int_0^T \left[ L(y, \dot{y}) - \mu \right] dt = p(x_0) - p(y(T)).$$

(5.15)

Using the convergence of $x(T)$ and $y(T)$ to $s$, the continuity of $p(\cdot)$, (5.14) and (5.15) we conclude that given an $\epsilon > 0$ then

$$\epsilon + \int_0^T L(x, \dot{x}) dt > \int_0^T L(y, \dot{y}) dt$$

for all large enough $T$, proving the overtaking optimality of $y(\cdot)$. \qed

**A remark.** A sufficient condition which guarantees that the uniform rate limit (5.13) exists in the case where $L(\cdot, \cdot)$ is jointly convex in $(x, v)$ is given in Leizarowitz [8].

**A remark.** In the case where $L(x, \cdot)$ is strictly convex as a function of $v$ for every fixed $x$, the differential inclusion (5.1) is in fact a differential equation. If, further, it is sufficiently regular, then it has a unique viable solution through every initial value in $K$. Asymptotic dynamic properties of this differential equation are then related to optimality properties of the control problem, as asserted in Theorems 5.2 and 5.3.

6. **An Application to the Scalar Case**

In this section we consider the scalar case $n = 1$. $L(\cdot, \cdot)$ is defined on $[a, b] \times \mathbb{R}^1$ where $[a, b] \subset \mathbb{R}^1$ is a finite interval in which $x(\cdot)$ is constrained to take values. In this case we can give explicit and rather simple expressions for $\mu$ and $p(\cdot)$. In fact, we have
\[(6.1) \quad \mu = \min_{a \leq x \leq b} L(x, 0) \]

while \(dp(x)/dx\) is the negative value of the slope of the tangent line to the graph of \(L(x, \cdot)\) which goes through the point \((0, \mu)\). Once explicit expressions for \(\mu\) and \(dp/dx\) are known, the set valued function \(x \mapsto G(x)\) is also known and can be used to compute the optimal trajectories.

We are looking for a good solution of the Bellman equation

\[(6.2) \quad \min_v \left[ L(x, v) + p'(x)v \right] = \mu . \]

A solution \((p, \mu)\) of (6.2) will be a good solution if the following will hold:

(i) \(p(\cdot)\) is a \(C^1\) function on \([a, b]\).

(ii) Denote by \(g(x)\) the set of points \(v \in \mathbb{R}^1\) for which the minimum in the left hand side of (6.2) is attained. Then there is a \(\delta > 0\) such that

\[
a \leq x \leq a + \delta \quad \text{and} \quad v \in g(x) \text{ imply } v > 0 ,
\]

\[
b - \delta < x \leq b \quad \text{and} \quad v \in g(x) \text{ imply } v < 0 .
\]

This condition guarantees the viability of solutions of \(\dot{x}(t) \in g(x(t))\).

It follows from (6.2) that \(L(x, v) + p'(x)v \geq \mu\) for every \(v\), in particular \(\mu \leq L(x, 0)\). Therefore \(\mu\) must satisfy

\[(6.3) \quad \mu \leq \min_{a \leq x \leq b} L(x, 0). \]

We claim that a strict inequality in (6.3) can't hold, if we wish \(p'(x)\) to be continuous. To see this, observe that equation (6.2) is equivalent to saying that \(-p'(x)\) is the slope of a tangent to the graph of \(v \rightarrow L(x, \cdot)\) which goes through the point \((0, \mu)\). If a strict inequality holds in (6.3), then for every \(x \in [a, b]\) there are two such tangent lines, one of a slope \(\ell_+(x)\) and the other of a slope \(\ell_-(x)\), with \(\ell_+(x) > \ell_-(x)\).

Then in order to have a continuous function \(x \mapsto p'(x)\) we must choose the tangent
lines in such a way that either all of them are of a slope $\ell_+(x)$ or all of them are of a slope $\ell_-(x)$ for all $x \in [a, b]$. But then we will have either $g(x) \subset (\alpha, \infty)$ for all $x \in [a, b]$, or $g(x) \subset (-\infty, -\alpha]$ for all $x \in [a, b]$, for some $\alpha > 0$. Clearly, there are no viable solutions for $\dot{x} \in g(x(t))$ in $[a, b]$ in either case.

