SOURCE IDENTIFICATION FOR THE HEAT EQUATION WITH MEMORY

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Source Identification for the Heat Equation with Memory

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Abstract

We consider source identification problems for the heat equation with memory on an interval and on graphs without cycles (trees). We propose a stable efficient identification algorithm which reduces essentially to solving linear integral Volterra equations of the second kind.

Keywords: Heat equation with memory, source identification

AMS classification: 35Q93, 45K05, 93B30

1. Introduction

In this paper we consider source identification problems for the heat equation with memory on intervals and graphs. On an interval the problem is described by the equation

$$u_t(x,t) - \int_0^t Q(t-s)u_{xx}(x,s)\,ds = f(t)g(x),\ 0 < x < l,\ 0 < t < T; \tag{1}$$

with boundary and initial conditions

$$u(0,t) = u(l,t) = 0,\ \ u(x,0) = 0. \tag{2}$$

The functions N, $f \in H^1(0,T)$ are known, and we assume that $f(0) \neq 0$ and $Q(0) > 0$. The function $g \in L^2(0,l)$ is unknown and has to be recovered from the observation $\mu(t) := u_x(0,t),\ t \in [0,T]$. We will prove that this problem is solvable for $T \geq l/\sqrt{Q(0)}$.

The heat equation with memory was proposed by Cattaneo (1948) and, in a more general form, by Gurtin and Pipkin (1968). After differentiation, equation (1) with right-hand side fg set to zero takes a form of the viscoelasticity equation:

$$u_{tt}(x,t) - Q(0)\,u_{xx}(x,t) - \int_0^t Q'(t-s)\,u_{xx}(x,s)\,ds = 0. \tag{3}$$

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Both equations (1) and (3) possess a finite speed of the wave propagation equal to $\sqrt{Q(0)}$. Heat and wave equations with memory arise in many problems of physics and engineering. Controllability problems for these equations were actively studied recent time (see, e.g. [1, 2, 3] and references therein), source identification problems for these equations on an interval were considered in [4, 5]. We propose a quite different approach to source identification problems for the heat and wave equations with memory closely related to boundary control problems. For the wave equation without memory a method based on similar ideas was described in [6]. The case of equations with memory presented in the current paper is more complicated and requires using results and techniques developed in papers [2, 3]. The main advantage of our approach is its locality: to recover an unknown coefficient on a part of the interval we use an information relevant only to this subinterval. This allows us to develop a very efficient identification algorithm which is much more simple than algorithms proposed in [4, 5]. More of that, starting with identification problems on an interval we extend our approach to equations on star graphs and arbitrary trees.

2. Identification problem on an interval

Without loss of generality we assume that $Q(0) = 1$ (it can be achieved by a simple change of variable $x \mapsto \sqrt{Q(0)}x$).

Solution of the IBVP (1), (2) can be presented in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \quad \phi_n(x) = \sqrt{\frac{2}{l}}\sin\frac{\pi n x}{l}.$$

Substituting to (1), (2) leads to the following equations for $a_n(t)$:

$$a_n'(t) + \omega_n^2 \int_0^t Q(t-s)\, a_n(s)\, ds = f(t) \int_0^l g(x)\phi_n(x)\, dx, \;\; a_n(0) = 0, \quad n \in \mathbb{N}.$$

It follows that the coefficients a_n can be written as

$$a_n(t) = \kappa_n \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau, \quad \text{where } \kappa_n = \int_0^l g(x)\phi_n(x)\, dx, \;\; \omega_n = \frac{\pi n}{l},$$

and $s_n(t)$ satisfies the IVP

$$s_n'(t) + \omega_n^2 \int_0^t Q(t-\tau)\, s_n(\tau)\, d\tau = 0, \;\; s_n(0) = 1.$$

