FRIENDLY EQUILIBRIA

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ABSTRACT
In game theory there are many discussions about the concept of rationality. Theoretically these discussions are concerned with an appropriate mathematical concept which derives from the concept itself. In this paper we present a new way of looking rationality which is concerned with an external view of the rules of games. We prove the existence of these new friendly equilibrium and perturbed friendly equilibria.

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1 – RESULTS

Here we are going to introduce the general tools of the game. Consider an n-person game in normal form

$$\Gamma = \{\Sigma_i, A_i \quad i \in N\}$$

where the strategy sets are non-empty and finite. The mixed extension is

$$\tilde{\Gamma} = \{\Sigma_i, E_i \quad i \in N\}$$

Now for a player consider a finite sequence of players

$$f^1(i) = i, f^2(i), ..., f^{k_i}(i)$$

which we call the "successor friends" of player $i \in N$. Then from an intuitive point of view it is natural to assume that the player $i \in N$ is "rational": whenever his payoff is the same he is going to act in such way that first he will maximize the payoff of his first successor friend $f^1(i)$. In the case that the payoff of this successor friend $f^1(i)$ is equal again he will act to maximize the payoff of his second successor friend $f^2(i)$ and so on until the last $f^{k_i}(i)$.

Consider for $y = (y_i, y_{-i}) \in \underset{i \in N}{x} \tilde{\Sigma}_i$ the sets

$$\psi_i^1(y) = \{x_i \in \tilde{\Sigma}_i : E_i(x_i, y_{-i}) \geq E_i(z_i, y_{-i}) \quad \forall z_i \in \Sigma_i\}$$
$$\psi_i^2(y) = \{x_i \in \psi_i^1(y) : E_{f^2(i)}(x_i, y_{-i}) \geq E_{f^2(i)}(z_i, y_{-i}) \ \forall z_i \in \psi_i^1(y)\}$$
$$\vdots$$
$$\psi_i^{k_i}(y) = \{x_i \in \psi_i^{k_{i-1}}(y) : E_{f^{k_i}(i)}(x_i, y_{-i}) \geq E_{f^{k_i}(i)}(z_i, y_{-i}) \ \forall z_i \in \psi_i^{k_{i-1}}(y)\}$$

Formally we define a friendly equilibrium point if $\bar{x}_i \in \psi_i^{k_i}(\bar{x}) \quad \forall i = 1, ..., n$.

Theorem 1: Given any game Γ and $f^2(i), ..., f^{k_i}(i)$, if ψ_i^k is upper semi-continuous for each $i \in N$, there always exists a friendly equilibrium.

Proof: given any point $y \in \underset{i \in N}{x} \tilde{\Sigma}_i$ consider the set of points

$$\psi_i^1(y) = \{x_i \in \tilde{\Sigma}_i : E_i(x_i, y_{-i}) \geq E_i(z_i, y_{-i}) \quad \forall z_i \in \tilde{\Sigma}_i\}$$
$$\psi_i^2(y) = \{x_i \in \psi_i^1(y) : E_{f^2(i)}(x_i, y_{-i}) \geq E_{f^2(i)}(z_i, y_{-i}) \ \forall z_i \in \psi_i^1(y)\}$$
$$\vdots$$
$$\psi_i^{k_i}(y) = \{x_i \in \psi_i^{k_{i-1}}(y) : E_{f^{k_i}(i)}(x_i, y_{-i}) \geq E_{f^{k_i}(i)}(z_i, y_{-i}) \ \forall z_i \in \psi_i^{k_{i-1}}(y)\}$$

since the expectations are multi-linear then each $\psi_i^{k_i}(y)$ is non-empty convex and compact. Moreover, by the continuity of the expected functions and the hypothesis such, $\psi_i^{k_I}$ are upper semi-continuous. Therefore using Kakutani's fixed point applied to $\psi(y) = \underset{i \in Ni}{x} \psi_i^{k_i}(y)$ there exists a fixed point $x \in \psi(x)$. Such point is a friendly equilibrium (q.e.d.).

As a simple observation we would like to point out that in the case that $k_i = 1$ for each $i \in N$ our new notion coincides with the fundamental Nash equilibrium points.

