BORELL’S GENERALIZED PRÉKOPA-LEINDLER INEQUALITY: A SIMPLE PROOF

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Borell’s generalized Prékopa-Leindler inequality: A simple proof

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Abstract

We present a simple proof of Christer Borell’s general inequality in the Brunn-Minkowski theory. We then discuss applications of Borell’s inequality to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

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1 Introduction

Let us denote by \(\text{supp}(f)\) the support of a function \(f\). In [6] Christer Borell proved the following inequality (see [6, Theorem 2.1]), which we will call the Borell-Brunn-Minkowski inequality.

**Theorem 1** (Borell-Brunn-Minkowski inequality). Let \(f, g, h : \mathbb{R}^n \to [0, +\infty)\) be measurable functions. Let \(\varphi = (\varphi_1, \ldots, \varphi_n) : \text{supp}(f) \times \text{supp}(g) \to \mathbb{R}^n\) be a continuously differentiable function with positive partial derivatives, such that \(\varphi_k(x, y) = \varphi_k(x_k, y_k)\) for every \(x = (x_1, \ldots, x_n) \in \text{supp}(f), y = (y_1, \ldots, y_n) \in \text{supp}(g)\). Let \(\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)\) be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality

\[
\prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\prod_{k=1}^n \rho_k, \prod_{k=1}^n \eta_k)
\]

(1)

holds for every \(x \in \text{supp}(f)\), for every \(y \in \text{supp}(g)\), for every \(\rho_1, \ldots, \rho_n > 0\) and for every \(\eta_1, \ldots, \eta_n > 0\), then

\[
\int h \geq \Phi \left( \int f, \int g \right).
\]

C. Borell proved a slightly more general statement, involving an arbitrary number of functions. For simplicity, we restrict ourselves to the statement of Theorem 1.

Theorem 1 yields several important consequences. For example, applying Theorem 1 to indicators of compact sets (i.e. \(f = 1_A, g = 1_B, h = 1_{\varphi(A,B)}\)) yields the following generalized Brunn-Minkowski inequality.

**Corollary 2** (Generalized Brunn-Minkowski inequality). Let \(A, B\) be compact subsets of \(\mathbb{R}^n\). Let \(\varphi = (\varphi_1, \ldots, \varphi_n) : A \times B \to \mathbb{R}^n\) be a continuously differentiable function with positive partial derivatives, such that \(\varphi_k(x, y) = \varphi_k(x_k, y_k)\) for every \(x = (x_1, \ldots, x_n) \in A, y = (y_1, \ldots, y_n) \in B\). Let \(\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)\) be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality

\[
\prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\prod_{k=1}^n \rho_k, \prod_{k=1}^n \eta_k)
\]

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holds for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$, then
\[
|\varphi(A, B)| \geq \Phi(|A|, |B|),
\]
where $| \cdot |$ denotes Lebesgue measure and $\varphi(A, B) = \{\varphi(x, y) : x \in A, y \in B\}$.

The classical Brunn-Minkowski inequality (see e.g. [23], [13]) follows from Corollary 2 by taking $\varphi(x, y) = x + y$, $x \in A, y \in B$, and $\Phi(a, b) = (a^{1/n} + b^{1/n})^n$, $a, b \geq 0$. Although the Brunn-Minkowski inequality goes back to more than a century ago, it still attracts a lot of attention (see e.g. [20], [11], [14], [18], [9], [10], [12], [15], [17]).

Theorem 1 also allows us to recover the so-called Borell-Brascamp-Lieb inequality. Let us denote by $M_s^\lambda(a, b)$ the $s$-mean of the real numbers $a, b \geq 0$ with weight $\lambda \in [0, 1]$, defined as
\[
M_s^\lambda(a, b) = ((1 - \lambda) a^s + \lambda b^s)^{1/s}
\]
if $s \not\in \{-\infty, 0, +\infty\}$,
\[
M_{-\infty}^\lambda(a, b) = \min(a, b), \quad M_0^\lambda(a, b) = a^{1-\lambda} b^{\lambda}, \quad M_{+\infty}^\lambda(a, b) = \max(a, b).
\]
We will need the following Hölder inequality (see e.g. [16]).