In [1, 2] it was proved that the functions $s_n(t)$ are asymptotically close to $\cos\omega_n t$ and form a Riesz sequence in $L^2(0,T)$ for $T \geq l$. Therefore we can justify the following formal presentations of the observation:

$$\mu(t) = u_x(0,t) = \sum_{n=1}^{\infty} \phi_n'(0) \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau \int_0^l g(x)\, \phi_n(x)\, dx = \int_0^l g(x)\, w(x,t)\, dx\,, \tag{4}$$

$$\text{where } \; w(x,t) = \sum_{n=1}^{\infty} \left[\phi_n'(0) \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau \right] \phi_n(x).$$

One can check (similar calculations are performed in [3]) that the function w is a solution to the IBVP

$$w_{tt}(x,t) = w_{xx}(x,t) + \int_0^t Q'(t-s)\, w_{xx}(x,s)\, ds \; = 0; \;\; 0 < x < l, \; t > 0, \tag{5}$$

$$w(0,t)=h(t),\ w(l,t)=0;\ w(x,0)=0,\ w_t(x,0)=0, \tag{6}$$

where h is uniquely determined by f : $h(t)+\int_0^t Q(t-s)\,h(s)\,ds=f(t)$.

It is not difficult to prove (see [7, Sec. 1.2] for details) that for $0<t<l$ the function w can be presented in the form

$$\begin{cases} w(x,t)=h(t-x)+(Bh)(x,t), & x<t, \\ 0, & x>t, \end{cases} \tag{7}$$

where Bh is a more regular term comparing with h, and $(Bh)(x,t)=0$ for $x\geq t$. In particular, w satisfies the equality

$$w(t-0,t)=h(0)=f(0). \tag{8}$$

Therefore, formula (4) can be written in the form

$$\mu(t)=\int_0^t g(x)w(x,t)\,dx\ =f(0)\,G(t)-\int_0^t w_x(x,t)\,G(x)\,dx,\ \ 0\leq t\leq l, \tag{9}$$

where $G(x)=\int_0^x g(\xi)d\xi$. This is a second kind Volterra equation for $G(x)$. Solving it, we find $G(x)$ and then, $g(x)$.

We can summarize our results as follows.

Theorem 1. *For any $f\in H^1(0,T)$, the observation $u_x(0,t), t\in[0,T]$, belongs to $H^1(0,T)$. The function g is recovered by solving the Volterra equation of a second kind (9) on the interval $[0,T]$, and $T=l$ is generally the minimal identification time. The identification is stable, more exactly, any every $T\leq l$, the following estimates are valid:*

$$c\|u_x(0,\cdot)\|_{H^1(0,T)}\leq\|g\|_{L^2(0,T)}\leq C\|u_x(0,\cdot)\|_{H^1(0,T)}\,, \tag{10}$$

with positive constants c, C independent of g.

3. A star graph

Our method can be extended to source identification problems for the heat equations with memory on graphs. First we consider a star graph Γ consisting of N edges e_j identified with intervals $[0,l_j]$, $j=1,\ldots,N$, connected at an internal vertex γ_0 which we identify with the set of the left end points of the intervals. A star graph with $N=6$ is presented on Fig. 1. The boundary vertices γ_j are identified with the right end points of the corresponding intervals. The following initial boundary value problem is considered on each interval:

$$\begin{cases} \partial_t u_j(x,t)-\int_0^t Q(t-s)\,\partial_x^2 u_j(x,s)\,ds\ =f_j(t)g_j(x), & 0<x<l_j,\ 0<t<T, \\ u_j(l_j,t)=0, & 0<t<T, \\ u_j(x,0)=0, & 0<x<l_j. \end{cases} \tag{11}$$

At the internal vertex we impose the standard Kirchhoff–Neumann matching conditions

$$\begin{cases} u_1(0,t)=\ldots=u_N(0,t), & 0<t<T, \\ \sum_{j=1}^N \partial_x u_j(0,t)=0, & 0<t<T. \end{cases} \tag{12}$$

In mechanical systems, the first condition expresses the continuity of the solution, the second is the balance of forces (Newton's second law).

