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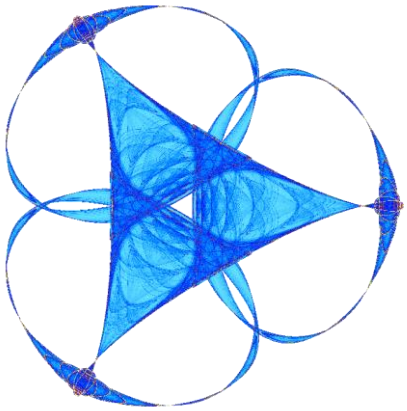

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THE SOLUTION OF THE MULTIPLE MINIMA IN AN ARBITRARY DIMENSION

By Ezio Marchi *

Abstract

The solution of the multiple minima appear in many, many places of the scientific literature. There are many theories and computational approaches. But in our knowledge until now there is not a solution from a theoretical point of view. By this we mean that a technical formulation and the corresponding solution did not exist, up today. In this paper we solve it by taking into consideration the envelope. We construct recursively from a dimension to another the envelope. In each step we solve a multiple minima and so on. In this way by moving the "slides" obtained from the original functions, constructively we archive the goal. A technical fact is solving in each step a multiple minima of one dimension and on this way we move to the global minima. This method is completely new, and indeed we solve the theoretical problem. For them we reach the minimum of the envelope and at the same time to the minimum absolute of the function.
From now on one has to take into consideration the construction of the global envelope.There exist several studies about the envelopes. Bibliography articles about the matter of the envelope are to be very important and for the previous matter are going to be very important. We relate as bibliography various references. Finally the tool used here was the Kakutani's fix point theorem.

Introduction

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Empirical conformational energy functions are used to try to compute the three-dimensional structures of polypeptides and proteins.

The conformational energy surfaces of such molecules have many local minima, and conventional energy minimization procedures reach only a local minimum (near the starting point of the optimization algorithm) instead of the global minimum (the multiple-minima problem). Several procedures have been developed to surmount this problem. A summary is given here of five of these methods,(i) build-up,(ii) Monte Carlo-plus-minimization(MCM),(iii) relaxation of dimensionality,(iv) pattern-recognition-based importance-sampling minimization(PRISM), and surface, leaving only the potential well containing the global minimum, these and other procedures have been applied to a variety of polypeptide structural problems. These include the computation of the structures of open-chain and cyclic peptides, fibrous proteins and globular proteins. Present efforts are being devoted to scaling up these procedures from small polypeptides to proteins, to try to compute the three-dimensional structure of a protein from its amino sequence.

Ever since Anfinsen demonstrated that proteins fold spontaneously to achieve their native conformation, attempts have been made to try to compute the three-dimensional structure of a native protein as the one for which the free energy of the system (protein plus solvent) is a minimum. Empirical potential functions and procedures for generations and minimization procedures are available, a difficult problem that had to be surmounted arose from the presence of many local minima in the conformational energy surface (the multiple-minima problem).[1]

The multiple-minima problem is a severe one in the use of energy minimization in calculations of molecular conformation. Current minimization algorithms are very efficient in finding local minima but, having found a local minimum, are then trapped in the potential well of that minimum. Various approaches to the multiple-minima problem have been suggested in several recent papers. The build-up procedure is essentially an efficient and systematic search of the conformational energy space by a judicious choice of starting conformations from which to carry out minimizations. The Monte Carlo approach avoids minimization altogether and attempts to sample conformational energy space efficiently to locate the energetically favorable regions. In the annealing approach, when the minimization becomes trapped in a local minimum, the temperature of the system is raised and a Monte Carlo procedure is carried out to allow the system to escape from the local potential well.

We present a commentary of a method for relaxing a system, not by raising the temperature but by raising the dimensionality of the space. The idea is that, in higher dimensional space, there are many more degrees of freedom in which the atoms can move about, making it easier to adjust to a low-energy conformation. Many potential barriers in three dimensions will not exist in higher dimensions. A method is presented for starting from a very-low-energy high-dimensional conformation and obtaining a low-energy there-dimensional structure from it by gradual contraction of the dimensionality. The method may also be used for escaping from a three-dimensional minimum, raising the dimensionality, and then contraction it back to the re dimensions. [2]

A deterministic algorithm designed to search for the global minimum of a potential energy functional in the conformational analysis of molecules was proposed. The algorithm is based on the deformation of the original potential energy hypesurface in such a way as to obtain only a single minimum which, in most cases, is related to the global one. This single minimum can easily be attained from any starting point of the modified hypersurface by standard local minimization procedures. The position of this minimum with respect to the global one in the original hypersurface may have been changed during deformation; therefore, reversing procedure is applied in which the global minimum is usually attained by gradually reversing the deformation.

