REDUCED TU INTERVAL GAMES

By

Ezio Marchi and Martín Matons

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436
Phone: 612-624-6066 Fax: 612-626-7370
URL: http://www.ima.umn.edu

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By

EZIO MARCHI

Universidad Nacional de San Luis-Conicet
Ejército de los Andes 950, San Luis, 5700, Argentina.
emarchi1940@gmail.com

MARTÍN MATONS

Universidad Tecnológica Nacional-FRM-Conicet
Cnel. Rodriguez 273, Mendoza, 5500, Argentina.
mmatons@unsl.edu.ar

Abstract. Recently, Tijs et al. [9],[10],[11] introduced the notion of interval game which generalized the classical concept of TU games. Here we add a further restriction over the profit that some coalitions might obtain. We define and describe the reduced core, giving same properties of it. We also provide a standard sufficient and necessary condition for the non-emptiness of the reduced core.

Key words. Interval games, reduced core, minimal and balanced coalition

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1. Introduction

In some recent publications of Tijs et al., they study properties of interval valued games which have been introduced by Yager and Kreinovich [14]. Tijs et al. consider as preliminaries the concept of interval valued games as a pair $\langle N, v \rangle$, where $v : 2^N \to I(\Re)$, $I(\Re)$ is the set of closed intervals in the real numbers, 2^N is the set of all subsets of $N = \{1,...,n\}$. Moreover $v(\emptyset) = [0,0]$, where $\emptyset$ is the empty set. For each $S \in 2^N$ the worth set $v(S) = [\underline{v}(S), \bar{v}(S)]$, with $\underline{v}(S) \leq \bar{v}(S)$ are the extreme points of the interval, is the set that the coalition S will expect to get its rewards.

In this paper we extend the notion of interval valued games to multi-interval valued games (MVG^N), which generalized them, not only the definition but also with many properties. By definition a MVG^N is a $\langle N, v, w \rangle$, where $N = \{1,...,n\}$ is the set of players, $v : 2^N \to I(\Re)$, $w : Q \subset 2^N \to I(\Re)$, and Q is a subfamily of coalitions in 2^N. The interval $v(S) = [\underline{v}(S), \bar{v}(S)]$ is the range that the coalition will obtain by itself, that is to say $\sum_{i \in S} x_i \geq v(S) \in [\underline{v}(S), \bar{v}(S)]$, where x_i is the outcome of player $i \in N$. On the other hand $\sum_{i \in S} x_i \leq w(S) \in [\underline{w}(S), \bar{w}(S)]$, are the upper bounds of the possible outcomes. Marchi and Matons [3] introduced a second value $w(S)$, for each coalition $S \in Q$, that will indicate the upper bound of the values to be obtained by the coalition S. Consider a pair of functions $\lambda_S : 2^N - \{\emptyset\} \to \Re^+$, $\tilde{\lambda}_S : Q - \{\emptyset\} \to \Re^+$. If the pair $(\lambda_S, \tilde{\lambda}_S)$ fulfills $\sum_{\substack{S \in P(N) \\ i \in S}} \lambda_S e^S - \sum_{\substack{S \in Q \\ i \in S}} \tilde{\lambda}_S e^S = e^N$ then is called a balanced map. Here e^S is the characteristic function of the coalition S: $e_i^S = \begin{cases} 1 & i \in S \\ 0 & i \in N - S \end{cases}$.

2. The Model

Let $\langle N, v, w\rangle$ be a MVG^N, the function $\tilde{v}: 2^N \rightarrow \Re$ is a selection of v if $\tilde{v}(S) \in v(S)$ for each coalition $S \subset N$, and $Sel(v) = \{\tilde{v} / \tilde{v}(S) \in v(S), \forall S \in 2^N\}$. In the same way we define a selection $\tilde{w}$ for w. Consider the imputation set introduced in Marchi-Matons given by:

$$I(\tilde{v}, \tilde{w}) = \left\{ x / \sum_{\substack{S \in 2^N \\ i \in S}} e_S x_i - \sum_{\substack{S \in Q \\ i \in S}} e_S x_i = e_N, \forall S \subset N \right\}$$

and the multi-valued imputation set:

$I(v, w) = \cup\{I(\tilde{v}, \tilde{w}) / \tilde{v} \in Sel(v) \wedge \tilde{w} \in Sel(w)\}$.

