BOUNDARY INVERSE PROBLEMS FOR NETWORKS OF VIBRATING STRINGS WITH ATTACHED MASSES

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Abstract. We consider inverse problems for tree-like networks of strings with point masses loaded at the internal vertices. We prove the identifiability of varying coefficients of the string equations along with the complete information on the graph, i.e. the loaded masses, the lengths of the edges and the topology (connectivity) of the graph. The results are achieved using the Titchmarch-Weyl function for the spectral problem and the Steklov-Poincaré operator for the dynamic wave equation on the tree. The general result is obtained by the leaf peeling method which reduces the inverse problem layer-by-layer from the leaves to the clamped root of the tree.

AMS 2010 Classification: 35L05, 35R30, 34B20

1. Introduction

This paper concerns inverse problems for differential equations on quantum graphs. By quantum graphs we understand differential operators on geometric graphs coupled by certain vertex matching conditions.

Network-like structures play a fundamental role in many problems of science and engineering. The classical problem here that comes from applications is the problem of oscillations of the flexible structures made of strings, beams, cables, and struts. These models describe bridges, space-structures, antennas, transmission-line posts, steel-grid reinforcements and other typical objects of civil engineering. More recently, the applications on a much smaller scale came into focus. In particular, hierarchical materials like ceramic or metallic foams, percolation networks and carbon and graphene nano-tubes, and graphene ribbons have attracted much attention.

Papers discussing differential and difference equations on graphs have been appearing in various areas of science and mathematics since at least 1930s, but in the last two decades their number grew enormously. Quantum graphs arise as natural models of various phemonema in chemistry (free-electron theory of conjugated molecules), biology (genetic networks, dendritic trees), geophysics, environmental science, disease control, and even in Internet (Internet or network tomography). In physics, interest to quantum graphs arose, in particular, from applications to nano-electronics and quantum waveguides. On the other hand, quantum graph theory gives rise to numerous challenging problems related to many areas of mathematics from combinatorics to PDEs and spectral theory. Work on quantum graph theory and applications has truly interdisciplinary character, and a series of meetings on this topic stimulated collaboration of researchers from different areas of science, engineering and mathematics. A number of surveys and collections of papers on quantum graphs appeared last years, and the first book on this topic by Berkolaiko and Kuchment [5] contains an excellent list of references.

Inverse theory constitutes an important part of this rapidly developing area of applied mathematics — analysis on graphs. It is tremendously important for all aforementioned applications. However, these theories have not been sufficiently developed. Inverse problems for DEs on graphs appear to be much more complicated than similar problems on an interval (see, e.g. [4] and references therein).

A new effective leaf-peeling method for solving inverse problems for differential equations on graphs without cycles has been proposed in [4] and developed further in [1, 2]. The main goal of the present paper is to extend this method to DEs on graphs with attached point masses.

Let $\Gamma = E \cup V$ be a finite compact metric graph without cycles, where $E = \{e_j\}_{j=1}^N$ is a set of edges and $V = \{\nu_j\}_{j=1}^{N+1}$ is a set of vertices. We recall that a graph is called a *metric graph* if every edge $e_j \in E$ is identified with an interval (a_{2j-1}, a_{2j}) of the real line with a positive length $l_j = |a_{2j-1} - a_{2j}|$, and a graph is

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a tree if it has no cycles. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$.

Let $\{\gamma_1, \dots, \gamma_m\} = \partial\Gamma \subset V$ be the boundary vertices, i.e. if the index (or multiplicity) of a vertex, $id(\nu)$, is the number of edges incident to it, then $\partial\Gamma = \{\nu \in V | id(\nu) = 1\}$. A nonnegative mass M_ν is attached to each vertex $\nu \in V \setminus \partial\Gamma$.

On Fig. 1 we give an example of a star graph (a graph with one internal vertex). Such graphs play an important role in the leaf peeling method described below in Sec. 3. A tree with $m = 9$ and $N = 12$ is presented on Fig. 2.