It is interesting to remark that the existence of a friendly equilibrium is equivalent to the existence of a solution $\bar{x}$, λ 's of the non-linear system

$$
\begin{array}{ll}
\lambda_i - E_i\left(\sigma_i, \bar{x}_{-i}\right) = 0 & \forall \sigma_i \in S(\bar{x}_i) \\
\lambda_i - E_i\left(\sigma_i, \bar{x}_{-i}\right) \geq 0 & \forall \sigma_i \in \sum_i \\
\lambda_{f^1(i)} - E_{f^1(i)}\left(\sigma_i, \bar{x}_{-i}\right) = 0 & \forall \sigma_i \in S(\bar{x}_i) \\
\lambda_{f^1(i)} - E_{f^1(i)}\left(\sigma_i, \bar{x}_{-i}\right) \geq 0 & \forall \sigma_i \in \psi_i^1(\bar{x}) \\
\lambda_{f^2(i)} - E_{f^2(i)}\left(\sigma_i, \bar{x}_{-i}\right) = 0 & \forall \sigma_i \in S(\bar{x}_i) \\
\lambda_{f^2(i)} - E_{f^2(i)}\left(\sigma_i, \bar{x}_{-i}\right) \geq 0 & \forall \sigma_i \in \psi_i^2(\bar{x}) \\
\vdots & \\
\lambda_{f^k(i)} - E_{f^k(i)}\left(\sigma_i, \bar{x}_{-i}\right) = 0 & \forall \sigma_i \in S(\bar{x}_i) \\
\lambda_{f^k(i)} - E_{f^k(i)}\left(\sigma_i, \bar{x}_{-i}\right) \geq 0 & \forall \sigma_i \in \psi_{(i)}^{k_i - 1}(\bar{x}) \\
\sum_{\sigma_i \in \Sigma_i} \bar{x}_i(\sigma_i) = 1 & \forall i \in N \\
\bar{x}_i(\sigma_i) \geq 0 & \forall i \in N, \ \forall \sigma_i \in \Sigma
\end{array}
$$

where $S(x_i)$ denotes the discrete support of x_i. The proof of the equivalence is rather easy and is left to the reader.

2 – PERTURBED FRIENDLY EQUILIBRIA

Now we will extend the concept of friendly equilibria to the theory of perturbed equilibria. For this, we will follow the approaches of Selten (1975) and Myerson (1978).

Consider a point $\bar{x}_i \in \underset{i \in N}{x} \tilde{\tilde{\Sigma}}_i$ such that it satisfies

b$_1$) $E_i(\sigma_i, \bar{x}_{-i}) < E_i(\bar{\sigma}_i, \bar{x}_{-i}) \Rightarrow \bar{x}_i(\sigma_i) = 0$

$$\theta_i^1(\bar{x}) = \left\{ \sigma_i \in \Sigma_i : E_i(\sigma_i, \bar{x}_{-i}) = \max_{\bar{\sigma}_i \in \Sigma_i} E_i(\bar{\sigma}_i, x_{-i}) \right\}$$

b$_2$) $E_{f^2(i)}(\sigma_i, \bar{x}_{-i}) < E_{f^2(i)}(\sigma_i, \bar{x}_{-i}), \ \sigma_i, \bar{\sigma}_i \in \theta_i^1(\bar{x}) \Rightarrow \bar{x}_i(\sigma_i) = 0$

$$\theta_i^r(\bar{x}) = \left\{ \sigma_i \in \theta_i^{r-1}(\bar{x}) : E_{f^r(i)}(\sigma_i, \bar{x}_{-i}) = \max_{\bar{\sigma}_i \in_i \theta^{r-1}} E(\tilde{\sigma}_i, x_{-i}) \right\} \quad \forall i$$

$\vdots$

b$_{r+1}$) $E_{f^{r+1}(i)}(\sigma_i, \bar{x}_{-i}) < E_{f^{r+1}(i)}(\sigma_i, \bar{x}_{-i}), \ \sigma_i, \bar{\sigma}_i \in \theta_i^r(\bar{x}) \Rightarrow \bar{x}_i(\sigma_i) = 0$

Theorem 2: A point is a friendly equilibrium point if and only if it fulfills B) with $r = k_i - 1$ for each $i = 1,..,n$.

Proof: We prove it by induction on $k_i, \forall i$. For the case $k_i = 1$ for each i, this is very well known. Assume that it is valid for $k_i = r_i$, then we will show that it is also valid for $k_i = r_i + 1$.