The problem is to recover unknown functions g_j, $j=1,\ldots,N$, from the observations

$$\mu_j(t):=\partial_x u_j(l_j,t),\ j=1,\ldots,N-1,\ t\in[0,T].$$

Note, that we use the observations at all but one boundary vertices.

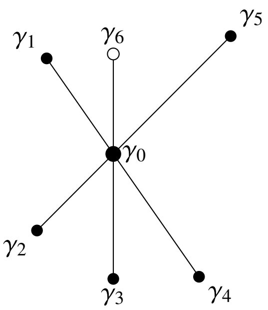

Fig. 1: A star graph

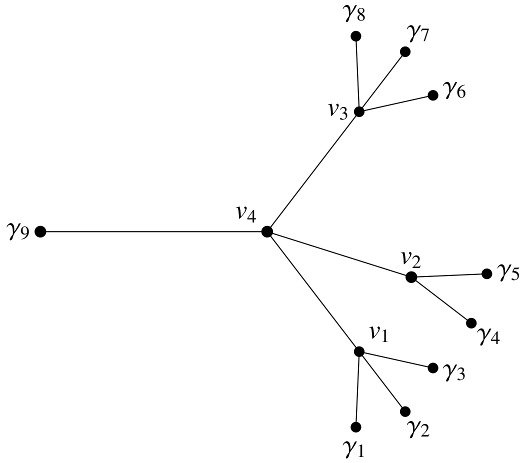

Fig. 2: A metric tree

On the first step we recover the functions g_j, $j = 1, \dots, N-1$, using correspondingly observations $\mu_j(t)$, $j = 1, \dots, N-1$, with the help of a similar algorithm to that described in Sec. 2. It can be done in the time interval of length $\max_{j=1,\dots,N-1}\{l_j\}$. Indeed, to recover g_j we use the solution w of the wave equation with memory on the graph with zero right-hand side and the boundary conditions $w(l_j, t) = f_j(t)$, $w(l_i, t) = 0$ for $i \neq j$. (See the definition of a similar function in (17) below.) This function is certainly different from the function used for one interval in (5), (6), but our identification procedure requires only the observation at the point l_j in the time interval $[0, l_j]$. Boundary observation in this time interval "does not feel" the other edges of the graph, therefore, the identification algorithm is the same as for one interval.

Then, since the g_j, $j = 1, \dots, N-1$ are known, we can consider the IBVP on the star graph where instead of the first line of (11) we have

$$\begin{cases} \partial_t v_j(x,t) - \int_0^t Q(t-s)\,\partial_x^2 v_j(x,s)\,ds = f_j(t) g_j(x), \quad j \neq N, \\ \partial_t v_N(x,t) - \int_0^t Q(t-s)\,\partial_x^2 v_N(x,s)\,ds = 0. \end{cases}$$

Subtracting solution of this problem from solution of (11), we reduce our identification problem to the case where all g_j except g_N are equal to zero:

$$\begin{cases} \partial_t u_j - \int_0^t Q(t-s)\,\partial_x^2 u_j(x,s)\,ds = 0, \; j \neq N, & 0 < x < l_j, \; 0 < t < T, \\ \partial_t u_N - \int_0^t Q(t-s)\,\partial_x^2 u_N(x,s)\,ds = f_N(t) g_N(x), & 0 < x < l_N, \; 0 < t < T, \\ u_j(l_j, t) = 0, & 0 < t < T, \\ u_j(x, 0) = 0, & 0 < x < l_j \end{cases} \tag{13}$$

with the Kirchhoff–Neumann conditions at $x = 0$.