**Lemma 3** (Generalized Hölder inequality). Let $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $1/\beta + 1/\gamma = 1/\lambda$. Then, for every $a, b, c, d \geq 0$ and $\lambda \in [0, 1]$,
\[
M_\lambda^\alpha(a, b) M_\lambda^\beta(c, d) \leq M_\lambda^\gamma(f(x), g(y))
\]
holds for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, then
\[
\int_{\mathbb{R}^n} h \geq M_{1/\lambda}^{\lambda(1)} \left( \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).
\]

Corollary 4 follows from Theorem 1 by taking $\varphi(x, y) = (1 - \lambda)x + \lambda y$, $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, and $\Phi(a, b) = M_{1/\lambda}^{\lambda(1)} (a, b)$, $a, b \geq 0$. Indeed, using Lemma 3, one obtains that for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, and for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$,
\[
h(\varphi(x, y)) \Pi_{k=1}^n \frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k
\]
\[
\geq M_{\gamma}^\lambda(f(x), g(y)) \Pi_{k=1}^n (1 - \lambda) \rho_k + \lambda \eta_k
\]
\[
\geq M_{\gamma}^\lambda(f(x), g(y)) \Pi_{k=1}^n (1 - \lambda) \rho_k + \lambda \eta_k
\]
\[
= \Phi(f(x)) \Pi_{k=1}^n (1 - \lambda) \rho_k + \lambda \eta_k.
\]

Corollary 4 was independently proved by Borell (see [6, Theorem 3.1]), and by Brascamp and Lieb [8].

Another important consequence of the Borell-Brunn-Minkowski inequality is obtained when considering $\varphi$ to be nonlinear. Let us denote for $p = (p_1, \ldots, p_n) \in [-\infty, +\infty]^n$, $x = (x_1, \ldots, x_n) \in [0, +\infty]^n$ and $y = (y_1, \ldots, y_n) \in [0, +\infty]^n$,
\[
M_p^\lambda(x, y) = (M_{p_1}^\lambda(x_1, y_1), \ldots, M_{p_n}^\lambda(x_n, y_n)).
\]

**Corollary 5** (nonlinear extension of the Brunn-Minkowski inequality). Let $p = (p_1, \ldots, p_n) \in [0, 1]^n$, $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$, $\lambda \in [0, 1]$, and $f, g, h : [0, +\infty)^n \rightarrow [0, +\infty)$ be measurable functions. If the inequality
\[
h(M_p^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y))
\]
holds for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, then
\[
\int_{[0, +\infty)^n} h \geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^{\lambda(n)} \left( \int_{[0, +\infty)^n} f, \int_{[0, +\infty)^n} g \right).
\]
Corollary 5 follows from Theorem 1 by taking $\varphi(x, y) = M_p^\lambda(x, y), x \in \text{supp}(f), y \in \text{supp}(g)$, and $\Phi(a, b) = M_p^\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)(a, b), a, b \geq 0$. Indeed, using Lemma 3, one obtains that for every $x \in \text{supp}(f), y \in \text{supp}(g)$, and for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$,

$$h(\varphi(x, y))\prod_{k=1}^n\left( \frac{\partial \varphi}{\partial x_k}\rho_k + \frac{\partial \varphi}{\partial y_k}\eta_k \right) = h(M_p^\lambda(x, y))\prod_{k=1}^n M_{\rho_k}^{\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)} - \Phi(\varphi(x, y))\prod_{k=1}^n \rho_k^{\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)} - 1 \Phi(x_k^{\rho_k}, y_k^{\rho_k}, \eta_k^{\rho_k})$$

$$\geq M_p^\lambda(f(x), g(y))\prod_{k=1}^n M_{\rho_k}^{\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)} - \Phi(x_k^{\rho_k}, g(y))\prod_{k=1}^n \rho_k^{\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)} - 1 \Phi(x_k^{\rho_k}, y_k^{\rho_k}, \eta_k^{\rho_k})$$

$$\geq M_p^\lambda(f(x), g(y))\prod_{k=1}^n \rho_k^{\lambda\left(\sum_{i=1}^n (\rho_i)\lambda^{-1} - 1\right)} - 1 \Phi(f(x)\prod_{k=1}^n (\rho_k), y_k^{\rho_k}, \eta_k^{\rho_k})$$

$$= \Phi(f(x)\prod_{k=1}^n (\rho_k)\Phi(1, 0)).$$

In the particular case where $p = (0, \ldots, 0)$, Corollary 5 was rediscovered by Ball [1]. In the general case, Corollary 5 was rediscovered by Uhrin [24].