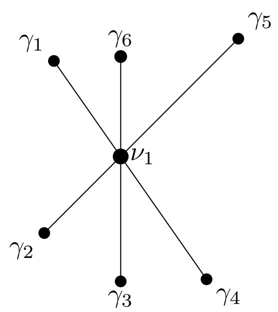

Fig. 1: A star graph

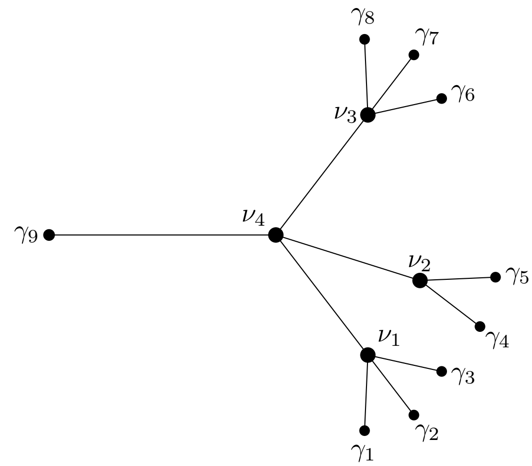

Fig. 2: A metric tree

Our initial boundary value problem is

$$u_{tt} - u_{xx} + q(x)u = 0 \text{ in } \{\Gamma \setminus V\} \times (0, T) \tag{1.1}$$

$$\begin{aligned} &\textstyle\sum_{e_j \sim \nu} \partial u_j(\nu, t) = M_\nu u_{tt}(\nu, t) \text{ at each vertex } \nu \in V \setminus \partial\Gamma, \text{ and } t \in [0, T] \\ &u(\cdot, t) \text{ is continuous at each vertex, for } t \in [0, T] \end{aligned} \tag{1.2}$$

$$u = f \text{ on } \partial\Gamma \times [0, T] \tag{1.3}$$

$$u|_{t=0} = 0 \text{ in } \Gamma. \tag{1.4}$$

In (1.2) (and below) $\partial u_j(\nu)$ denotes the derivative of u at the vertex ν taken along the edge e_j in the direction outwards the vertex. Also, $e_j \sim \nu$ means edge e_j is incident to vertex ν, and the sum is taken over all edges incident to ν. Since $\partial\Gamma$ consists of m vertices, f can be naturally identified with a function acting from $[0, T]$ to $\mathbb{R}^m$.

The metric graph Ω determines naturally the Hilbert space of square integrable functions $\mathcal{H} = L^2(\Omega)$. We define the space $\mathcal{H}^1$ of continuous functions v on Ω such that $v_{|_e} \in H^1(e)$ for every $e \in E$.

The f appearing in (1.3) is the (boundary) control for the problem (1.1)-(1.4), and a solution to (1.1)-(1.4) will be denoted u^f. One can prove that for $f \in \mathcal{F}^T := L^2([0, T]; \mathbb{R}^m)$, the generalized solution v^f of (1.1)-(1.4) belongs to $C([0, T]; \mathcal{H})$ (see Theorem 1 below), and the *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}$, given by $W^T f := u^f(\cdot, T)$ is bounded.

The *response operator* (Steklov-Poincaré operator) for the system, $R^T = \{R^T_{ij}\}^m_{i,j=1}$, defined on $\mathcal{F}^T$ is defined by

$$(R^T f)(t) = \partial u^f(\cdot, t)|_{\partial\Gamma}\ , \qquad 0 < t < T\ . \tag{1.5}$$

Our *dynamic* inverse problem is to recover the unknown coefficient $q(x)$ on each edge of the graph from the response operator R^T. We also can recover the graph topology, all $M_\nu, \nu \in V \setminus \partial\Gamma$, and the lengths of all the edges. We can actually do this with the reduced operator $\{R^T_{ij}\}^{m-1}_{i,j=1}$. That is, the method has the flexibility of not needing the control and observation at one of the boundary vertices. We prove the dynamic inverse problem has a unique solution for sufficiently large T (see Theorem 2 below) and give a constructive method for finding it.