In the same way the multi-valued core or simply core is:

$$C(\tilde{v}, \tilde{w}) = \left\{ x \in I(\tilde{v}, \tilde{w}) / \sum_{i \in N} x_i = \tilde{v}(N), \sum_{\substack{S \in 2^N \\ i \in S}} x_i \geq \tilde{v}(S), \sum_{\substack{S \in Q \\ i \in S}} x_i \leq \tilde{w}(S), \tilde{v} \in Sel(v), \tilde{w} \in Sel(w) \right\}$$

and $C(v, w) = \cup\{C(\tilde{v}, \tilde{w}) / \tilde{v} \in Sel(v) \wedge \tilde{w} \in Sel(w)\}$.

We say that a MVG^N is strongly balanced when for each balanced map $(\lambda, \tilde{\lambda})$ it holds $\sum_{\substack{S \in 2^N \\ i \in S}} \lambda_S \bar{v}(S) - \sum_{\substack{S \in Q \\ i \in S}} \tilde{\lambda}_S \underline{w}(S) \leq \underline{v}(N)$, and we say that a MVG^N is strongly unbalanced if it satisfies $\sum_{\substack{S \in 2^N \\ i \in S}} \lambda_S \underline{v}(S) - \sum_{\substack{S \in Q \\ i \in S}} \tilde{\lambda}_S \bar{w}(S) > \bar{v}(N)$. Then it follows the next result

Proposition 1: Let $\langle N, v, w\rangle$ be a MVG^N, then the next three assertions are equivalent:

i). For each selection $\tilde{v} \in Sel(v)$ and $\tilde{w} \in Sel(w)$ the MVG^N $\langle N, \bar{v}, \bar{w}\rangle$ is balanced.

ii). For each selection $\tilde{v} \in Sel(v)$ and $\tilde{w} \in Sel(w)$ the core $(\bar{v}, \bar{w})$ is non-empty.

iii). The MVG^N $\langle N, v, w\rangle$ is strongly balanced.

Proof:

i) iff ii) is proved in Marchi-Matons

i) implies iii) We will prove it by the absurd, for that suppose the MVG^N is strongly unbalanced, then:

$$\sum_{\substack{S \in 2^N \\ i \in S}} \lambda_S \tilde{v}(S) - \sum_{\substack{S \in Q \\ i \in S}} \tilde{\lambda}_S \tilde{w}(S) \geq \sum_{\substack{S \in 2^N \\ i \in S}} \lambda_S \underline{v}(S) - \sum_{\substack{S \in Q \\ i \in S}} \tilde{\lambda}_S \bar{w}(S) > \bar{v}(N) \geq v(N).$$

Therefore the MVG^N $\left\langle N,\bar{v},\bar{w}\right\rangle$ is not balanced for any selection pair $(\tilde{v},\tilde{w})$.

iii) implies i) Take MVG^N $\left\langle N,v,w\right\rangle$ strongly balanced, then

$$\sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S\tilde{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S\tilde{w}(S)\leq\sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S\bar{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S\underline{w}(S)\leq\bar{v}(N)\leq\tilde{v}(N)$$

from which we derive that for each selection $\tilde{v}\in Sel(v)$ and $\tilde{w}\in Sel(w)$ the MVG^N $\left\langle N,\bar{v},\bar{w}\right\rangle$ is balanced.

(q.e.d)

As Tijs et al. introduced the sum of intervals, we remind that a interval n-tuple $I=\{I_1,...,I_n\}$, where $I_i\in I(\Re)$ is a real close interval and we denote by $I(\Re)^N$ the set of all intervals $I=\{I_1,...,I_n\}$. According with Moore[13], we have for each $S\in 2^N-\{\emptyset\}$, $\sum_{i\in S}I_i=\left[\sum_{i\in S}\underline{I}_i,\sum_{i\in S}\bar{I}_i\right]\in I(\Re)$. The multi-interval imputation set $I(v,w)$ of the MVG^N $\left\langle N,v,w\right\rangle$ is defined by $I(v,w)=\left\{(I_1,...,I_n)\in I(\Re)^N/\sum_{i\in N}I_i=v(N),w(i)\preceq I_i\ \forall i\in N\right\}$. Tijs et al. showed that this definition generalizes the imputation concept in the cooperative game theory. We now define the core:

$$C(v,w)=\left\{(I_1,...,I_n)\in I(\Re)^N/\sum_{i\in N}I_i=v(N),\sum_{\substack{i\in S\\ S\in 2^N}}I_i\succeq v(S),\sum_{\substack{i\in S\\ S\in Q}}I_i\preceq w(S),\forall i\in N\right\}.$$