Notice that the condition on $p$ in Corollary 5 is less restrictive in dimension 1. It reads as follows:

**Corollary 6** (nonlinear extension of the Brunn-Minkowski inequality on the line). Let $p \leq 1$, $\gamma \geq -p$, and $\lambda \in [0, 1]$. Let $f, g, h : [0, +\infty) \to [0, +\infty)$ be measurable functions such that for every $x \in \text{supp}(f), y \in \text{supp}(g)$,

$$h(M_p^\lambda(x, y)) \geq M_p^\lambda(f(x), g(y)).$$

Then,

$$\int_0^{+\infty} h \geq M_p^\lambda(1 + \gamma)^{-1} \left( \int_0^{+\infty} f, \int_0^{+\infty} g \right).$$

A simple proof of Corollary 6 was recently given by Bobkov et al. [4].

In section 2, we present a simple proof of Theorem 1, based on mass transportation. In section 3, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang. We also prove an equivalence between the log-Brunn-Minkowski inequality and its possible extensions to convex measures (see section 3 for precise definitions).

## 2 A simple proof of the Borell-Brunn-Minkowski inequality

In this section, we present a simple proof of Theorem 1.

**Proof of Theorem 1.** The proof is done by induction on the dimension. To prove the theorem in dimension 1, we use a mass transportation argument.

**Step 1:** (In dimension 1)
First let us see that if $\int f = 0$ or $\int g = 0$, then the result holds. Let us assume, without loss of generality, that $\int g = 0$. By taking $\rho = 1$, by letting $\eta$ go to 0 and by using continuity and homogeneity of $\Phi$ in the condition (1), one obtains

$$h(\varphi(x, y))\frac{\partial \varphi}{\partial x} \geq \Phi(f(x), 0) = f(x)\Phi(1, 0).$$

It follows that, for fixed $y \in \text{supp}(g)$,

$$\int h(z)dz \geq h(\varphi(\text{supp}(f), y))h(z)dz = \int_{\text{supp}(f)} h(\varphi(x, y))\frac{\partial \varphi}{\partial x}dx \geq \int f\Phi(1, 0) = \Phi(\int f, \int g).$$
A similar argument shows that the result holds if \( \int f = +\infty \) or \( \int g = +\infty \). Thus we assume thereafter that \( 0 < \int f < +\infty \) and \( 0 < \int g < +\infty \).

Let us show that one may assume that \( \int f = \int g = 1 \). Let us define, for \( x,y \in \mathbb{R} \) and \( a,b \geq 0 \),

\[
\tilde{f}(x) = f \left( \Phi \left( \int f, 0 \right) x \right) \Phi(1,0), \quad \tilde{g}(x) = g \left( \Phi \left( 0, \int g \right) x \right) \Phi(0,1),
\]

\[
\tilde{h}(x) = h \left( \Phi \left( \int f, \int g \right) x \right),
\]

\[
\tilde{\varphi}(x,y) = \frac{\varphi(\Phi(\int f,0)x, \Phi(0,\int g)y)}{\Phi(\int f, \int g)}, \quad \tilde{\Phi}(a,b) = \Phi \left( a \frac{\int f}{\Phi(\int f, \int g)}, b \frac{\int g}{\Phi(\int f, \int g)} \right).
\]

Let \( x \in \text{supp}(\tilde{f}) \), \( y \in \text{supp}(\tilde{g}) \), and let \( \tilde{\rho}, \tilde{\eta} > 0 \). One has,

\[
\tilde{h}(\tilde{\varphi}(x,y)) \left( \frac{\partial \tilde{\varphi}}{\partial x} \tilde{\rho} + \frac{\partial \tilde{\varphi}}{\partial y} \tilde{\eta} + \right) \geq \Phi \left( f(\Phi(\int f,0)x) \frac{\Phi(\int f,0)}{\Phi(\int f, \int g)} \tilde{\rho}, g(\Phi(0,\int g)y) \frac{\Phi(0,\int g)}{\Phi(\int f, \int g)} \tilde{\eta} \right)
\]

\[
= \Phi(\tilde{f}(x)\tilde{\rho}, \tilde{g}(y)\tilde{\eta}).
\]