Applying formally the Fourier–Laplace transform

$$g \mapsto \int_0^\infty g(t)e^{i\omega t}dt$$

to the equations (1.1)–(1.3) we obtain the following boundary value problem depending on a complex parameter $\lambda = \omega^2$:

$$-\phi_{xx}(x,\lambda) + q(x)\,\phi(x,\lambda) = \lambda\,\phi(x,\lambda) \text{ on } \{\Gamma \setminus V\}, \tag{1.6}$$

$$\begin{aligned}&\textstyle\sum_{e_j \sim \nu} \partial\phi_j(\nu,\lambda) = -\lambda\, M_\nu \phi(\nu,\lambda) \text{ at each vertex } \nu \in V \setminus \partial\Gamma,\\ &\phi(\cdot,\lambda) \text{ is continuous at each vertex,}\end{aligned} \tag{1.7}$$

The system of differential equations (1.6), (1.7) with zero Dirichlet boundary condition has only a trivial solution for $\lambda \notin \mathbb{R}$. Therefore, for any $\alpha \in \mathbb{C}^m$, this system of equations has a unique solution, $\phi^\alpha(x,\lambda)$, satisfying non-zero boundary conditions:

$$\phi^\alpha(\gamma_j,\lambda) = \alpha_j, \quad j = 1, 2, ..., m, \quad \alpha = \text{ col}\,\{\alpha_1, \dots, \alpha_m\}, \tag{1.8}$$

The $m \times m$ matrix $\mathcal{M}(\lambda)$ defined by $\mathcal{M}(\lambda)\,\alpha = \partial\phi^\alpha|_{\partial\Gamma}$ is called the Titchmarsh–Weyl matrix function, or the TW-function. The TW-function is known also as the (spectral) Dirichlet-to-Neumann map. The TW-function $\mathcal{M}(\lambda)$ known for $\Im\lambda > 0$ will play the role of the spectral data for solving boundary inverse problems on graphs.

2. Main results

In the case of a string with loaded masses it was noticed [6, 3] that the wave transmitted through a mass is more regular than the incoming wave. A similar effect takes place for networks of strings. Let d_{ij} be the number of nonzero loaded masses on the path from edge e_i to the boundary vertex γ_j and $d_i := \min_{j=1,\dots,m} d_{ij}$.

Theorem 1. *If $f \in \mathcal{F}^T$ then for any $t \in [0,T]$, $u^f(\cdot,t) \in \mathcal{H}$ and $u^f \in C([0,T];\mathcal{H})$. Furthermore, for each $e_i \in E$, $u^f|_{e_i} \in C([0,T]; H^{d_i}(e_i))$.*

The proof of the theorem is based on the analysis of the waves incoming to, transmitted through and reflected from an inner vertex, taking into account the conditions (1.2). For a simplest graph of serially connected strings with attached masses such a result was obtained in [3].

The next theorem describes the solution of the dynamic inverse problem.

Theorem 2. *Let $T_* = 2\max_{j\neq m} dist\{\gamma_j,\gamma_m\}$. The operator $\{R_{ij}^T\}_{i,j=1}^{m-1}$ known for $T > T_*$ uniquely determines q on Γ, $\{M_\nu : \nu \in V \setminus \partial\Gamma\}$, $\{l_j : j = 1,\dots,N\}$ and the graph topology. If the topology is known, all other parameters can be found from the main diagonal $\{R_{ii}^T\}_{i=1}^{m-1}$ of the reduced response operator.*

We also extend to our networks the leaf peeling method proposed in [4] and develop a constructive algorithm solving the inverse problem.

A spectral analog of Theorem 2 reads as follows.

Theorem 3. *The reduced TW matrix function $\{\mathcal{M}_{ij}(\lambda)\}_{i,j=1}^{m-1}$ known for $\Im\lambda > 0$ uniquely determines q on Γ, $\{M_\nu : \nu \in V \setminus \partial\Gamma\}$, $\{l_j : j = 1,\dots,N\}$ and the graph topology. If the topology is known, all other parameters can be found from the main diagonal $\{\mathcal{M}_{ii}(\lambda)\}_{i=1}^{m-1}$ of the reduced TW matrix function.*

3. Proof of Theorem 3

The response operator R^T and TW-function $\mathcal{M}(\lambda)$ are connected with each other by the Fourier–Laplace transform (see, e.g. [4]). Therefore, knowledge of $\mathcal{M}(\lambda)$ allows finding R^T for all $T > 0$, and knowledge of R^T for all $T > 0$ allows finding $\mathcal{M}(\lambda)$.

In this section we prove Theorem 3. We will give a brief description of an algorithm which allows us to recalculate the TW matrix function from the original graph to a smaller graph by "pruning" boundary edges. Ultimately, it allows us to reduce the original inverse problem on the graph to the inverse problem on a single interval.

