FLows of stochastic dynamical systems:

Nontriviality of the Lyapunov spectrum.

BY

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Abstract

We give conditions under which the flow of a stochastic dynamical system, defined on a compact Riemannian manifold, has nontrivial Lyapunov spectrum. This nontriviality seems to be related to the flow having strictly positive entropy, and to mixing the manifold in a strong sense. The approach is based on a Furstenberg-type theorem for the system, and various versions of the Stroock-Varadhan Support Theorem.

1. Introduction

In this paper we give conditions under which the flow of a stochastic dynamical system \((X, \sigma)\), defined on a compact Riemannian manifold \(M\) of dimension \(m\), has nontrivial Lyapunov spectrum. Flows of stochastic systems are studied in [2] (see also references therein, especially [9], [12]), and their Lyapunov spectra are studied in [3], [14]. See [8] for a survey of these and many more results about Lyapunov spectra for stochastic systems. We will assume throughout that the stochastic flow a.s. preserves the normalised Riemannian volume \(\mu\) on \(M\); this implies that the average of the Lyapunov spectrum (with multiplicities) is zero. By the spectrum being nontrivial we mean that we do not have the entire spectrum being zero; this is equivalent to the highest element of the spectrum being strictly positive. This nontriviality seems to be an important property because it is likely to be related to the stochastic flow having strictly positive entropy, and to the flow mixing the manifold up in some sense (See [14] for a definition of entropy in this context and a proof that it exists.)

Throughout, we will take \(X\) of \((X, \sigma)\) to be a smooth bundle morphism
\[ M \times \mathbb{R}^{n+1} \to TM, \text{ i.e. a smooth map such that for each } x \in M, \text{ the map } X(x, -) \text{ is a linear map } \mathbb{R}^{n+1} \to T_xM. \] Also we take \( z \) to be the process \( t \mapsto (B_t^1, \ldots, B_t^N) \), where \( B_t \equiv (B_t^1, \ldots, B_t^N) \) is a Brownian motion in \( \mathbb{R}^N \), and we denote the probability space associated to \( B_t \) by \((\Omega, \mathcal{F}, \mathbb{P})\). Also we will put \( X(-, e_i) = Y^i \) for \( i = 1, \ldots, n \), and \( X(-, e_{n+1}) = \Lambda \), where \( \{e_1, \ldots, e_{n+1}\} \) is the standard basis in \( \mathbb{R}^{n+1} \). Thus, \( Y^1, \ldots, Y^n, \Lambda \) are smooth vector fields on \( M \). Finally, we will denote by \( \xi_t(\omega) \) the flow of \((X, z)\). This is defined to be a diffeomorphism of \( M \) such that for any \( x \in M \) the random trajectory \( \xi_t(\omega)x \) is the solution to \((X, z)\) starting from \( x \), i.e.

\[
d\xi_t(\omega)x = X(\xi_t(\omega)x) \circ dz_t(\omega)
\]

\[
= \Lambda(\xi_t(\omega)x)dt + \sum_{i=1}^{n} Y^i(\xi_t(\omega)x) \circ dB_t^i(\omega).
\]

(n.b. throughout we use Stratonovitch integrals.)

As we have said, we assume throughout that the stochastic flow \( \xi_t(\omega) \) preserves the normalised Riemannian volume \( \rho \) of \( M \), i.e. for a.e. \( \omega \) we have \( \xi_t(\omega)\rho = \rho \) for all \( t > 0 \). This is equivalent to the conditions \( \text{div } \Lambda(x) = 0 \), \( \text{div } Y^i(x) = 0 \) for \( i = 1, \ldots, n \), for all \( x \in M \). (To see this consider the stochastic equation for \( \log \det [T\xi_t(\omega)x] \) of [1] (Theorem 3.1), or use the Stroock-Varadhan support Theorem for the flow \( \xi_t(\omega) \) (see Lemma 3.1 below.)

The condition also implies that \( \rho \) is a stationary (invariant) measure for the Markov process associated to \((X, z)\), with transition probabilities \( \rho_t(x, B) = P(\omega : \xi_t(\omega)x \in B) \) (\( B \) - Borel set in \( M \)). We must assume moreover that \( \rho \) is ergodic for this Markov process; this will be ensured for instance if for each \( x \in M \) the vector fields evaluated at \( x \) of the Lie algebra generated by \( \{Y^1, \ldots, Y^n\} \) (denoted by \( L(Y^1, \ldots, Y^n)(x) \)) spans \( T_xM \). See [11].