Notice that the functions \( \tilde{\varphi} \) and \( \tilde{\Phi} \) satisfy the same assumptions as the functions \( \varphi \) and \( \Phi \) respectively, and that \( \int \tilde{f} = \int \tilde{g} = 1 \). If the result holds for functions of integral one, then

\[
\int \tilde{h}(w)dw \geq \tilde{\Phi}(1,1) = 1.
\]

The change of variable \( w = z/\Phi(\int f, \int g) \) leads us to

\[
\int h(z)dz \geq \Phi \left( \int f, \int g \right).
\]

Assume now that \( \int f = \int g = 1 \). By standard approximation, one may assume that \( f \) and \( g \) are compactly supported positive Lipschitz functions (relying on the fact that \( \Phi \) is continuous and increasing in each coordinate, compare with [2, page 343]). Thus there exists a non-decreasing map \( T : \text{supp}(f) \to \text{supp}(g) \) such that for every \( x \in \text{supp}(f) \),

\[
f(x) = g(T(x))T'(x),
\]

see e.g. [3], [25]. Since \( T \) is non-decreasing and \( \partial \varphi/\partial x, \partial \varphi/\partial y > 0 \), the function \( \Theta : \text{supp}(f) \to \varphi(\text{supp}(f), T(\text{supp}(f))) \) defined by \( \Theta(x) = \varphi(x, T(x)) \) is bijective. Hence the change of variable \( z = \Theta(x) \) is admissible and one has,

\[
\int h(z)dz \geq \int_{\text{supp}(f)} h(\varphi(x, T(x))) \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} T'(x) \right) dx \geq \int_{\text{supp}(f)} \Phi(f(x), g(T(x))T'(x)) dx
\]

\[
= \int \Phi(f(x), f(x)) dx.
\]

Using homogeneity of \( \Phi \), one deduces that

\[
\int h \geq \Phi(1,1) \int f(x)dx = \Phi \left( \int f, \int g \right).
\]

**Step 2 :** (Tensorization)

Let \( n \) be a positive integer and assume that Theorem 1 holds in \( \mathbb{R}^n \). Let \( f, g, h, \varphi, \Phi \) satisfying the assumptions of Theorem 1 in \( \mathbb{R}^{n+1} \). Recall that the inequality

\[
h(\varphi(x,y)) \prod_{k=1}^{n+1} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x)) \prod_{k=1}^{n+1} \rho_k, g(y) \prod_{k=1}^{n+1} \eta_k,
\]

(2)
holds for every \( x \in \text{supp}(f), y \in \text{supp}(g) \), and for every \( \rho_1, \ldots, \rho_{n+1}, \eta_1, \ldots, \eta_{n+1} > 0 \). Let us define, for \( x_{n+1}, y_{n+1}, z_{n+1} \in \mathbb{R} \),

\[
F(x_{n+1}) = \int_{\mathbb{R}^n} f(x, x_{n+1})dx, \quad G(y_{n+1}) = \int_{\mathbb{R}^n} g(x, y_{n+1})dx, \quad H(z_{n+1}) = \int_{\mathbb{R}^n} h(x, z_{n+1})dx.
\]

Since \( \int f > 0, \int g > 0 \), the support of \( F \) and the support of \( G \) are nonempty. Let \( x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G) \), and let \( \rho_{n+1}, \eta_{n+1} > 0 \). Let us define, for \( x, y, z \in \mathbb{R}^n \),

\[
f_{x_{n+1}}(x) = f(x, x_{n+1})\rho_{n+1}, \quad g_{y_{n+1}}(y) = g(y, y_{n+1})\eta_{n+1}, \quad \varphi(x, y) = (\varphi_1(x_1, y_1), \ldots, \varphi_n(x_n, y_n)),
\]

\[
h_{\varphi_{n+1}}(z) = h(z, \varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right).
\]

Let \( x \in \text{supp}(f_{x_{n+1}}), y \in \text{supp}(g_{y_{n+1}}) \), and let \( \rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0 \). One has

\[
h_{\varphi_{n+1}}(\varphi(x, y)) \Pi_{k=1}^{n} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\int f_{x_{n+1}}(x)dx, \int g_{y_{n+1}}(y)dx).
\]

This yields that for every \( x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G) \), and for every \( \rho_{n+1}, \eta_{n+1} > 0 \),

\[
H(\varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right) \geq \Phi(F(x_{n+1}), G(y_{n+1})).
\]

Hence, applying Theorem 1 in dimension 1, one has

\[
\int h(x)dx \geq \Phi(\int F(x)dx, \int G(x)dx).
\]

This yields the desired inequality. \( \square \)

### 3 Applications to the log-Brunn-Minkowski inequality

In this section, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang [7].