Our next step is to obtain the spectral representation of the solution of (13). Let $\Phi_n = (\phi_{n,1}, \dots, \phi_{n,N})$ and ω_n^2, $n \in \mathbb{N}$, be eigenfunctions and eigenvalues of the following eigenvalue problem on the graph Γ :

$$\begin{cases} -\phi_j'' = \omega^2 \phi_j, \;\; 0 < x < l_j, \; j = 1, \dots, N, \\ \phi_j(l_j) = 0, \; \phi_1(0) = \ldots = \phi_N(0), \; \sum_{j=1}^N \phi_j'(0) = 0. \end{cases} \tag{14}$$

Applying the Fourier method, one can obtain the formula

$$u_j(x,t) = \sum_{n=1}^{\infty} \phi_{n,j}(x) \left[\int_0^{l_N} g_N(x)\,\phi_{n,N}(x)\,dx \right] \int_0^t f_N(\tau)\,s_n(t-\tau)\,d\tau, \tag{15}$$

Now we demonstrate how to recover g_N using any of the boundary observations $\mu_j,\ j = 1,\dots,N-1$, say, μ_1. Changing formally the order of summation and integration in (15) (it can be justified similarly to the case of one interval considered in the previous section) we get

$$\partial_x u_j(l_j,t) = \int_0^{l_N} g_N(x)\left[\sum_{n=1}^{\infty}\phi_{n,N}(x)\,\phi_n'(l_j)\int_0^t f_N(\tau)\,s_n(t-\tau)\,d\tau\right]dx. \tag{16}$$

In this formula the expression in the brackets (it will be denoted by $w_N(x,t)$) is the restriction to the edge e_N of the solution, $w(x,t)$, of the homogeneous wave equation with memory on Γ with the Dirichlet boundary control applied to the boundary vertex γ_1:

$$\begin{cases} \partial_t^2 w_j - \partial_x^2 w_j(x,t) - \int_0^t Q'(t-s)\,\partial_x^2 w_j(x,s)\,ds = 0, & j = 1,\dots,N,\\ w_1(l_1,t) = h_N(t),\ w_j(l_j,t) = 0,\ j\neq 1, & 0<t<T,\\ w_j(x,0) = \partial_t w_j(x,0) = 0, & 0<x<l_j,\ j=1,\dots,N, \end{cases} \tag{17}$$

$$h_N(t) := f_N(t) + \int_0^t Q(t-\tau)\,f_N(\tau)\,d\tau.$$

This solution satisfies also the Kirchhoff–Neumann matching conditions at $x = 0$.

Taking into account that the speed of the wave propagation in our system is one, we present the formula (16) as (compare with (9))

$$\mu_1(t) = \int_0^{t-l_1} g_N(x) w_N(x,t)\,dx,\quad l_1\le t\le l_1+l_N. \tag{18}$$

From this formula one can recover g_N in the time interval $l_1 \le t \le l_1 + l_N$. Indeed, for $0 < t < l_1$, $w_1(x,t) = h_N(t+x-l_1) + (B_1 h_N)(x,t)$ (see (7)) and $w_i(x,t) = 0,\ i \neq 1$. Using the Kirchhoff–Neumann matching conditions at $x = 0$, one can find the values of w_N at the front points: $w_N(t-l_1,t) = \frac{2}{N} f_N(0)$ for $l_1 < t \le l_N$ (compare with (8)). Substituting into (18) and putting $t = \tau + l_1,\ G_N(x) = \int_0^x g_N(\xi)\,d\xi,$ we obtain a second kind Volterra equation for $G_N(x)$:

$$\mu_1(\tau+l_1) = \frac{2}{N} f_N(0)\,G_N(\tau) - \int_0^{\tau} G_N(x)\,\partial_x w_N(x,\tau+l_1)\,dx,\quad 0<\tau<l_N. \tag{19}$$

From this equation one can now find $G_N(x)$ and, so, $g_N(x)$ for $0 \le x \le l_N$.