Our approach is based on the following result, which is given essentially in [6] (see also [14], [1]) (in fact this present paper can be regarded as a refinement of Section 3 of [6]):
Theorem 1 (Furstenberg's Theorem for nonlinear stochastic systems)

The largest Lyapunov exponent $\lambda$ for the stochastic system $(X, Z)$ is strictly positive unless there is a Borel probability measure $\nu$ on the projective bundle $\pi^{-1}(x)$, such that $\pi(\nu) = \rho$ ($\pi$: PM - bundle projection), and such that $\nu$ is preserved a.e. by the flow $\eta_t(\omega)$ on PM, induced naturally from $T\xi_t(\omega)$ on TM, i.e. for a.e. $\omega$ we have $\eta_t(\omega)\nu = \nu$ for all $t > 0$.

(n.b. the flow $\eta_t(\omega)$ is studied in detail in [4], but there it is defined on the sphere bundle $SM$ (unit spheres in tangent bundle $TM$), as the flow induced from the derivative $T\xi_t(\omega)$ by radial projection onto the unit sphere in each tangent space. Our version of $\eta_t(\omega)$ is obtained from that of [4] simply by taking PM to be SM with antipodal points in each sphere identified).

Proof

The discrete time version of the result is given in [6] Theorem 2.1. Thus for any $n \in \mathbb{N}$ there exists a Borel probability $\nu_n$ on PM such that

$\pi(\nu_n) = \rho$ and $\pi(\xi_t(\omega)\nu_n) = \nu_n$ for a.e. $\omega$. The set $\mathcal{M}$ of each such measures is weak* compact (see [5] Theorem 3.5), therefore we can find $\nu$ such that passing to a subsequence we have $\nu_n \to \nu$ as $n \to \infty$. Thus we can deduce that $\eta_t(\omega)\nu = \nu$ for a.e. $\omega$ if $t$ is a dyadic rational, i.e. for any $f \in C(PM)$ we have

$$\int f(v) d[\eta_t(\omega)\nu](v) = \int f(\eta_t(\omega)v) d\nu(v) = \int F(v) d\nu(v)$$  \hspace{1cm} (1)

if $t$ is a dyadic rational. But then (1) must be true a.s. for all $t > 0$, because a.s., $\eta_t(\omega)$ is continuous in $t$. //

Note The version of Furstenberg's Theorem given in [14], [1] is stronger than that of [6], in that it is not required that $\xi_t(\omega)$ preserves the measure $\rho$ a.s. Starting from the stronger version we could obtain stronger versions of the results of this paper. We work from [6] for simplicity and because it seems to contain the essence of the motivation.
Our approach is to give criteria for the non-existence of a measure \( \nu \) satisfying the conditions of Theorem 1. These criteria are control theoretic and fall into two groups (Sections 2 and 3), each relying on different extensions of the Stroock-Varadhan Support Theorem (see [15], [12]). It is convenient to define here the control system \((X, C)\) associated with \((X, z)\); this is the control system of Section 3 (through that of Section 2 needs more structure.) We take \(C\) to be the space of piecewise smooth maps ('controls') \([0, \infty) \to \mathbb{R}^n\), and for \(c \in C\) we define the flow \( \xi_t(c) \) of \((X, C)\) with control \(c\) to be the flow of the time dependent vector field

\[
x + \sum_{i=1}^{n} Y_i(x) c_t^i + A(x).
\]

We see that the control stands in place of the Brownian motion in the control system. One thinks of choosing a control \(c\) to steer the trajectory \( \xi_t(c) x \) along a desired path in \(M\): one can think of the stochastic dynamical system as essentially the same as the control system, except that the choice of control is made randomly.

The Stroock-Varadhan Support Theorem says that the support of the measure on \(C([0, \infty), M)\) induced from the solution \(\xi_{\cdot}(\omega)x\) is equal to the closure of the collection of paths \(t \to \xi_t(c)x: c \in C\).

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2. The Conditioned Approach

In this section we must insist that the stochastic system \((X, z)\) is non-degenerate, i.e. for each \(x \in \mathcal{M}\), the vectors \(Y_1(x), \ldots, Y^n(x)\) span \(T_x \mathcal{M}\).

Our approach is first to recall a criterion (Lemma 2.1) for the invariance of Theorem 1 for the measure \(\nu\) on \(\mathcal{M}\), in terms of its marginals (conditional measures) \(\nu_x \) in the fibers of \(\mathcal{M}\); then (using a conditioned version of the Stroock-Varadhan Support Theorem-Proposition 2.2) we translate this criterion into control theoretic terms (Lemma 2.3); finally we observe (Theorem 2.4) that by nondegeneracy (which implies that the associated control system can be steered along any smooth path in \(\mathcal{M}\)) we only have to verify the criterion for a pair of points in \(\mathcal{M}\).