Recall that a convex body in \( \mathbb{R}^n \) is a compact convex subset of \( \mathbb{R}^n \) with nonempty interior. Böröczky et al. conjectured the following inequality.

**Conjecture 7** (log-Brunn-Minkowski inequality). Let \( K, L \) be symmetric convex bodies in \( \mathbb{R}^n \) and let \( \lambda \in [0, 1] \). Then,

\[
|1 - \lambda| \cdot K \oplus_0 \lambda \cdot L \geq |K|^{1-\lambda}|L|^\lambda.
\]

Here,

\[
(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda}h_L(u)^\lambda \text{ for all } u \in S^{n-1} \},
\]

where \( S^{n-1} \) denotes the \( n \)-dimensional Euclidean unit sphere, \( h_K \) denotes the support function of \( K \), defined by \( h_K(u) = \max_{x \in K} \langle x, u \rangle \), and \( |\cdot| \) stands for Lebesgue measure.

Böröczky et al. [7] proved that Conjecture 7 holds in the plane. Using Corollary 5 with \( p = (0, \ldots, 0) \), Saroglou [21] proved that Conjecture 7 holds for unconditional convex bodies.
Theorem 9. Theorem generalizing Saroglou’s result discussed earlier. Let \( f : \mathbb{R}^n \to [0, +\infty) \) be a function, with Lebesgue measure and inequality holds for every symmetric log-concave measure \( \mu \). Let us denote \( \mu((1 - \lambda)A + \lambda B) \geq M_\lambda^\alpha(\mu(A), \mu(B)) \)

holds for all compact sets \( A, B \subset \mathbb{R}^n \) such that \( \mu(A)\mu(B) > 0 \) and for every \( \lambda \in [0, 1] \) (see [5], [6]). The 0-concave measures are also called log-concave measures, and the \(-\infty\)-concave measures are also called convex measures. A function \( f : \mathbb{R}^n \to [0, +\infty) \) is \( \alpha \)-concave, \( \alpha \in [-\infty, +\infty] \), if the inequality

\[
 f((1 - \lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))
\]

holds for every \( x, y \in \mathbb{R}^n \) such that \( f(x)f(y) > 0 \) and for every \( \lambda \in [0, 1] \).

Saroglou [22] recently proved that if the log-Brunn-Minkowski inequality holds, then the inequality

\[
 \mu((1 - \lambda) \cdot K \oplus_0 \lambda \cdot L) \geq \mu(K)^{1 - \lambda} \mu(L)\lambda
\]

holds for every symmetric log-concave measure \( \mu \), for all symmetric convex bodies \( K, L \) in \( \mathbb{R}^n \) and for every \( \lambda \in [0, 1] \).

An extension of the log-Brunn-Minkowski inequality for convex measures was proposed by the author in [19], and reads as follows:

**Conjecture 8.** Let \( p \in [0, 1] \). Let \( \mu \) be a symmetric measure in \( \mathbb{R}^n \) that has an \( \alpha \)-concave density function, with \( \alpha \geq -\frac{p}{n} \). Then for every symmetric convex body \( K, L \) in \( \mathbb{R}^n \) and for every \( \lambda \in [0, 1] \),

\[
 \mu((1 - \lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_\lambda^\alpha((\frac{p}{n + \frac{1}{\alpha}}) - 1)(\mu(K), \mu(L)).
\]

Here,

\[
 (1 - \lambda) \cdot K \oplus_p \lambda \cdot L = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq M_\lambda^\alpha(h_K(u), h_L(u)), \text{ for all } u \in S^{n-1} \}.
\]

In Conjecture 8, if \( \alpha \) or \( p \) is equal to 0, then \((n/p + 1/\alpha)^{-1}\) is defined by continuity and is equal to 0. Notice that Conjecture 7 is a particular case of Conjecture 8 when taking \( \mu \) to be Lebesgue measure and \( p = 0 \).