Now we introduced the following

Definition: consider a coalition family $S\in 2^N$ of subsets then we say that it is I-balanced if for each balanced map $(\lambda_S,\tilde{\lambda}_S)$, we have

$$\sum_{\substack{S\in P(N)\\ i\in S}}\lambda_S v(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S w(S)\preceq v(N).$$

The next result appears naturally

Proposition 2: Let $\left\langle N,v,w\right\rangle$ be a MVG^N strongly balanced then it is I-balanced.

Proof: Consider a MVG^N strongly balanced then we have

$$\sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S\underline{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S\underline{w}(S)\leq\sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S\bar{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S\underline{w}(S)\leq\underline{v}(N)$$

and

$$\sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S \bar{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S \bar{w}(S)\le \sum_{\substack{S\in 2^N\\ i\in S}}\lambda_S \bar{v}(S)-\sum_{\substack{S\in Q\\ i\in S}}\tilde{\lambda}_S \underline{w}(S)\le \underline{v}(N)$$

therefore the game is I-balanced.

(q.e.d)

The relation of the core of the games is given by

Proposition 3: Let $\langle N,v,w\rangle$ be a MVG^N, then the next two statements are equivalent

i). The core of $\langle N,v,w\rangle$ is non-empty.

ii). The game $\langle N,v,w\rangle$ is I-balanced.

Proof: We note that the core is non-empty if and only if the following equalities are simultaneously reached:

$$\underline{v}(N)=\min\left\{\sum_{i\in N}\underline{I}_i \,/\sum_{i\in S}\underline{I}_i\ge \underline{v}(S),\forall S\in 2^N-\{\varnothing\}\right\}$$

subject to:

$$\begin{aligned}&\sum_{i\in S}\underline{I}_i\ge \underline{v}(S) \qquad \forall S\in 2^N\\ &\sum_{i\in S}\underline{I}_i\le \underline{w}(S) \qquad \forall S\in Q\end{aligned} \qquad\qquad \text{(A)}$$

and

$$\bar{v}(N)=\min\left\{\sum_{i\in N}\bar{I}_i \,/\sum_{i\in S}\bar{I}_i\ge \bar{v}(S),\forall S\in 2^N-\{\varnothing\}\right\}$$

subject to:

$$\begin{aligned}&\sum_{i\in S}\bar{I}_i\ge \bar{v}(S) \qquad \forall S\in 2^N\\ &\sum_{i\in S}\bar{I}_i\le \bar{w}(S) \qquad \forall S\in Q\end{aligned} \qquad\qquad \text{(B)}$$

Then, by the duality theorem of the linear programming we have that (A) holds true if and only if

$$\max\sum_{i\in S}y_S\,\underline{v}(S)-\sum_{i\in S}\tilde{y}_S\,\underline{w}(S) \qquad \forall S\in 2^N$$

subject to:

$$\begin{aligned}&\sum_{i\in S}y_S e^S-\sum_{i\in S}\tilde{y}_S e^S=e^N \qquad \forall S\in 2^N\\ &y_S,\tilde{y}_S\ge 0\end{aligned} \qquad\qquad \text{(A dual)}$$

and analogously

$$\max \sum_{i\in S} y'_S \bar{v}(S) - \sum_{i\in S} \tilde{y}'_S \bar{w}(S) \quad \forall S \in 2^N$$

subject to:

$$\begin{aligned} &\sum_{i\in S} y'_S e^S - \sum_{i\in S} \tilde{y}'_S e^S = e^N \qquad \forall S \in 2^N \\ &y'_S, \tilde{y}'_S \geq 0 \end{aligned} \qquad \text{(B dual)}$$

On the other hand the system (A dual) is non empty if and only if

$$\sum_{i\in S} y_S \underline{v}(S) - \sum_{i\in S} \tilde{y}_S \underline{w}(S) \leq \underline{v}(N) \qquad \forall S \in 2^N$$

and in the same way, (B dual) is non empty if and only if

$$\sum_{i\in S} y'_S \bar{v}(S) - \sum_{i\in S} \tilde{y}'_S \bar{w}(S) \leq \bar{v}(N) \qquad \forall S \in 2^N$$

These last two inequalities together say that the game is I-balanced.