Lemma 2.1

Fix \(T > 0\). Suppose \(\nu\) is a Borel probability on \(\mathcal{M}\) such that \(\pi(\nu) = \rho\).

Then the following are equivalent:

1. \(\eta_T(\omega) \nu = \nu\) for a.e. \(\omega\).
2. Disintegrate \(\nu\) as \(\rho \otimes \{\nu_x \}_{x \in \mathcal{M}}\),

so that if \(B\) is Borel in \(\mathcal{M}\), then

\[
\nu(B) = \int \nu_x (B \cap \Pi_x \mathcal{M}) \, d \rho(x).
\]

Then for \(\rho\) - a.e. \(x\), and given \(x\) for \(p_T(x, -)\) a.e. \(y\) we have

\[
P(\eta_T(\nu_x) = \nu_y \mid \xi_T(x) = y) = 1.
\]

Proof

See [6] Proposition 2.2. //
Before stating Proposition 2.2 we must introduce the concepts involved. The result applies to any (non-explosive) stochastic system $(GX, Z)$ defined on a smooth finite dimensional fibre bundle $GM$ over $M$, which is a lift of the system $(X, Z)$. Denote the solution to $(GX, Z)$ starting from $v \in GM$ by $G_{\xi_t}^\omega v$. Being a lift is defined by either of the two equivalent conditions $\tau \circ G_{\xi_t}^\omega (v) = \xi_t^\omega (x) \left( \text{where } \tau (v) = x \right)$, or $T_x (GM, X)$, where $\tau : GM \to M$ is the bundle projection. The result characterises the support of the measure on $C([0, T], GM)$ (any $T > 0$) induced from the measure $\nu$ via the map $\omega \to (\text{path } t \to G_{\xi_t}^\omega v)$, and conditioned on the event $\xi_T (x) = y$. (Here $x, y \in M, v \in G_x M$.) We will denote this conditioned measure by $[G_{\xi}^\nu | \xi_T (x) = y]$; it is defined up to a $\mu_T (x, \cdot)$-null set of $y$'s in $M$, or equivalently (since $(X, Z)$ is nondegenerate) up to a $\nu$-null set of $y$'s. The characterisation is in terms of the control system $(GX, C)$ associated to $(GX, Z)$, but with a condition on the controls to accommodate the condition on the stochastic system.

**Proposition 2.2** (Conditioned Stroock-Varadhan Support Theorem).

Assume the stochastic system $(X, Z)$ is nondegenerate. Let $GM, (GX, Z)$ be as above, and take $T > 0$, $x \in M$, and $v \in G_x M$.

Then up to a null set of $y$'s in $M$ we have

$$\text{support } [G_{\xi}^\nu | \xi_T (x) = y] =$$

$$\text{Closure } (G_{\xi}^c v : c \in C \text{ and } \xi_T (c) x = y)$$

(2)

**Proof**

This is a manifold formulation of [7] Theorem 4.1. //

We apply Proposition 2.2 taking $GM$ to be the principal bundle over $M$, and $(GX, Z)$ to be the natural lift of $(X, Z)$ to $GM$. We will breifly explain these structures: note that in [6] we take the natural lift to the special principal bundle, which is almost the same as we have here, except that in [6] we insist
that all the linear maps have determinant 1. For \( x \in M \), the fibre \( G^x_M \) consists of the invertible linear maps \( v : R^m \to T_xM \), and we can identify \( v \in G^x_M \) with the frame \( (v(e_1), \ldots, v(e_m)) \) of tangent vectors in \( T_xM \), where \( \{e_1, \ldots, e_m\} \) is the standard basis in \( R^m \). We define \( G^X \) over a chart \( U \) for \( M \) by

\[
G^X : U \times GL(m) \to U \times GL(m) \times \mathbb{R}^m \times g^x(U); \\
(x, u) \to (x, v, x(n), DX(x) \cdot v).
\]

Here \( GL(m) \) denotes the invertible linear maps in \( R^m \), and \( g^x(U) \) denotes all the linear maps. Using [2] Remark 4.2 (b) we see from this definition that the flow \( G^x_t(\omega) \) of \( (G^X, z) \) is given by

\[
[G^x_t(\omega) v] e = T^x_t(\omega)(v(e)) \quad (\text{any } e \in R^m).
\]

Thus, for any \( v \in G^x_M \), \( h^x_t(\omega) \) a.s. incorporates the derivative \( T^x_t(\omega)x \) in the sense that

\[
[G^x_t(\omega) v] \cdot v^{-1} = T^x_t(\omega)x : T_xM \to T^x_t(\omega) M. \quad (3)
\]

The analogous result to (3) for the control system also holds – to write this we just replace \( \omega \) by \( c \).