We shall show now that there is a good solution for (6.2) with $\mu$ as in (6.1). This will prove our assertion regarding strict inequality in (6.3) since there is uniqueness in $\mu$ for good solutions. Then let $\mu$ be given by (6.1) and assume first that the minimum in (6.1) is attained at a unique point $a < c < b$. For every $x \in [a, b]$, $x \neq c$, there are two tangent lines for $v \rightarrow L(x, v)$ that go through the point $(0, \mu)$. We again denote their slopes by $\ell_+(x)$ and $\ell_-(x)$. We define $\ell : [a, b] \rightarrow \mathbb{R}^1$ as follows:

$$
\ell(x) = \begin{cases} 
\ell_+(x) & \text{if } a \leq x < c \\
\text{a slope of a tangent to } L(c, \cdot) \text{ on } v = 0 & \text{if } x = c \\
\ell_-(x) & \text{if } c < x \leq b 
\end{cases}
$$

It is easy to see that $\ell(\cdot)$ is continuous in $[a, c)$ and in $(c, b]$. Moreover, let $g(x)$ be the points $v \in \mathbb{R}^1$ where the following minimum is attained:

$$
(6.4) \quad \min_v \left[ L(x, v) - \ell(x)v \right].
$$

Then the set valued function $x \rightarrow g(x)$ is upper semicontinuous for every $x \in [a, b]$. This is clear for $x \neq c$, since then $\ell(\cdot)$ is continuous. We prove upper semicontinuity at $x = c$: Let $x_j \rightarrow c$, $v_j \in g(x_j)$, $v_j \rightarrow v$ and suppose that $v > 0$ and also $v \notin g(c)$. Then

$$
(6.5) \quad L(c, v/2) < \frac{1}{2} \left[ \mu + L(c, v) \right].
$$

Also for every $j$ we have

$$
L(x_j, v_j/2) \geq \mu + \ell_j \cdot \frac{v_j}{2} = \frac{1}{2} \left[ \mu + L(x_j, v_j) \right].
$$

Thus, letting $j \rightarrow \infty$ in $L(x_j, v_j/2) \geq \frac{1}{2} \left[ \mu + L(x_j, v_j) \right]$ and using the continuity of $L$ leads into a contradiction to (6.5). This proves the upper semicontinuity of $x \rightarrow g(x)$ in $[a, b]$. 

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The coercivity of \( v \mapsto \mathbf{L}(x, v) \) as assumed in Assumption A implies that there is an interval \([\alpha, \beta] \subset \mathbb{R}^1\) such that \( g(x) \subset [\alpha, \beta] \) for every \( x \in [a, b] \). This, with the upper semicontinuity of \( g(\cdot) \) guarantees existence of solutions of \( \dot{x}(t) \in g(x(t)) \) by standard results about differential inclusions. The fact that

\[
v \in g(a) \implies v > 0
\]

and

\[
v \in g(b) \implies v < 0
\]

then guarantees that this solution is viable in \([a, b]\) for every initial value \( x_0 \in [a, b] \).

The discussion above still holds if the minimum in (6.1) is obtained in more than one point. We then choose one such point \( c \) and proceed as before. This holds true also when \( c \) is a boundary point of \([a, b]\). We have thus proved

**Theorem 6.1.** Let \( \mu \) be as in (6.1) and \( g(x) \) the set of values \( v \in \mathbb{R}^1 \) for which the minimum in (6.4) is achieved. Then \( \mu \) is the minimal cost growth of trajectories constrained to \([a, b]\). Moreover, any viable solution of

\[
\dot{x}(t) \in g(x(t)) , \quad x(0) = x_0
\]

is a weak overtaking optimal trajectory. Such a solution exists for every \( x_0 \in [a, b] \).

A stronger result is obtained when the minimum in (6.1) is unique.

**Theorem 6.2.** Let \( \mu \) and \( g(\cdot) \) be as above and suppose that \( \mu \) is achieved in (6.1) at a unique point \( c \in [a, b] \). Then any solution of \( \dot{x}(t) \in g(x(t)), \ x(0) = x_0 \) is an overtaking optimal trajectory. Such a solution exists for every initial value \( x_0 \) in \([a, b]\).

**Proof:** The result will follow from Theorem 5.3 once we prove that \( x(t) \rightarrow c \) as \( t \rightarrow \infty \) in a uniform rate for all the solutions of \( \dot{x}(t) \in g(x(t)), \ x(0) = x_0 \). But this follows from the fact that for every \( \delta > 0 \) there is an \( \eta > 0 \) such that

\[
x \geq c + \delta \quad \text{and} \quad v \in g(x) \implies v \leq -\eta ,
\]

and also for every \( \delta > 0 \) there is a \( \nu > 0 \) such that

\[
x \leq c - \delta \quad \text{and} \quad v \in g(x) \implies v \geq \nu .
\]
7. Applications to Control Systems in $\mathbb{R}^n$

In this section we further demonstrate how our results can be used to compute optimal quantities.