(q.e.d)

3. About the families of coalitions

In this section we obtain some results concerning to the behaviour of families of coalitions I-balanced.

Proposition 4: Union of two families of I-balanced coalitions, is I-balanced

Proof: Consider two I-balanced collections $C = \{S_1,...,S_m\}$ and $D = \{T_1,...,T_n\}$ with balanced vectors $(y_{S_j}, \tilde{y}_{S_j})$ and $(z_{T_j}, \tilde{z}_{T_j})$ respectively. Let $C \cup D = \{R_1,...,R_q\}$ with $q \leq m+n$ and consider $Q_C = \{S_j / S_j \in C \cap Q\}$ and $Q_D = \{T_j / T_j \in D \cap Q\}$.

For each t, $0 < \mathrm{t} < 1$, we define:

$$w_{R_j} = \begin{cases} t\, y_{S_j} & R_j = S_j \in C - D \\ (1-t)\, z_{T_p} & R_j = T_p \in D - C \\ t\, y_{S_j} + (1-t)\, z_{T_p} & R_j = S_j = T_p \in C \cap D \end{cases} \qquad (3.1)$$

and

$$\tilde{w}_{R_j} = \begin{cases} t\, \tilde{y}_{S_j} & R_j = S_j \in Q_C - Q_D = Q_{C-D} \\ (1-t)\, \tilde{z}_{T_p} & R_j = T_p \in Q_D - Q_C = Q_{D-C} \\ t\, \tilde{y}_{S_j} + (1-t)\, \tilde{z}_{T_p} & R_j = S_j = T_p \in Q_C \cap Q_D = Q_{C\cap D} \end{cases} \qquad (3.2)$$

In order to prove the theorem we claim that the next equality holds true:

$$\sum_{R_j \in (C\cup D)} w_{R_j} v(R_j) - \sum_{R_j \in (C\cup D)} \tilde{w}_{R_j} w(R_j) \preceq v(N) \qquad \forall i \in N \qquad (3.3)$$

for if, we have

$C \cup D = (C - D) \cup (D - C) \cup (C \cap D)$

and

$Q_{C \cup D} = Q_{C-D} \cup Q_{D-C} \cup Q_{C \cap D}$

therefore we have:

$$\sum_{\substack{R_j \in (C-D) \\ i \in R_j}} w_{R_j} e^{R_k} + \sum_{\substack{R_j \in (D-C) \\ i \in R_j}} w_{R_j} e^{R_k} + \sum_{\substack{R_j \in (D \cap C) \\ i \in R_j}} w_{R_j} e^{R_k} - \sum_{\substack{R_j \in Q_{(C-D)} \\ i \in R_j}} \tilde{w}_{R_j} e^{R_k} - \sum_{\substack{R_j \in Q_{(D-C)} \\ i \in R_j}} \tilde{w}_{R_j} e^{R_k} - \sum_{\substack{R_j \in Q_{(D \cap C)} \\ i \in R_j}} \tilde{w}_{R_j} e^{R_k}$$

from here substituting the $w's$ and $\tilde{w}'s$ by mean of the equalities (3.1)-(3.2) we derive the equality (3.3).

(q.e.d.)

From here, by induction, the union of any number of I-balanced families is I-balanced.

Proposition 5: Let us consider two I-balanced families of subsets of N, $C = \{S_1,..., S_k\}$ and $D = \{S_1,..., S_k, S_{k+1},...S_m\}$, i.e. $C \subset D$, with I-balanced vectors $(y_{S_j}, \tilde{y}_{S_j})$ and $(z_{S_j}, \tilde{z}_{S_j})$ respectively, such that they fulfill:

a). $C \subset D, C \neq D$.

b). $S_J \in Q \cap D$ implies $S_j \in C$.

c). $\sum_{\substack{S \in C \\ i \in S}} y_S e^S - \sum_{\substack{S \in Q_C \\ i \in S}} \tilde{y}_S e^S = \sum_{\substack{S \in D \\ i \in S}} z_S e^S - \sum_{\substack{S \in Q_D \\ i \in S}} \tilde{z}_S e^S \qquad \forall i \in N$.

then there exists a I-balanced family B such that $B \cup C = D$ and $D \neq B$.