**Lemma 2.3**

Fix \( T > 0 \). Assume \( (X, z) \) is nondegenerate and take \( x \in M \). Also, for each \( y \in M \) choose a Borel probability measure \( \nu_y \) on \( |P_y| \).

Then for a.e. \( y \in M \) we have:

\[
P\{\eta_T(x) = y \mid T(x) = y \} = 1 \iff T(x) = y \}
\]

\[
T(c)x = y \quad \text{for all controls } c \text{ such that } T(c)x = y.
\]

**Proof.**

Take \( y \in M \) such that (2) is true, taking \( (G^X, z) \) to be the natural lift of
(X,z) to the principal bundle. Then for w such that \( \xi_T(w)x = y \) and any \( u \in G_x^M \), \([G\xi_T(w)v]^{-1}\) is an element of \( GL(T_xM, T_yM) \), and from (3) we see that

\[
([G\xi_T(w)v] \circ v^{-1})u_x = u_y \iff \eta_T(u_x) = u_y.
\]

Now, the subset of \( GL(T_xM, T_yM) \) given by \( \{ T : T u_x = u_y \} \) is closed, as also is the subset of \( G_y^M \) given by \( \{ S : (Sv)u_x = u_y \} \). (n.b. we denote the induced action \( P_xM \to P_yM \) of \( T \) again by \( T \)).

Therefore

\[
\{ S \in G_y^M : (Sv)u_x = u_y \} = \text{Support} [G\xi_T(v)|\xi_T(x) = y] \quad (4)
\]

\[
\iff P\{ (G\xi_T(v) \circ v^{-1})u_x = u_y | \xi_T(x) = y \} = 1
\]

\[
\iff P\{ \eta_T(u_x) = u_y | \xi(x) = y \} = 1.
\]

Also using (2), we have

\[
(L) \iff \\
([G\xi_T(c)v] \circ v^{-1})u_x = u_y \text{ for all controls } c \text{ such that } \xi_T(c)x = y
\]

\[
\iff \{ \eta_T(c)u_x = u_y \text{ for controls } c \text{ such that } \xi_T(c)x = y \},
\]

and the result follows. (n.b. to be strict, we should write the LHS of (4) as

\[
\{ \gamma \in C([0,T], GM) : \gamma_0 \in G_x^M, \gamma_T \in G_y^M \text{ and } (\gamma_T \circ \gamma_0^{-1})u_x = u_y \}
\]

Theorem 2.4

Assume \((X,z)\) is nondegenerate. Suppose there exist a pair \( x,y \in M \), and \( T > 0 \) for which we have the following property (which we call Property (5)):

There is no pair \( u_x, u_y \) of probability measures in \( P_xM, P_yM \) for which

\[
\eta_T(c)u_x = u_y \text{ for all controls } c \text{ such that } \xi_T(c)x = y.
\]

Then \( \lambda > 0 \).
Proof

We will deduce from these conditions that for any pair \( \tilde{x}, \tilde{y} \) in \( M \), and with \( T \) replaced by \( 3T \), we have Property (5). But if \( \lambda = 0 \), then by Theorem 1 we have a probability \( \nu \) on \( \mathcal{M} \) such that \( \pi(\nu) = \rho \) and \( \eta_{3T}(\omega)\nu = \nu \) for a.e. \( \omega \).

Taking the disintegration \( \rho \otimes \{ \nu_x \}_{x \in X} \) of \( \nu \) and applying Lemmas 2.1, 2.3 (with \( T \) replaced by \( 3T \)), we see that there are many pairs \( x, y \) for which Property (5) does not hold - take the pair of measures to be \( \nu_x, \nu_y \) - and we have a contradiction.