**Example 7.1.** We consider the following minimization of

$$
(7.1) \quad \int_0^T \left[ \frac{1}{2} |x(t)|^2 + f(x(t)) \right] dt , \quad T \to \infty
$$

where $x(\cdot)$ is an a.c. trajectory in $K \subset \mathbb{R}^n$, $f(\cdot)$ is a continuous function on $K$. We are looking for optimal trajectories for the cost expression (7.1), as $T \to \infty$. In this case we have $L(x,v) = f(x) + \frac{1}{2} v^2$ and the Bellman equation (3.2) is

$$
(7.2) \quad f(x) + \min_v \left[ \frac{1}{2} |v|^2 + v \cdot \nabla p(x) \right] = \mu
$$

or, equivalently,

$$
(7.3) \quad \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial p}{\partial x_i} \right)^2 = f(x) - \mu.
$$

The unique point $v$ where the minimum in (7.2) is achieved is $-\nabla p(x)$, hence the o.d.e. we have to consider is

$$
(7.4) \quad \dot{x}(t) = -\nabla p(x(t)) , \quad x(0) = x_0 .
$$

This is the characteristic equation of the p.d.e. (7.3). If we wish all the solutions of (7.4) with $x_0 \in K$ to remain in $K$ for all $t \geq 0$ then we must have, for a smooth solution $p(\cdot)$ of (7.3), that

$$
(7.5) \quad \nabla p(x) \cdot \hat{u}_x \geq 0 \quad \text{for all } x \in \partial K ,
$$

where $\hat{u}_x$ is the unit outward normal vector at $x \in \partial K$. In the present example a good solution is a Lipschitz continuous function $p(\cdot)$ which satisfies the boundary condition (7.5). In this example we have also that

$$
(7.6) \quad \mu = \min_{x \in K} f(x) .
$$
That \( \mu \) is not larger than \( \min_{x \in K} \ell(x) \) is clear from (7.3). If, however,

\[
\mu - \min_{x \in K} \ell(x) = -\epsilon < 0
\]

then

\[
\frac{d}{dt} p(x(t)) = \nabla p(x(t)) \cdot \dot{x}(t) = -\left| \nabla p(x(t)) \right|^2 \leq -\frac{1}{2} \epsilon \quad \text{a.e. in } [0, \infty)
\]

which contradicts the boundedness of \( p(\cdot) \) on \( K \).

A more general situation is

\[
\int_{0}^{T} \left[ g(x(t)) + \ell(x(t)) \right] dt, \quad T \to \infty
\]

where we assume that \( g(\cdot) \) is strictly convex on \( \mathbb{R}^n \) and \( g(v) / |v| \to \infty \) as \( |v| \to \infty \). The Bellman equation is

(7.7)

\[
- \min_{v} \left[ g(v) + v \cdot \nabla \ell(x) \right] = \ell(x) - \mu.
\]

The left hand side of (7.7) is not smaller than \(-g(0)\), implying that

(7.8)

\[
\mu \leq g(0) + \min_{x \in K} \ell(x).
\]

We claim that in fact equality holds in (7.8). To prove this let \((p(\cdot), \mu)\) be a good solution of (7.7), and let \( x(\cdot) \) be a trajectory such that

(7.9)

\[
\mu = \ell(x(t)) + g(\dot{x}(t)) + \dot{x}(t) \cdot \nabla p(x(t)), \quad \text{a.e. in } [0, \infty].
\]

Integrating the right hand side of (7.9) over \([0, T]\) and using the convexity of \( g(\cdot) \) we get

\[
\mu \geq \frac{1}{T} \int_{0}^{T} \ell(x(t)) dt + g \left( \frac{1}{T} (x(T) - x(0)) \right) + \frac{1}{T} \left[ p(x(T)) - p(x(0)) \right]
\]

from which it follows that \( \mu \geq g(0) + \min_{x \in K} \ell(x) \). We thus have proved the following:
Theorem 7.1. Let $g(\cdot)$ be convex on $\mathbb{R}^n$ and satisfy $g(v)/|v| \to \infty$ as $|v| \to \infty$. Let $\ell(\cdot)$ be continuous on $K$. Then the minimal cost growth rate for
$L(x, v) = g(v) + \ell(x)$ is given by
$$\mu = g(0) + \min_{x \in K} \ell(x).$$

For this $\mu$ there is a good solution $p(\cdot)$ for (7.7). Let $H(\cdot)$ be defined by
\begin{equation}
H(\eta) = \min_{v \in \mathbb{R}^n} [g(v) + v \cdot \eta].
\end{equation}

Assume that $g(\cdot)$ is strictly convex. Then the differential equation (3.4) is given by
\begin{equation}
\frac{dx}{dt} = \nabla H(\nabla p(x)).
\end{equation}

Proof: Only the last assertion has to be proved, and it follows from the fact that, for $H(\cdot)$ defined as in (7.10), $\nabla H(\eta)$ is equal to the value $v$ for which the minimum in the right hand side of (7.10) is attained.