For this, consider that $\bar{x}_i \in \psi_i^{r_{i-1}}(\bar{x})$ for each $i = 1,..,n$. Suppose that for i

$$E_{f^{r_i+1}(i)}(\sigma_i,\bar{x}_{-i}) < E_{f^{r_i+1}(i)}(\sigma_i,\bar{x}_{-i}),\ \ \sigma_i,\bar{\sigma}_i \in \theta_i^{r_i}(\bar{x}) \text{ and } \ \bar{x}_i(\sigma_i) > 0$$

Let $\tilde{x}_i \in \tilde{\Sigma}_i$ be the point defined as

$$\tilde{x}_i(\tilde{\sigma}_i) = \begin{cases} \bar{x}_i(\sigma_i) + \tilde{x}_i(\sigma_i) & \tilde{\sigma}_i = \bar{\sigma}_i \\ 0 & \tilde{\sigma}_i = \sigma_i \\ \bar{x}_i(\sigma_i) & \tilde{\sigma}_i \in \theta_i^{r}(\bar{x}) \quad \tilde{\sigma}_i \neq \sigma_i, \bar{\sigma}_i \end{cases}$$

then

$$\begin{aligned} E_{f^{r_i+1}(i)}(\tilde{x}_i,\bar{x}_{-i}) &= \sum_{\tilde{\sigma}_i \in \Sigma_i} x_i(\tilde{\sigma}_i) E_{f^{r_i+1}(i)}(\tilde{\sigma}_i,\bar{x}_{-i}) \\ &= \sum_{\tilde{\sigma}_i \in \theta_i^{r_i}(\bar{x})} \tilde{x}_i(\tilde{\sigma}_i) E_{f^{r_i+1}(i)}(\tilde{\sigma}_i,\bar{x}_{-i}) \\ &> \sum_{\tilde{\sigma}_i \in \theta_i^{r_i}(\bar{x})} \bar{x}_i(\tilde{\sigma}_i) E_{f^{r_i+1}(i)}(\tilde{\sigma}_i,\bar{x}_{-i}) = E_{f^{r_i+1}(i)}(\bar{x}_i,\bar{x}_{-i}) \end{aligned}$$

which is impossible since $\bar{x}_i \in \psi_i^{r_{i+1}}(\bar{x}_i,\bar{x}_{-i})$ and $\overline{\theta_i^{r_i}(\bar{x})}$ is identified with $\psi_i^{r_i}(\bar{x})$.
Inversely, consider a point $\bar{x}$ such that

$$E_{f^{r_i+1}(i)}(\sigma_i,\bar{x}_{-i}) < E_{f^{r_i+1}(i)}(\bar{\sigma}_i,\bar{x}_{-i}),\ \sigma_i,\bar{\sigma}_i \in \theta_i^{r_i}(\bar{x}) \Rightarrow \bar{x}_i(\sigma_i) = 0$$

for each i, then $\bar{x}_i(\sigma_i) > 0$ only if $\sigma_i \in \theta_i^{r_i}(\bar{x})$ and

$$E_{f^{r_i+1}(i)}(\sigma_i,\bar{x}_{-i}) = \max_{\tilde{\sigma}_i \in \theta_i^{r_i}(y)} E_{f^{r_i+1}(i)}(\tilde{\sigma}_i,\bar{x}_{-i})$$

therefore $\bar{x}_i \in \psi_i^{r_{i+1}}(\bar{x}_i)$ for each i.

Then following the ideas of Selten (1975), we define a perfect friendly equilibrium as follows: given $\varepsilon > 0$, a point $x^{-\varepsilon} \in \underset{i \in N}{X} \tilde{\Sigma}_i$ is called a $\in$-perfect friendly equilibria if

$$\left.\begin{aligned} &E_i(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_i(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \\ &E_{f^2(i)}(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_{f^2(i)}(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}), \quad \sigma_i,\bar{\sigma}_i \in \theta_i^1(\bar{x}^{\varepsilon}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \\ &\vdots \\ &E_{f^{k_i}(i)}(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_{f^{k_i}(i)}(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}), \quad \sigma_i,\bar{\sigma}_i \in \theta_i^{k_{i-1}}(\bar{x}^{\varepsilon}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \end{aligned}\right\} \forall i$$

A perfect friendly equilibrium is a limit point of a sequence $\{x^{\varepsilon}\}_{\varepsilon \downarrow 0}$ where x^{ε} is an ε-perfect friendly rational equilibria.

Similarly following the ideas of Myerson (1978) we define ε-proper friendly equilibria as

$$\left.\begin{aligned} &E_i(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_i(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \bar{x}_i^{\varepsilon}(\bar{\sigma}_i) \\ &E_{f^2(i)}(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_{f^2(i)}(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}), \quad \sigma_i,\bar{\sigma}_i \in \theta_i^1(\bar{x}^{\varepsilon}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \bar{x}_i^{\varepsilon}(\bar{\sigma}_i) \\ &\vdots \\ &E_{f^{k_i}(i)}(\sigma_i,\bar{x}^{\varepsilon}_{-i}) < E_{f^{k_i}(i)}(\bar{\sigma}_i,\bar{x}^{\varepsilon}_{-i}), \quad \sigma_i,\bar{\sigma}_i \in \theta_i^{k_{i-1}}(\bar{x}^{\varepsilon}) \Rightarrow \bar{x}_i^{\varepsilon}(\sigma_i) \le \varepsilon \bar{x}_i^{\varepsilon}(\bar{\sigma}_i) \end{aligned}\right\} \forall i$$