Proof: For $t > 0$, we define:

$$w_{S_j} = \begin{cases} (1+t) z_{S_j} - t y_{S_j} & S_j \in C. \\ (1+t) z_{S_j} & S_j \in D - C. \end{cases}$$

$$\tilde{w}_{S_j} = \begin{cases} (1+t) \tilde{z}_{S_j} - t \tilde{y}_{S_j} & S_j \in Q_C \cap Q_D \\ (1+t) \tilde{z}_{S_j} & S_j \in Q_D \textit{ and } S_j \notin Q_C \end{cases}$$

For t small, it results that w_{S_j} and $\tilde{w}_{S_j}$ are strictly positive for all j. Furthermore, it holds true for each $i \in N$:

$$\sum_{\substack{S_j \in D \\ i \in S_j}} w_{S_j} e^{S_j} - \sum_{\substack{S_j \in Q_D \\ i \in S_j}} \tilde{w}_{S_j} e^{S_j} =$$

$$\sum_{\substack{S_j \in C \\ i \in S_j}} [(1+t) z_{S_j} - t y_{S_j}] e^{S_j} + \sum_{\substack{S_j \in D-C \\ i \in S_j}} (1+t) z_{S_j} e^{S_j} - \sum_{\substack{S_j \in Q_C \\ i \in S_j}} [(1+t) \tilde{z}_{S_j} - t \tilde{y}_{S_j}] e^{S_j} - \sum_{\substack{S_j \in Q_{D-C} \\ i \in S_j}} (1+t) \tilde{z}_{S_j} e^{S_j} =$$

$$(1+t)\left(\sum_{\substack{S_j\in D\\ i\in S_j}} z_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_D\\ i\in S_j}}\tilde{z}_{S_j}e^{S_j}\right)+(-t)\left(\sum_{\substack{S_j\in C\\ i\in S_j}} y_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_C\\ i\in S_j}}\tilde{y}_{S_j}e^{S_j}\right)=(1+t)e^N-te^N=e^N$$

Then $(w_{S_j},\tilde{w}_{S_j})$ is an I-balanced vector for D. Besides, there exist an j, $1\le j\le k$, such that $z_{S_j}<y_{S_j}$. Assume the contrary, then for each j, $1\le j\le k$, $z_{S_j}\ge y_{S_j}$. Therefore for each $i\in N$:

$$e^N=\sum_{\substack{S_j\in C\\ i\in S_j}} y_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_C\\ i\in S_j}}\tilde{y}_{S_j}e^{S_j}=\sum_{\substack{S_j\in C\cap Q_C\\ i\in S_j}} y_{S_j}e^{S_j}+\sum_{\substack{S_j\in C-Q_C\\ i\in S_j}} y_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_C\\ i\in S_j}}\tilde{y}_{S_j}e^{S_j}\le$$

$$\sum_{\substack{S_j\in C\cap Q_C\\ i\in S_j}} y_{S_j}e^{S_j}+\sum_{\substack{S_j\in C-Q_C\\ i\in S_j}} z_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_C\\ i\in S_j}}\tilde{y}_{S_j}e^{S_j}<\sum_{\substack{S_j\in C\cap Q_C\\ i\in S_j}} z_{S_j}e^{S_j}+\sum_{\substack{S_j\in C-Q_C\\ i\in S_j}} z_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_C\\ i\in S_j}}\tilde{y}_{S_j}e^{S_j}<$$

$$\sum_{\substack{S_j\in D\\ i\in S_j}} z_{S_j}e^{S_j}-\sum_{\substack{S_j\in Q_D\\ i\in S_j}}\tilde{z}_{S_j}e^{S_j}=e^N$$

which is a contradiction.