So fix a pair \( x, y \) in \( M \) for which Property (5) holds, and take any pair \( \tilde{x}, \tilde{y}, \tilde{z} \), and suppose Property (5) does not hold for \( \tilde{x}, \tilde{y}, 3T \). Thus there is a pair \( \mu_x, \mu_y \) of probability measures on \( P_xM, P_yM \) such that for any \( \tilde{c} : [0,3T] \to \mathbb{R}^n \) with \( \xi_{3T}(\tilde{c})\tilde{x} = \tilde{y} \), we have \( \eta_{3T}(\mu_x) = \mu_y \). Fix controls \( a, b : [0,T] \to \mathbb{R}^n \) such that \( \xi_T(a)x = x, \xi_T(b)y = y \) (the existence of such \( a, b \) follows from the non-degeneracy of \( \langle X, Z \rangle \)), and for any control \( c : [0,T] \to \mathbb{R}^n \) with \( \xi_T(c)x = y \), denote by \( (a,c,b) \) the control \( [0,3T] \to \mathbb{R}^n \) given by\( d(a,c,b)/dt = da_T/dt \) for \( 0 < t < T \), \( d(a,c,b)/dt = dc_{t,T}/dt \) for \( T < t < 2T \), \( d(a,c,b)/dt = dc_{T-2T}/dt \) for \( 2T < t < 3T \). (Thus, \( (a,c,b) \) is just \( a, c, b \) applied successively. n.b. the time derivative of the control is the important thing.) Then \( \xi_{3T}((a,c,b)) \tilde{x} = \tilde{y} \) and so by our assumption we have \( \eta_{3T}((a,c,b))\mu_x = \mu_y \). But we deduce from this that \( \eta_T(c)\mu_x = \mu_y \), where \( \mu_x = \eta_T(a)\mu_x \), and \( \mu_y = [\eta_T(b)]^{-1}\mu_y \), and this contradicts the assertion that \( x, y, T \) satisfy Property (5). //

Corollary 2.5

We can strengthen Property (5) in Theorem 2.4 to:

There is no pair \( \mu_x, \mu_y \) such that \( S\mu_x = \mu_y \) for all \( S \) in the set

\[
\text{Closure } \{ T\xi_T(c)x : c \in C \text{ with } \xi_T(c)x = y \}
\]

in \( G_L(T_xM, T_yM) \).

(Here we denote the induced action \( \mathcal{P}_xM \to \mathcal{P}_yM \) of \( S \) again by \( S \).) //
The set (5) corresponds to the support of the measure on $SL(m)$ in [10], which determines the random matrix product. Having found (5) we could apply the techniques of [10], Theorem 3.6 to obtain criteria for $\lambda > 0$ in terms of noncompactness and irreducibility of (5). We will be content with the following criterion, which can sometimes be verified by looking at the lift of $(X,z)$ to the principal bundle (see [6] Section 3):

**Corollary 2.6**

Assume $(X,z)$ is nondegenerate and suppose there exist $x,y \in M$ and $T > 0$ for which the set (5) has an interior in $SL(T_xM, T_yM)$. Then $\lambda > 0$. (n.b. the linear maps of (5) always have determinant 1.)

**Proof**

Apply [6] Lemma 3.1 to Corollary 2.5. //

The following is a local criterion for $\lambda > 0$:

**Theorem 2.7.**

Assume $(X,z)$ is nondegenerate, has zero drift, i.e. $A \equiv 0$, and take any $x \in M$. Suppose there is no probability measure $\mu_x$ on $\mathcal{P}_xM$ such that

$$(\exp S)\mu_x = \mu_x$$

for all linear maps $S : T_xM \to T_xM$ in the set

$$\text{Closure } \{ W(x) : y \in \mathcal{L}(y^1, \ldots, y^n) \text{, and } Y(x) = 0 \}.$$  

Then $\lambda > 0$.

(Here, by $\exp S$ we mean $I + S + S^2/2! + \ldots$, and $\mathcal{L}(y^1, \ldots, y^n)$ denotes the Lie algebra generated by $y^1, \ldots, y^n$.)

**Proof**

Fix $T > 0$ and note that if we take $c \equiv 0$ then $\eta(c) \equiv \text{Id}$, and so for any probability $\mu_x$ on $\mathcal{P}_xM$ we have $y_1(c)\mu_x = \mu_x$. Therefore, applying Corollary 2.5 with $x = y$, it suffices to show that there is no $\mu_x$ such that $T_\varepsilon\mu_x = \mu_x$. \[\]
for all $T \xi$ on $TM$ induced from $\xi$ in the set

\[ \text{Closure}_{C^1} \{ \xi_T(c) : c \in C \text{ and } \xi_T(c)x = x \} \]  

(7) (i.e. closure in the $C^1$ topology.)