Clearly an ε-proper friendly equilibrium is an ε-perfect friendly equilibrium. A point $\bar{x} \in \underset{i\in N}{X} \tilde{\Sigma}_i$ is called a proper friendly equilibria if x is a limit point of a sequence $\{x^{\varepsilon}\}_{\varepsilon \downarrow 0}$ where x^{ε} is an ε-proper friendly equilibrium.

When $k_i = 1$ the M's corresponding concepts become the perfect and proper ones respectively.

Theorem 3: Every normal form game possesses at least one proper friendly equilibrium if $\forall i$, F_i is upper semi-continuous.

Proof: In order to prove this existence theorem let us observe that for given positive integers $n_1, n_2, ..., n_{r_i}$ it always holds true

$$\prod_{s=2}^{r_i} n_s - 1 = \prod_{s=3}^{r_i} n_s (n_2 - 1) + \prod_{s=4}^{r_i} n_s (n_3 - 1) + \prod_{s=5}^{r_i} n_s (n_4 - 1) + \ldots + n_{r_i} (n_{r_i - 1} - 1) + (n_{r_i - 1} - 1)$$

This can be proven by induction on r_i, which we skip. Therefore it is clear that

$$\prod_{s=2}^{r_i} n_s - 1 \geq = \prod_{s=3}^{r_i} n_s a_2 + \prod_{s=4}^{r_i} n_s a_3 + \ldots + n_{r_i} a_{r_i - 1} + a_{r_i} \qquad \text{for } 0 \leq a_r \leq n_r - 1 \quad \forall r$$

Now given $\varepsilon \in (0,1)$, define

$$\eta_i = \frac{\varepsilon^{\prod_{i=1}^{k_i} m_{f^s(i)} + \prod_{i=2}^{k_i} m_{f^s(i)} + \prod_{i=3}^{k_i} m_{f^s(i)} + \ldots + m_{f(i)}^{r_i} m_{f(i)}^{r_i - 1} + m_{f(i)}^{r_i}}}{m_i}$$

where $m_i = |\Sigma_i|$ is the cardinality of Σ_i, and let

$$S_i(\eta_i) = \{x_i \in \tilde{\Sigma}_i : x_i(\sigma_i) \geq \eta_i \quad \forall \sigma_i \in \Sigma_i\}$$

For $i \in N$ define the correspondence F_i from $\prod_{i\in N} S_i(\eta_i)$ by:

$F_i(x) = \{y_i \in S_i(\eta_i)$:

if $E_i(\sigma_i, x_{-i}) < E_i(\bar{\sigma}_i, x_{-i}) \Rightarrow y_i(\sigma_i) \leq \varepsilon y_i(\bar{\sigma}_i)$ for all $\sigma_i, \bar{\sigma}_i$

if $E_{f^2(i)}(\sigma_i, x_{-i}) < E_{f^2(i)}(\bar{\sigma}_i, x_{-i}), \quad \sigma_i, \bar{\sigma}_i \in \theta_i^1(x) \Rightarrow y_i(\sigma_i) \leq \varepsilon y_i(\bar{\sigma}_i)$

$\vdots$

if $E_{f^{k_i}(i)}(\sigma_i, x_{-i}) < E_{f^{k_i}(i)}(\bar{\sigma}, x_{-i}), \quad \sigma_i, \bar{\sigma}_i \in \theta_i^{k_{i-1}}(x) \Rightarrow y_i(\sigma_i) \leq \varepsilon y_i(\bar{\sigma}_i)$

We have that $F_i(x)$ is closed and convex for every x and the mapping F_i is upper semi-continuous.

Now we will prove that $F_i(x) \neq \Phi$. Consider

$$\nu_{f^r(j)}(x, \sigma_j) = \left| \left\{ \bar{\sigma}_j \in \Sigma_j : E_{f^r(j)}(\sigma_j, x_{-j}) < E_{f^r(j)}(\bar{\sigma}_j, x_{-j}) \quad \sigma_j, \bar{\sigma}_j r - \in \theta_j^{r-1}(x) \right\} \right|$$

for all $j \in N$ and $\sigma_j \in \Sigma_j$. Clearly $\nu_j(x, \sigma_j) \leq m_j - 1$.