Let $\bar{t}=\min\left\{\frac{z_{S_j}}{y_{S_j}-z_{S_j}}/\,y_{S_j}>z_{S_j}, S_j\notin Q\right\}$

and consider the set

$$C'=\left\{S_j\in C/(1+\bar{t})z_{S_j}=\bar{t}\,y_{S_j}\right\}.$$

Clearly C' is nonempty subcollection of C. Define the collection $B=D-C'$, then we have $D\ne B$ and $B\cup C=D$. The vector $(w_{S_j},\tilde{w}_{S_j})$, for $t=\bar{t}$, is an I-balanced vector for B. (q.e.d)

4. Minimal Collections

In this section we are going to study same properties of the structure of minimal collections.

Definition: A I-balanced collection B with I-balanced vector $(y_{S_j},\tilde{y}_{S_j})$ is minimal if $B'=\{S\in B/S\notin Q\}$ does not have a proper subcollection B'' such that $B''\cup(B\cap Q)$ is I-balanced collection with I-balanced vector $(z_{S_j},\tilde{z}_{S_j})$, $S_j\in B''\cup(B\cap Q)$, which verifies

$$\sum_{\substack{S\in B\cap Q\\ i\in S}} y_S e^S-\sum_{\substack{S\in B\cap Q\\ i\in S}}\tilde{y}_S e^S=\sum_{\substack{S\in B\cap Q\\ i\in S}} z_S e^S-\sum_{\substack{S\in B\cap Q\\ i\in S}}\tilde{z}_S e^S \quad \forall i\in N \tag{4.1}$$

Proposition 6: Any I-balanced collection is union of minimal I-balanced collections.

Proof: We will prove it by induction on the number of subsets of the family. For $m = 1$, the only member in the collection is {N} and it is minimal. The theorem is true.

Assume that the theorem is true for all collections with $(m-1)$ or less elements. Let D be an I-balanced collection with m elements. If D is minimal there is nothing to prove. If not, consider the collection $D' = \{S \in D / S \notin Q\}$, which contains a proper subcollection D'' such that $D'' \cup (D \cap Q)$ is I-balanced collection satisfying the hypothesis of proposition 5. Therefore there exists a proper subcollection $B \subset D$ such that $B \cup D'' \cup (D \cap Q) = D$. Since B and D'' are proper subcollections of D, they have at most $(m-1)$ elements and therefore each one of them can be express as union of minimal subcollections and the proposition is true.

(q.e.d.)

Next we will introduce a new notion. We said that an I-balanced collection C, has a distinguished I-balanced vector, if for each pair $(\lambda_S, \tilde{\lambda}_S)$ and $(\gamma_S, \tilde{\gamma}_S)$ of balancing vectors such that

$$\sum_{\substack{i \in S \\ S \in C \cap Q}} \lambda_S e^S - \sum_{\substack{i \in S \\ S \in C \cap Q}} \tilde{\lambda}_S e^S = \sum_{\substack{i \in S \\ S \in C \cap Q}} \gamma_S e^S - \sum_{\substack{i \in S \\ S \in C \cap Q}} \tilde{\gamma}_S e^S \qquad (4.2)$$

it holds true $\lambda_S = \gamma_S$ for each $S \in C - Q$.

Proposition 7: Under the conditions of proposition 5 an I-balanced collection has a distinguished I-balanced vector if and only if it is minimal.

Proof: The proposition 5 says that an I-balanced vector is distinguished only for I-balanced minimal collections. Reciprocally, assume that D does not have a distinguished balanced vector, then there exist vectors $(y_S, \tilde{y}_S)$ and $(z_S, \tilde{z}_S)$ verifying (4.2) but $y_S \neq z_S$ for some $S \in C - Q$. Without lost of generality we assume that $y_S > z_S$, choosing $w = (1+\bar{t})z_S - \bar{t}\, y_S$ where

$$\bar{t} = \min\left\{ \frac{z_S}{y_S - z_S} / \, y_S > z_S, S \in D - Q \right\}.$$