To prove the result we will show that if $Y \in \mathcal{L}(Y^1, \ldots, Y^n)$ and $Y(x) = 0$, then $\phi^{\dot{Y}}_t \in (7)$, where by $\phi^X_t$ for a vector field $X$, we mean the flow of $X$ through time $t$. (n.b. if $Y(x) = 0$ then $T_Y \xi(x) \equiv \exp t_Y : T_x M + T_x M$.)

Now it is well known that if $Y \in \mathcal{L}(Y^1, \ldots, Y^n)$, then $\phi^Y_T \in \text{Closure}_{C^1} \{ \xi_T(c) : c \in C \}$. To see this note that for vector fields $X, Z$, and putting $[X, Z] = W$, then easily from the definition we have in the $C^1$-topology that

\[ \phi^{W^1} = \lim_{n \to \infty} (\phi^{-1})^n \circ X^{\circ n} \circ \phi^{-1} \circ Z^{\circ n} \circ \phi^{-1}^{X^1} \circ Z^{\circ n} \circ \phi^{-1}^{X^1} \]

Thus it suffices to deal with the conditions $Y(x) = 0$ and $\xi_T(c)x = x$.

So take $\varepsilon > 0$. We must find $c \in C$ such that $d_{C^1}(\phi^Y_T, \xi_T(c)) < \varepsilon$ (distance in the $C^1$-topology.) and $\xi_T(c)x = x$, and the proof will then be complete. Our approach to this is as follows: Note that since $\phi^Y_T \in \text{Closure}_{C^1} \{ \xi_T(c) : c \in C \}$ we can find $\tilde{c} \subseteq C$ such that $d_{C^1}(\phi^Y_T, \xi_T(\tilde{c}))$ and hence $d(\xi_T(c)x, x)$ are arbitrarily small. We will find $c$ by taking such a $\tilde{c}$ and speeding it up by a factor $T/\tau$ (any fixed $\tau \in (0, T)$), i.e. replacing it by $c^T : [0, T/\tau] \to \mathbb{R}^n$ given by $c^T_t = \tilde{c}_{\tau/\tau}^{\tau/\tau}$, and then extending $c^T$ to $[0, T]$ so that as $t$ increases from $0$ to $T$, $\tilde{x}$ goes to $x$.

Put $\phi^Y_t = \text{flow through time } t$ along $Y(e)$ ($\equiv \sum_i \alpha^i_t Y^i$ if $e = \sum_i \alpha^i e_i$ $e \in \mathbb{R}^n$) and define $\psi : \mathbb{R}^n + TM$ by $\psi(e) = \phi^Y_T(x)$. Note that $\psi$ is smooth and $T \psi(0) : \mathbb{R}^n + T_x M$ is given by $e \to Y(x)e$. Therefore $T \psi(0)$ is surjective (since $(X, Z)$ is nondegenerate) and so $\psi$ is locally a projection, and we can find a smooth inverse $\psi^{-1}$ to $\psi$ in a neighborhood of $x$, with $\psi^{-1}(x) = 0$. Note also that $\psi^c$ is a continuous function of $e$ to the $C^1$-topology. Therefore if
we denote \( \tilde{\phi}_t^{-1}(\tilde{x}) \) by \( \tilde{\phi}_t^x \) then \( \tilde{\phi}_t^x \) is a smooth map \( M + M \) given by the flow of \( \nu(\phi_t^{-1}(x)) \), \( \phi_t^x(x) = x \), and \( \phi_t^x \) is continuous in \( \tilde{x} \) at \( \tilde{x} = x \) and \( \phi_t^x = \text{Id} \). So given \( \varepsilon > 0 \) as above there exists \( \delta > 0 \) such that if \( \tilde{x} \in M \) and \( d(\tilde{x}, x) < \delta \) then \( d(\phi_t^x, \text{Id}) < \varepsilon/2 \). Take \( \tilde{c} \in C \) such that \( d(\tilde{c}(\xi_t(c), \phi_t^x) \). To obtain \( c \), speed \( \tilde{c} \) up as above and then take \( \dot{c}_t \) to be constant and equal to \( -\phi_t^{-1}(\tilde{x}) \) for \( t \in [T-\tau, T \), where \( \tilde{x} = \xi_T(\tilde{c})x \). //

**Corollary 2.8.**

Suppose the set \( (6) \) of Theorem 2.7 has an interior in the space of skew-adjoint (antisymmetric in an orthonormal basis) linear maps in \( T_x M \). Then \( \lambda > 0 \).

**Proof.**

Note that if \( (6) \) has an interior then \( \exp (6) \) has an interior in \( S_L(T_x M) \).