By using Corollary 6, we will prove that Conjecture 7 implies Conjecture 8, when \( \alpha \leq 1 \), generalizing Saroglou’s result discussed earlier.

**Theorem 9.** If the log-Brunn-Minkowski inequality holds, then the inequality

\[
 \mu((1 - \lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_\lambda^\alpha((\frac{p}{n + \frac{1}{\alpha}}) - 1)(\mu(K), \mu(L))
\]

holds for every \( p \in [0, 1] \), for every symmetric measure \( \mu \) in \( \mathbb{R}^n \) that has an \( \alpha \)-concave density function, with \( 1 \geq \alpha \geq -\frac{p}{n} \), for every symmetric convex body \( K, L \) in \( \mathbb{R}^n \) and for every \( \lambda \in [0, 1] \).

**Proof.** Let \( K_0, K_1 \) be symmetric convex bodies in \( \mathbb{R}^n \) and let \( \lambda \in (0, 1) \). Let us denote \( K_\lambda = (1 - \lambda) \cdot K_0 \oplus_\lambda \cdot K_1 \) and let us denote by \( \psi \) the density function of \( \mu \). Let us define, for \( t > 0 \), \( h(t) = |K_\lambda \cap \{ \psi \geq t \}|, f(t) = |K_0 \cap \{ \psi \geq t \}| \) and \( g(t) = |K_1 \cap \{ \psi \geq t \}| \). Notice that

\[
 \mu(K_\lambda) = \int_{K_\lambda} \psi(x)dx = \int_{K_\lambda} \psi(x)dx = \int_0^{+\infty} |K_\lambda \cap \{ \psi \geq t \}| = \int_0^{+\infty} h(t)dt.
\]

Similarly, one has

\[
 \mu(K_0) = \int_0^{+\infty} f(t)dt, \quad \mu(K_1) = \int_0^{+\infty} g(t)dt.
\]
Let $t, s > 0$ such that the sets $\{ \psi \geq t \}$ and $\{ \psi \geq s \}$ are nonempty. Let us denote $L_0 = \{ \psi \geq t \}$, $L_1 = \{ \psi \geq s \}$ and $L_\lambda = \{ \psi \geq M_\lambda^n(t,s) \}$. If $x \in L_0$ and $y \in L_1$, then $\psi((1-\lambda)x + \lambda y) \geq M_\lambda^n(\psi(x), \psi(y)) \geq M_\lambda^n(t,s)$. Hence,

$$L_\lambda \supset (1-\lambda)L_0 + \lambda L_1 \supset (1-\lambda) \cdot L_0 \oplus_p \lambda \cdot L_1,$$

the last inclusion following from the fact that $p \leq 1$. We deduce that $K_\lambda \cap L_\lambda \supset ((1-\lambda) \cdot K_0 \oplus_p \lambda \cdot K_1) \cap ((1-\lambda) \cdot L_0 \oplus_p \lambda \cdot L_1) \supset (1-\lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)$.

Hence,

$$h(M_\lambda^n(t,s)) = |K_\lambda \cap L_\lambda| \geq |(1-\lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)| \geq M_\lambda^n(f(t), g(s)),\$$

the last inequality is valid for $p \geq 0$ and follows from the log-Brunn-Minkowski inequality by using homogeneity of Lebesgue measure (see [7, beginning of section 3]). Thus we may apply Corollary 6 to conclude that

$$\mu(K_\lambda) = \int_0^{+\infty} h \geq M_\lambda^n\left(\frac{1}{1+\alpha}\right)^{-1} \left(\int_0^{+\infty} f, \int_0^{+\infty} g\right) = M_\lambda^n\left(\frac{1}{1+\alpha}\right)^{-1}(\mu(K_0), \mu(K_1)).$$

Since the log-Brunn-Minkowski inequality holds true in the plane, we deduce that Conjecture 8 holds true in the plane (with the restriction $\alpha \leq 1$). Notice that Conjecture 8 holds true in the unconditional case as a consequence of Corollary 5 (see [19]).

References


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