Let the point

$$y_i(\sigma_i) = \left[\varepsilon^{\prod_{s=2}^{k_i} m_{f^s(i)} \nu_i(x,\sigma_i) + \prod_{s=3}^{k_i} m_{f^s(i)} \nu_{f^2(i)}(x,\sigma_i) + \cdots} \right.$$

$$\left. + m_{f^{r_i}(i)} \nu_{f^{r_i-1}(i)}(x,\sigma_i) + \nu_{f^{r_i}(i)}(x,\sigma_i) \right] \Big/ \sum_{\bar{\sigma}_i \in \Sigma_i} \varepsilon^{\prod_{s=2}^{k_i} m_{f^s(i)} \nu_i(x,\sigma_i) +}$$

$$+ \prod_{s=3}^{k_i} m_{f^s(i)} \nu_{f^2(i)}(x,\sigma_i) + \ldots + m_{f^{r_i}(i)} \nu_{f^{r_i-1}(i)}(x,\sigma_i) + \nu_{f^{r_i}(i)}(x,\sigma_i)$$

Clearly $y_i(\sigma_i) \geq \eta_i$, $\sigma_i \in \Sigma_i$.
Now if $E_i(\sigma_i, x_{-i}) < E_i(\bar{\sigma}_i, x_{-i})$ then $\nu_i(x,\sigma_i) \geq \nu_i(x,\bar{\sigma}_i) + 1$ and

$$y_i(\sigma_i) \leq \frac{\varepsilon\varepsilon^{\left(\prod_{s=2}^{k_i} m_{f^s(i)^{-1}}\right) + \prod_{s=3}^{k_i} m_{f^s(i)} \nu_{f^2(i)} + \ldots + \nu_{f^{r_i}(i)}(x,\sigma_i) + \prod_{s=2}^{k_i} m_{f^s(i)} \nu_i(x,\bar{\sigma}_i)}}{\sum_{\bar{\sigma}_i \in \Sigma_i} \cdots} \leq \tag{1}$$

$$\frac{\varepsilon\varepsilon^{\prod_{s=2}^{k_i} m_{f^s(i)} \nu_i(x,\bar{\sigma}_i) + \prod_{s=3}^{k_i} m_{f^s(i)} \nu_{f^2(i)}(x,\bar{\sigma}_i) + \ldots + \nu_{f^{r_i}(i)}(x,\bar{\sigma}_i)}}{\sum_{\bar{\sigma}_i \in \Sigma_i} \cdots} = \tag{2}$$

$$= \varepsilon y_i(\bar{\sigma}_i)$$

since $\nu_{f^s(j)}(x,\sigma_i) \leq m_{f^s(j)^{-1}}$ and the observation given above.

Now if $E_{f^{r_i}(i)}(\sigma_i, x_{-i}) < E_{f^{r_i}(i)}(\bar{\sigma}_i, x_{-i})$, $\sigma_i, \bar{\sigma}_i \in \theta_i^{r-1}(x)$ then all

$$\nu_{f^s(\ )}(x,\sigma_i) = \nu_{f^s(i)}(x,\bar{\sigma}_i) \qquad \forall s \leq r-1$$

Then performing the same operation as before instead in, in , it is easy to see that

$$y_i(\sigma_i) \leq \varepsilon y_i(\bar{\sigma}_i).$$

Let F be the n-tuple $(F_1,\ldots,F_n)$. Then it satisfies the condition of the Kakutani Fixed Point Theorem (Kakutani, 1941) and the fixed point of F is an ε-proper friendly equilibrium. Now for $\varepsilon_k \to 0$ each ε we have an ε-proper friendly equilibrium, thus making $\varepsilon_k \to 0$, it is always possible to find a converging subsequence since the space is compact. The limit point of such subsequence is a friendly equilibrium point. The proof is complete, then the theorem follows.

As a simple consequence of this result we have

Corollary: Every normal form game has at least one perfect friendly equilibrium.

Finally we would like to say that if the $f^r(i)$ do not belong to N and are given externally, we have a external new notion of equilibrium. And we would like to emphasize that the previous material might be generalized accordingly for the continuous case. In this situation the upper semi-continuity condition appears more natural.

BIBLIOGRAPHY

Kakutani, S. (1941). A generalization of Brewer fixed point theorem. Duke Math. J. 451-459.

Marchi, E. (1991-1992). μ-subgame perfect equilibrium (to appear).

Myerson, R. B. (1978). Refinements of the Nash equilibrium concept. Int. Game Theory 7, 73-80.

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. Int. J. Game Theory 4, 25-55.

Van Damme (1987). Stability and perfection of Nash Equilibria. Springer-Verlag, Berlin – New York.

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