Now apply [6] Lemma 3.1, as in Corollary 2.6. //

3. **The Flow Approach**

This section is based on the Stroock-Varadhan Support Theorem for flows, which we state as Lemma 3.1. We do not have to assume that \( (X, Z) \) is non-degenerate, but merely that the vector fields are divergence-free and that the normalised Riemannian volume \( \nu \) is ergodic for the associated Markov process. Also we do not need the conditioning technique of Section 2, though to study the flow one must look at lifts of \( (X, Z) \) involving high order derivatives, or the induced system on the infinite dimensional manifold Diff\(^k\)(M) of \( C^k \) diffeomorphisms of \( M \) for suitable \( k > 1 \). (see [2], [12][9]).

In this section we will take \( C \) to be the space of piecewise smooth controls for \( (X, Z) \) over all positive time, i.e. \( C = \{ \text{Piecewise smooth paths } c : [0, \omega + K^0] \} \). Also, we will denote by \( \xi(P) \) the measure on \( C([0, \omega], \text{Diff}^k(M)) \) (any given \( k > 1 \)) induced from \( P \) on \( \omega \) via the map \( \omega + \text{path } \{ t + \xi_t(\omega) \} \).
Lemma 3.1 (Stroock-Varadhan Support Theorem for flows)

For any $k \geq 1$ we have \[ \text{Support}_{C^k} \{ \xi(P) \} = \text{Closure}_{C^k} \{ \xi(c) : c \in C \} \quad (8) \]

Proof.

See [13]. As they remark, the results can be obtained by applying the usual Support Theorem to the system obtained by taking a high order derivatives of $(x,z)$.

(n.b $(8)$ is equivalent to 
\[ \text{Support}_{C^{k-1}} \{ T\xi(P) \} = \text{Closure}_{C^{k-1}} \{ T\xi(c) : c \in C \} \]
and similarly for higher order derivatives.)  //

Lemma 3.2

Take any Borel measure $\nu$ on $\mathcal{PM}$. Then the following are equivalent:

(1) For a.e. $w \in \Theta$ we have $n_t(w)\nu = \nu$ for all $t > 0$.

(2) For any $c \in C$ we have $n_t(c)\nu = \nu$ for all $t > 0$.

Proof.

This follows directly from Lemma 3.1 with $k = 1$, if we note that the set 
\{ $\xi : T\xi \nu = \nu$ \} is closed in the $C^1$-topology. (Here the induced action of $T\xi$ on $\mathcal{PM}$ is denoted again by $T\xi$.)  //

Theorem 3.3.

Suppose there is no Borel probability $\nu$ on $\mathcal{PM}$ for which $\pi(\nu) = \rho$ and for any control $c \in C$ we have $n_t(c)\nu = \nu$ for all $t > 0$. Then $\lambda > 0$.

Proof.

Apply Lemma 3.2 to Theorem 1.  //
Example 3.4 (See [13])

Take $M$ to be the 2-torus $\mathbb{T}^2 \equiv (\mathbb{R}/2\pi \mathbb{Z})^2 \equiv S^1 \times S^1$. Also take $A = 0$ in the stochastic system $(x,z)$, take $n = 4$, and take the noise vector fields $Y^1, \ldots, Y^4$ to be given by

\[
Y^1(x,y) = (0, \sin x), \\
Y^2(x,y) = (0, \cos x), \\
Y^3(x,y) = (\sin y, 0), \\
Y^4(x,y) = (\cos y, 0).
\]

Then easily, the flow preserves the normalised Riemannian volume $\rho$, and $\rho$ is ergodic for the associated Markov process (since the system is nondegenerate - in fact the Markov process is Brownian motion on the torus).

We will show that there is no probability $\nu$ on $\mathbb{P}M \equiv \mathbb{P}^1 \times \mathbb{T}^2$ for which $\pi(\nu) = \rho$ and $\eta^i_t(\nu) = \nu$ for all $t > 0$, and $i = 1, \ldots, 4$, where $\eta^i_t$ is the flow on $\mathbb{P} \mathbb{T}^2$ induced from the flow on $\mathbb{T}^2$ of $Y^i$. It will then follow by Theorem 3.3 that $\lambda > 0$ for this example. The idea is to find, for each $i$, all the probability measures on $\mathbb{P} \mathbb{T}^2$ which project to $\rho$ and which are invariant for the flow $\eta^i_t$, and then to note that there is no measure common to all $i = 1, \ldots, 4$.