Our reduction algorithm combines both spectral and dynamical approaches, i.e. uses $\mathcal{M}(\lambda)$ and R^T. As we mentioned above, the TW matrix function determines the response operator for the system (1.1)-(1.4). Therefore, under the conditions of Theorem 3 the entries $R^T_{ij}, i, j = 1, \dots, m-1$ are known for $T > 0$.

Step 1. Knowledge of R^T_{jj} for sufficiently large T allows to recover the length of the edge $e \in E$ incident to γ_j, the potential q on e and the mass M_ν, where $\nu \in V \setminus \partial\Gamma$ is an inner vertex to which e is incident. We can also recover $id(\nu)$, the total number of edges incident to ν. The proof of these statements is based on the analysis of the waves incoming to, transmitted through and reflected from vertex ν. Similar analysis was presented in [4] without the loaded masses; it is based on the boundary control method in inverse theory.

Step 2. We determine the boundary edges which have a common end point using the non-diagonal entries R^T_{ij} of the response operator. Since the speed of wave propagation in the system (1.1)-(1.4) equals one, two boundary edges, say, e_i and e_j, incident to the boundary edges γ_i and γ_j with the lengths l_i and l_j have a common end point if and only if

$$R^T_{ij} = \begin{cases} 0 & \text{for } T < l_i + l_j \\ \neq 0, & \text{for } T > l_i + l_j. \end{cases} \tag{3.1}$$

Definition of a sheaf. We consider a subgraph of Γ which is a star graph consisting of *all* edges incident to an internal vertex v. This star graph is called a sheaf if all but one its edges are the boundary edges of Γ. It is known that any tree has at least two sheaves.

Step 3. Leaf peeling. We consider now a sheaf consisting, say, of several boundary edges $e_1, \dots, e_p,\ p < m$, incident to boundary vertices $\gamma_1, \dots, \gamma_p$ are connected at the vertex ν_s (see, e.g. vertices $\gamma_1, \gamma_2, \gamma_3, \nu_1$ on Fig. 2). From Step 1 we know the potential on these edges, their lengths and the index of the vertex ν_s.

The index of the vertex ν_s is $p+1$ and there is exactly one internal edge incident to ν_s. We denote by $\widetilde{\mathcal{M}}(\lambda)$ the TW matrix function associated with the reduced graph $\widetilde{\Gamma}$, i.e. the original graph Γ without the boundary edges $e_1, \dots, e_p$ and vertices $\gamma_1, \dots, \gamma_p$.

We denote by $\widetilde{\mathcal{M}}_{0i}(\lambda)$, $\widetilde{\mathcal{M}}_{i0}(\lambda)$ and $\widetilde{\mathcal{M}}_{00}(\lambda)$ the entries of $\widetilde{\mathcal{M}}(\lambda)$ related to the "new" boundary point ν_{s_0} of the graph $\widetilde{\Gamma}$. The other entries of $\widetilde{\mathcal{M}}(\lambda)$ are denoted by $\widetilde{\mathcal{M}}_{ij},\ i, j = p+1, \dots, m$. We demonstrate now how to find the entries of $\widetilde{\mathcal{M}}(\lambda)$.

First we recalculate the entries $\widetilde{\mathcal{M}}_{00}(\lambda)$ and $\widetilde{\mathcal{M}}_{0i}(\lambda)$, $i = p+1, \dots, m-1$. Choose a boundary point, say γ_1, of the star-subgraph. Let $\phi(x, \lambda)$ be the solution to the problem (1.6), (1.7) subject to the boundary conditions

$$\phi(\gamma_1, \lambda) = 1, \quad \phi(\gamma_j, \lambda) = 0, \quad j = 2, \dots, m-1, m. \tag{3.2}$$

We notice that on the boundary edge e_1 the function ϕ solves the Cauchy problem:

$$-\phi'' + q(x)\phi = \lambda\phi, \quad x \in e_1, \tag{3.3}$$

$$\phi(\gamma_1, \lambda) = 1,\ \phi'(\gamma_1, \lambda) = \mathcal{M}_{11}(\lambda). \tag{3.4}$$