Therefore w is an I-balanced vector for the collection $B = \{S \in D - Q / (1+\bar{t})z_S > \bar{t}\, y_S\} \cup (D \cap Q)$ by virtue of proposition 5. Since B is a proper subcollection of D, and

$$\sum_{\substack{S \in B \cap Q \\ i \in S}} w_S e^S - \sum_{\substack{S \in B \cap Q \\ i \in S}} \tilde{w}_S e^S = \sum_{\substack{S \in D \cap Q \\ i \in S}} w_S e^S - \sum_{\substack{S \in D \cap Q \\ i \in S}} \tilde{w}_S e^S =$$

$$\left(\sum_{\substack{S\in D\cap Q\\ i\in S}} z_S e^S - \sum_{\substack{S\in D\cap Q\\ i\in S}} \tilde{z}_S e^S\right) + \bar{t}\left(\sum_{\substack{S\in D\cap Q\\ i\in S}} z_S e^S - \sum_{\substack{S\in D\cap Q\\ i\in S}} \tilde{z}_S e^S\right) - \bar{t}\left(\sum_{\substack{S\in D\cap Q\\ i\in S}} y_S e^S - \sum_{\substack{S\in D\cap Q\\ i\in S}} \tilde{y}_S e^S\right).$$

By (3.3), D is not minimal.

(q.e.d)

Proposition 8: Consider a vector $(y_S, \tilde{y}_S)$ that satisfies (A)-(B), then it is an I-balanced vector for a given I-balanced minimal collection.

Proof: Consider a vector $(y_S, \tilde{y}_S)$ fulfilling the hypothesis of the proposition; such vector is I-balanced for the collection $C = \{S \notin Q / y_S > 0\} \cup Q$. Assume that C is not minimal, then there exists a proper subcollection B of C with balanced vector $(z_S, \tilde{z}_S)$. If $z_S > 0$, $S \notin Q$, then for small values of t, both $w = (1-t)y + tz$ and $w' = (1-t)y - tz$ satisfy (A) and (B). Moreover $w \neq w'$ over $C - B$. But $y = \frac{1}{2}(w + w')$ which implies that y is not an extremal of (A)-(B).

(q.e.d)

5. Conclusion

In this paper we generalize the concept of interval game introduced by Moore [13] and studied extensively by Tijs et al. [9][10][11]. We have introduced an upper function for the study of the characteristic function, in a similar way that was done by Cantisani, Matons and Marchi [2][3].

The results obtained in this paper generalize the notion of core for the classical theory of TU-games. We derive some extension of classical and existential theorems regarding the basic Shapley-Bondareva and Tijs et al.

6. Appendix

A TU-modified game is introduced as follows:

Let N be finite set of players such that $\#(\mathrm{N}) = \mathrm{n}$. Let $\mathrm{A} = (\gamma_{iS}, \delta_{iS})$ be a matrix of order nm with $m \leq (2^n - 1)(2^n - 2)$, and for each $i \in N$ and $S \in 2^N - \{\varnothing\}$, γ_{iS} and δ_{iS} are positive. Here 2^{N} indicates the family of all the subsets of N and $\varnothing$ be the empty set.

Given the characteristic function v defined for a family $F \subseteq 2^N$ and the ceiling function w defined for some family $Q \subset 2^N$ and $w(S) \geq v(S)$ we denote with $\mathrm{G} = (\mathrm{N}, \mathrm{v}, \mathrm{w})$ a TU-modified game.

The $Q_{\gamma\delta}$ *Core* and $Q_{\gamma\delta}$ *balanceness*.

Let N be the set of players and consider two families of non-empty subsets of N, F and Q, with $Q \subset F$. We define the modified core as the set of imputations $(x_1,...,x_n)$ such that:

$$\sum_{i=1}^{n} \gamma_{iN} x_i = v(N)$$

$$\sum_{i \in S} \gamma_{iS} x_i \geq v(S) \qquad S \in F$$

$$\sum_{i \in S} \delta_{iS} x_i \leq w(S) \qquad S \in Q$$

In the case that $\mathrm{v(N)} < \mathrm{w(S)}$, for each $S \in Q$, and $\mathrm{F} = \mathrm{P(N)}$ we have the usual Core of the TU games with Billera´s weights. In the case that for every $\mathrm{S} \in \mathrm{Q}, \mathrm{v(S)} = \mathrm{w(S)}$ we obtain the weighted Core introduced by Cantisani and Marchi.