Similarly, we can consider the case of the observation on the whole boundary. The sharp identification time is generally smaller in this situation. We summarize our results in the form of

Theorem 2. *For a star graph described by equations (11), (12), a stable reconstruction of sources g_j is possible using N or $N-1$ boundary observations. In the first case the sharp identification time is $1/2\ \max_{i,j=1,\dots,N,\, i\neq j}\{l_i + l_j\}$, in the second —* $\max_{j=1,\dots,N-1}\{l_j + l_N\}$ *(assuming no observation at l_N).*

Remark 1. It can be proved that generally a stable source reconstruction for a star graph is impossible if we use less than $N-1$ boundary observations. However, a uniqueness result for this inverse problem may take place for smaller number of boundary observation. This number must be greater than or equal to the multiplicity of the spectrum of the operator $-d^2/dx^2$ on the graph Γ with Dirichlet boundary conditions and Kirchhoff–Neumann matching conditions.

4. A tree

Let Γ be a finite compact metric graph without cycles (tree), $E = \{e_j\}_{j=1}^N$ is the set of its edges and $V = \{v_j\}_{j=1}^{N+1}$ is the set of vertices. We denote the set of exterior vertices by $\{\gamma_1,\dots,\gamma_m\} = \partial\Gamma \subset V$. This set plays the role of the graph boundary. A tree with $m = 9$ is presented on Fig. 2.

On such a graph, we consider the heat equation with memory with Dirichlet boundary conditions at boundary vertices and Kirchhoff–Neumann matching conditions at internal vertices. The observations will be the set of normal derivatives $\{\partial u_j\}$ evaluated at all, $\gamma_j \in \partial\Gamma$, or at all but one boundary vertices, $\gamma_j \in \partial\Gamma_m := \partial\Gamma \setminus \{\gamma_m\}$, during the time interval $[0, T]$. The problem is to find the functions $g_j(x),\ j = 1, \ldots, N$, from the observation and determine the sharp T.

We consider a subgraph of Γ which is a star graph consisting of all edges incident to an internal vertex v. This star graph is called a *sheaf* if all but one its edges are the boundary edges adjacent to the boundary vertices of $\partial\Gamma_m$. It is known that any tree contains at least one sheaf.

Using the results of Section 3 we are able to solve the identification problem on an arbitrary tree. First, applying the techniques of the previous section we find the functions g_i on the sheaves, and we can further consider these functions to be zero there. Then we repeat the procedure to move further, on each step we use the observations on the original boundary to recover the functions g_i on the sheaves of the reduced graph (i.e. the graph Γ without the sheaves). On the final step we come to the edge incident to the root $\{\gamma_m\}$.

For the tree presented on Fig. 2 we use the boundary observations at the vertices $\gamma_1, \gamma_2, \gamma_3$ to recover the functions g_j on the sheaf with vertices $\gamma_1, \gamma_2, \gamma_3, v_4$ and v_1. Similarly, we recover the functions g_j on the sheaf with vertices γ_4, γ_5, v_4 and v_2 from the boundary observations at the vertices γ_4, γ_5 and on the sheaf with vertices $\gamma_6, \gamma_7, \gamma_8, v_4$ and v_3 — from the boundary observations at the vertices $\gamma_6, \gamma_7, \gamma_8$. Then we can assume that all functions g_j are zero on these sheaves and find g_j on the edge incident to the root γ_9 from the observation at any of the edges $\gamma_1, \ldots, \gamma_8$.

The identification problem can be solved using this algorithm at a finite number of steps. Our results can be formulated as follows.

Theorem 3. *For the heat equation with memory on a tree, a stable reconstruction of sources g_j is possible from the derivatives of the solution evaluated at all or all but one boundary vertices. In the first case the sharp identification time is one half of the tree diameter, in the second it is* $\max_{j=1,\ldots,m-1}$ *dist*$\{\gamma_m,\ \gamma_j\}$ *(assuming no observation at γ_m).*

Analog of Remark 1 is valid for trees: for a stable reconstruction we need not less than $m - 1$ boundary observations, for uniqueness of the identification the number of observations have to be not less than the multiplicity of the spectrum of the Laplacian on the graph.

Our approach works also for arbitrary graphs (with cycles). If the graph has cycles, boundary observations do not guarantee unique solvability of the source identification problem. We need also additional observations at the internal vertices. It will be a topic of a forthcoming paper.

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