First take $i = 1$. Note that by the Krylov Bogoliouboff Theory (see [16] Chapter 13), we can split $\mathbb{P} \mathbb{T}^2$ up into disjoint components which are invariant for $\eta^1_t$ and for which $\eta^1_t$ restricted to the component is uniquely ergodic, i.e. has just one invariant and hence ergodic measure. Then any invariant measure on $\mathbb{P} \mathbb{T}^2$ can be expressed as a convex combination of the measure on these components. In fact for each $x \in S^1$ the subset $\mathbb{P} \times \{(x) \times S^1\} \equiv \mathbb{P}^1 \times \{(x,y) : y \in S^1\}$ of $\mathbb{P}^1 \times \mathbb{T}^2 \equiv \mathbb{P}M$ is invariant for $\eta^1_t$, and unless $x = \pi/2$ or $3\pi/2$, the measure $\delta_{\text{NS}} \otimes \rho_x$ is uniquely ergodic for $\eta^1_t$. (Here, $\rho$ is the normalised Lebesque measure on $\{x\} \times S^1$ and $\delta_{\text{NS}}$ is the unit mass on the NS pole of $\mathbb{P}^1$, if $\mathbb{P}^1$ is regarded as $S^1$ with antipodal points identified. To see that $\delta_{\text{NS}} \times \rho_x$ is uniquely ergodic note that the derivative of $\xi^1_t$ is given by the matrix
1,
\[1, 0\] (cost 1). For \( x = \pi/2 \) or \( 3\pi/2 \) the ergodic measures are \( \delta_v \otimes \rho_x \) for each \( v \in \mathbb{P}^1 \). Thus the only invariant measure \( \nu \) on \( \mathbb{P}^1 \times \mathbb{T}^2 \) for which \( \pi(\nu) = \rho \), is given by \( \nu = \delta_{\text{NS}} \otimes \rho \) (n.b. we do not have to worry about \( x = \pi/2 \) or \( 3\pi/2 \) because \( \{\pi/2, 3\pi/2\} \times S^1 \) has \( \rho \)-measure zero in \( \mathbb{T}^2 \).)

By a similar argument, the same measure \( \delta_{\text{NS}} \otimes \rho \) is the only one on \( \mathbb{P}^1 \mathbb{T}^2 \) which projects to \( \rho \) and which is invariant for \( \eta_i^2 \). Also, the only measure on \( \mathbb{P} \mathbb{M} \) which projects to \( \rho \) and is invariant for \( \eta_i^3 \) and \( \eta_i^4 \) is \( \delta_{\text{EW}} \otimes \rho \), and so we see that there is no measure common to \( i = 1, \ldots, 4 \).

Notes

Baxendale (private communication) has calculated \( \lambda \) for Example 3.4 explicitly using the formula of [4]. Note that this example still satisfies our criterion for \( \lambda > 0 \) if we throw in more divergence-free noise vector fields.

Example 3.5 (Random twisting of a sphere)

Take \( M = S^2 \) and choose a diameter (axis). Take the drift vector field \( A \) of \( (X, Z) \) to be zero, and take the noise vector fields \( Y^1, Y^2, \ldots, Y^n \) to give rotations of \( S^2 \) about the axis, with speeds depending on the angle between the radius at a point in \( S^2 \), and the axis (i.e. (smooth) twists), and determined by functions \( f^1, \ldots, f^n : (\pi, \pi) \rightarrow \mathbb{R} \). Assume that for each angle \( \theta \) we have \( F^i(\theta) \neq 0 \) for some \( i \), and that the set \( \bigcup \{ \theta : (F^i)^{-1} = 0 \} \) has Lebesgue measure. By an argument similar to that of Example 3.4 we see that the only measure on \( \mathbb{P}S^2 \) which is invariant for the flows induced from \( Y^1, \ldots, Y^n \) and which projects to normalised Lebesgue measure in \( S^2 \), is concentrated on the EW pole in each fibre, if we take the axis to be the NS axis.

Now take a different axis, and a similar collection \( Y^1, \ldots, Y^\ell \) of vector fields corresponding to this axis. Then the system on \( S^2 \) with noise vector fields \( Y^1, \ldots, Y^n, \tilde{Y}^1, \ldots, \tilde{Y}^\ell \) has \( \lambda > 0 \), since no measure on \( \mathbb{P}S^2 \) is invariant for the induced flow of all of these vector fields.
Note

If the flow of a vector field on a manifold $M$ does not have closed orbits, then it might be rather difficult to find the measure which are invariant for the induced flow on $PM$. 
References