Example 7.2. Assume that $\ell(\cdot)$ and $g(\cdot)$ are both spherically symmetric. We consider $K$ to be the unit ball. Then we are looking for a good solution $p(\cdot)$ which is also spherically symmetric. More precisely, in this case there is a function $h(\cdot)$, $h : \mathbb{R}^+ \to \mathbb{R}^1$ so that $H(\cdot)$ in (7.10) is given by
$$H(\eta) = h(|\eta|).$$

The differential equation (7.7) becomes
\begin{equation}
h\left(\frac{dp}{dr}\right) + \ell(r) = \mu
\end{equation}

where $r \to p(r)$ is a spherically symmetric good solution for $\mu = g(0) + \min_{0 \leq r \leq 1} \ell(r)$.

Because of the symmetric structure of $g(\cdot)$ it can be assumed that $g(0) = 0$ is the minimal value of $g(\cdot)$. Then zero is also the maximal value of $h(\cdot)$, attained at $\eta = 0$. To construct a solution for (7.12) in case $\min_{0 \leq r \leq 1} \ell(r)$ is attained in a unique point $0 < r_0 < 1$ we employ the same method as in Section 6 for the scalar system. Then it is also clear that all the solutions of (7.11) in the present example will
converge, as $t \to \infty$, to the sphere $|x| = r_0$, and therefore there exists an overtaking optimal trajectory for every initial value. It is explicitly given as the solution of (7.11).

**Example 7.3.** Let the function $h : \mathbb{R}^n \to \mathbb{R}^n$ be such that

\begin{equation}
(7.13) \quad v \to g(v) = v \cdot h(v) \text{ is strictly convex}
\end{equation}

and $h(\cdot)$ is $C^1$ so we consider

\[ \int_0^T \left[ x(t) \cdot \left( \frac{\partial h}{\partial x} (\cdot)(-x) \right) x(t) + \dot{x}(t) \cdot h(x(t)) \right] dt \]

as $T \to \infty$. Then we have

\[ L(x, v) = x \cdot \left( \frac{\partial h}{\partial x} (-x) \right) x + v \cdot h(v) \]

and the Bellman equation is

\begin{equation}
(7.14) \quad x^T \cdot \left( \frac{\partial h}{\partial x} (-x) \right) x + \min_v \left[ v \cdot h(v) + v \cdot \nabla p(x) \right] = \mu
\end{equation}

By (7.13), if we choose

\[ p(x) = -x \cdot h(-x) , \]

then the minimum in (7.14) is attained at $v = -x$. To check that the differential equation (7.14) is indeed satisfied by $p(\cdot)$ we compute (recall (7.13))

\[ g(-x) + x \nabla g(-x) = g(-x) + x \cdot h(-x) - x \cdot \left( \frac{\partial h}{\partial x} (-x) \right) x = -x \left( \frac{\partial h}{\partial x} (-x) \right) x , \]

which shows that $\mu = 0$ together with $p(x) = g(-x)$ form a good solution. All the solutions of the differential equation $\dot{x} = -x$ converge to zero as $t \to \infty$, hence these solutions are the overtaking optimal trajectories for the optimization problem.

**Example 7.4.** We take in Example 7.3

\[ h(x) = |x|^{m-2} x \]

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for a natural number $m > 2$, $x \in \mathbb{R}^n$. Then we have

$$g(v) = v \cdot h(v) = |v|^m.$$ 

We compute $\frac{\partial h}{\partial x_j}$ where

$$h_i = \left( \sum_{k=1}^{n} x_k^2 \right)^{\frac{m}{2} - 1} x_i,$$

$$\frac{\partial h_i}{\partial x_j} = \left( \sum_{k=1}^{n} x_k^2 \right)^{\frac{m}{2} - 1} \delta_{ij} + \left( \frac{m}{2} - 1 \right) \left( \sum_{k=1}^{n} x_k^2 \right)^{\frac{m}{2} - 2} \cdot 2x_i x_j.$$

Therefore

$$x \cdot (\frac{\partial h}{\partial x} (-x)x = \left( \sum_{k=1}^{n} x_k^2 \right)^{m/2} + (m - 2) \left( \sum_{k=1}^{n} x_k^2 \right)^{m/2} = (m - 1) \left( \sum_{k=1}^{n} x_k^2 \right)^{m/2}.$$ 

The minimization problem is thus of

$$\int_0^T \left[ |\dot{x}(t)|^m + \lambda |x(t)|^m \right] dt, \quad T \to \infty$$

since by a suitable change of variable $t \to \alpha t$ we can obtain any coefficient $\lambda$ for $|x(t)|^m$. The result of the previous example shows that the overtaking optimal trajectories are given by $x(t) = x_0 \exp \left[ - \left( \frac{\lambda}{m-1} \right)^1/m t \right].$

References