On the other edges of the star subgraph it solves

$$-\phi'' + q(x)\phi = \lambda\phi, \quad x \in e_i, \quad i = 2, \dots, p, \tag{3.5}$$

$$\phi(\gamma_i, \lambda) = 0,\ \phi'(\gamma_i, \lambda) = \mathcal{M}_{1i}(\lambda), \quad i = 2, \dots, p. \tag{3.6}$$

Since the potential on the edges $e_1, \dots, e_p$ is known, we can solve the Cauchy problems (3.3), (3.4) and (3.5), (3.6) and use the matching conditions (1.7) at the internal vertex ν_s to recover the values $\phi(\nu_s, \lambda)$ and $\phi'(\nu_s, \lambda)$ on the "new" boundary edge at the "new" boundary point ν_s. Thus we obtain:

$$\begin{aligned} \widetilde{\mathcal{M}}_{00}(\lambda) &= \tfrac{\phi'(\nu_s, \lambda)}{\phi(\nu_s, \lambda)}, \\ \widetilde{\mathcal{M}}_{0i}(\lambda) &= \tfrac{\mathcal{M}_{1i}(\lambda)}{\phi(\nu_s, \lambda)}, \quad i = p+1, \dots, m. \end{aligned} \tag{3.7}$$

We recall that here $\Im\lambda \neq 0$, and so, $\phi(\nu_s, \lambda) \neq 0$. Otherwise, λ would be an eigenvalue of a selfadjoint operator.

To find $\widetilde{\mathcal{M}}_{i0}(\lambda)$ and $\widetilde{\mathcal{M}}_{ij}(\lambda)$, $i = p+1, \dots, m-1$ we fix γ_i $(i > p)$ and consider the solution $\psi(x, \lambda)$ to (1.6), (1.7) with boundary conditions

$$\psi(\gamma_i, \lambda) = 1, \quad \psi(\gamma_j, \lambda) = 0, \quad j \neq i. \tag{3.8}$$

The function ψ solves then the following Cauchy problems on the edges $e_1, \dots, e_p$:

$$-\psi'' + q(x)\psi = \lambda\psi, \quad x \in e_j, \quad j = 1, \dots, p, \tag{3.9}$$

$$\psi(\gamma_j, \lambda) = 0,\ \psi'(\gamma_j, \lambda) = \mathcal{M}_{ij}(\lambda). \tag{3.10}$$

Since we know the potential on the edges $e_1, \dots, e_p$, we can solve the Cauchy problems (3.9), (3.10) and use the conditions at the internal vertex ν_s to recover the values $\psi(\nu_s, \lambda)$ and $\psi'(\nu_s, \lambda)$ at the "new" boundary edge with the "new" boundary point ν_s.

Now we consider the following linear combination of the solutions ϕ and ψ:

$$\varphi(x, \lambda) = \psi(x, \lambda) - \frac{\psi(\nu_s, \lambda)}{\phi(\nu_s, \lambda)}\phi(x, \lambda)\,. \tag{3.11}$$

It is easy to check that on the subgraph $\widetilde{\Gamma}$ the function φ satisfies the boundary conditions

$$\varphi(\gamma_i, \lambda) = 1,\ \varphi(\gamma_j, \lambda) = 0,\ j \neq i\,. \tag{3.12}$$

Thus from (3.11) we obtain that

$$\begin{aligned} \widetilde{\mathcal{M}}_{i0}(\lambda) &= \psi'(\nu_s, \lambda) - \psi(\nu_s, \lambda)\widetilde{\mathcal{M}}_{00}(\lambda), \\ \widetilde{\mathcal{M}}_{ij}(\lambda) &= \mathcal{M}_{ij}(\lambda) - \psi(\nu_s, \lambda)\widetilde{\mathcal{M}}_{0j}(\lambda). \end{aligned} \tag{3.13}$$

To recover all elements of the reduced TW matrix function we need to use this procedure for all $i, j = p+1, \dots, m-1$.

We conclude that the (reduced) TW-function for the graph Γ determines the (reduced) TW-function for the graph $\widetilde{\Gamma}$. The inverse problem is reduced to the inverse problem for a smaller graph. Since the graph $\widetilde{\Gamma}$ is finite, this procedure may be continued, but it ends after a finite number of steps.

The proofs of Theorems 1 and 2 will be presented in a forthcoming paper.

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