We have the following characterization:

Theorem 2.1: The $Q_{\gamma\delta}\,Core$ of a modified TU game $\mathrm{G} = (\mathrm{N}, \mathrm{v}, \mathrm{w})$ is non-empty if and only if the linear program

$$\min_{x} \sum_{i=1}^{n} \gamma_{iN} x_i = z^* \tag{6.1}$$

subject to

$$\sum_{i \in S} \gamma_{iS} x_i \geq v(S) \qquad S \in F \tag{6.2}$$

$$\sum_{i \in S} \delta_{iS} x_i \leq w(S) \qquad S \in Q \tag{6.3}$$

has a minimum $\mathrm{z}^* \leq \mathrm{v(N)}$.

The reader would see that if $\mathrm{v(S)} > \mathrm{w(S)}$ for some $\mathrm{S} \in \mathrm{Q}$, then the core is empty. Next we have, by the duality theorem of linear programming, the following result:

Theorem 2.2: The dual problem of (6.1)-(6.3) is

$$\max_{y,y'} \left(\sum_{S \in F} y_S v(S) - \sum_{S \in Q} y'_S \, w(S) \right) = q^* \tag{6.4}$$

subject to

$$\sum_{\substack{S \in F \\ i \in S}} \gamma_{iS} y_S - \sum_{\substack{S \in Q \\ i \in S}} \delta_{iS} y'_S = \gamma_{iN} \qquad i \in N \tag{6.5}$$

$$y_S, y'_S \geq 0 \tag{6.6}$$

As a consequence of the previous considerations we also have:

Theorem 2.3: A necessary and sufficient condition in order that the TU modified game $\mathrm{G} = (\mathrm{N}, \mathrm{v}, \mathrm{w})$ has non empty $Q_{\gamma\delta}\, Core$ is that for each vector (y_S, y'_S), $\mathrm{S} \in \mathrm{F}$, satisfying (6.4)-(6.6) verifies

$$\sum_{S \in F} y_S v(S) - \sum_{S \in Q} y'_S\, w(S) \leq v(N).$$

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8. Bibliography

[1] Bondareva O. N. [1962] "Theory of the core in an n-person game", *Vestnik Leningradskü Universitet 13*, 141-142 (in Russian).

[2] Marchi E. and Cantisani M. [2004] "The weighted core with distinguished coalitions", *International Game Theory Review*, Vol. 6, No. 2, 239-246.

[3] Marchi E. and Matons M. [2008] "TU-Reduced Games and Their Cores". International Journal of Applied Mathematics, Game Theory and Algebra. Vol. 17 Issue 5/6.

[4] Marchi E. and Auriol I. [2002] "Quasi-assignment cooperative games", *International Game Theory Review*, Vol. 4, No. 2, 172-182.

[5] Owen G. [1982] *Game Theory* (Academic Press).

[6] Scarf L. [1970] "The core of an n-person game", *Econometrica 35*, 50-69.

[7] Shapley L. [1970} "On balanced games without side payments", *Mathematical Programming*, ed. Hu, 261-290.

[8] Shapley L. and Shubik M. [1972] "Assignment game I. The core", *Int. J. Game Theory I*, 111-130.

[9] Tijs S., Alparslan-Gök, S.Z. and Branzei, R. [2008] "Cores and Stable Sets for Interval – Valued Games", Discussion Paper.

[10] Tijs S., Alparslan-Gök, S.Z. and Branzei, R. [2008] "Convex Interval Games", Discussion Paper.

[11] Tijs S., Alparslan-Gök, S.Z. and Miquel, S. [2008] "Cooperation Under Interval Uncertainty", Discussion Paper.

[12] Tijs S., Dimitrov, D. and Branzei, R. [2008], "Models in Cooperative Game Theory", Springer-Verlag Berlin

[13] Moore, R. [1995] "Methods and Applications of Interval Analysis", SIAM Studies in Applied Mathematics.

[14] Yager, R. and Kreinovich, V. [2000] "Fair division under interval uncertainty, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 8, 611-618.

[15] Billera L. J. [1970] "Some theorems on the core of an n-person game without side payments", *SIAM. J. Appl. Math. 18,* 567-579.

[16] Billera L. J. [1971] "Some recent results in n-person game theory", *Mathematical Programming,* 1, 58-67.

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