The hypersurface is deformed with the aid of the diffusion (or heat conduction) equation, with the original shape of the hypersurface having the meaning of the initial concentration (or temperature) distribution. The algorithm functions efficiently in one and two-dimensional problems of chemical interest as well as in many dimensions for some test functions. Suggestions for extending it to higher dimensions for systems of chemical interest are provided and a possible application to molecular dynamics is indicated. The significance of the proposed method extend beyond its application in chemistry.

The multiple-minima problem is the most formidable one in the conformational analysis of macromolecules. The number of minima appearing in the energy hypersurface typically varies as 3^m for hydrocarbon-like molecules or as ⟦10⟧^m for polypeptides, m being the number of monomer units. This means that a study based on some systematic or random exploration of the whole energy hypersurface in order to locate the global minimum quickly becomes impossible as the number of minima increases, even when applying rapidly computable atom-atom potentials and using the most powerful computers available today. Actually, at present, m=5 is a practical upper limit for polypeptides, whereas proteins contain of the order of 100 residues. Some other methods proposed in the literature do not

search the whole space, but their application is limited to polypeptides and then only to chains containing not more than 20 amino acid residues.[3]

All this material has been obtained from the important contributions quoted in the bibliography.

Mian topics

Let us consider a non- empty rectangle
with an interior point in the rectangle $I = [a,b]x\,[c,d]\ belonging\ in\ R^2$ then $a < b, c < d.$
There is a real continuous function defined on it. This function in the applications of the physicists, biotechnologists, technologies, geologies, etc. represents the enough free energy, energies or potential.
From an applied point of view it is important to reach the absolute minimum of such a function, since as it such minimum represents a possible point of the stable point or in a word, the most stable configuration. From a strict mathematical point of view is important to reach such a point.
For technical reasons it is assumed that such a minimum point is interior of I
In other words $(\bar{x}\ , \bar{y}) \in \dot{I} = (a,b)\text{x}\ (c,d)$
Now, we revised some factors of the case where the function is defined in rectangles.
In other words, in such a way we are in a two dimensional

$$f : [a,b]\ x\ [c,d] \to R$$

In the previous paper [4] we stated the fact that the convex envelope reaches the minimum and equals the minimum of the function under consideration, this in the case of in any dimension. Moreover, this matter is regarding as a difficult and important problem of the multiple minima
There is an aspect which is of importance. The fact that is important in the problem of one dimension, and it is going to be important for the study of two dimensions. This is that in one dimension if the function is “sharp” at the minimum as is illustrated in the figure 1, the “separation” hyperplane to the

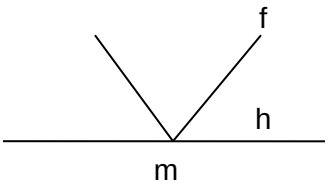

fig1

set formed by all the points(x,y) whose y is greater or equal to f (x) and to the corresponding envelope: one to be stressed is that the point m is an interior and not in the boundary. Here we do draw the partial “slide” envelope since it depends en the whole range or domain.

A second consideration that you have to take into account is when the case is geometrically given in the fig 2.

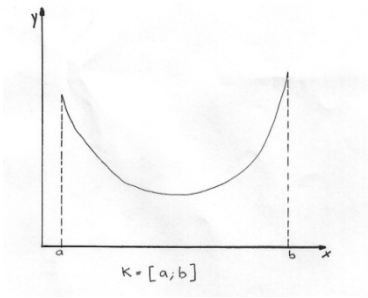

Fig 2

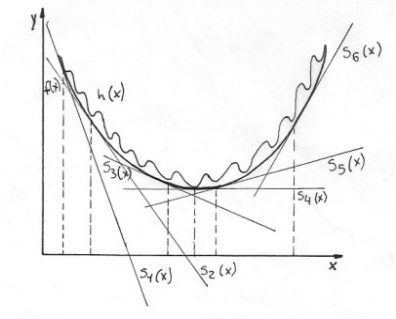

Fig 3

Then in all case we have a separation g hyperplane in a horizontal way and parallel to the axis. It is important to keep in mind that the slide function is similar to the one expressed in the previous case of only one dimension.

The main results

Without any less of generality we might consider a real continuous function f (x, y) ϵR defined on the square I=$[a_1, b_1]$ x $[a_2, b_2]$, with the property that the minimum it is reach in the interior

Ĭ = (a_1, b_1) x (a_2, b_2).

Then here we are going to study in very good way its minimum.

Now let is take the slides function for any fixed $\bar{y} \in [a_2, b_2]$

$$f_{\bar{y}}\,(x,\bar{y}) : [a_1, b_1] \longrightarrow R$$
$$: x1 \longrightarrow f_{\bar{y}}(x,\bar{y})$$

and on the other direction

$$f_{\bar{x}}\,(\bar{x},y) : [a_2, b_2] \longrightarrow R$$
$$: y\,1 \longrightarrow f\,(\bar{x},y)$$

Then the convex envelope of a real function in a open interval can be constructed in two ways namely: by taking the supreme or maximum of all convex function from below. The second method is taking all the straight lives. In one dimension they provide the same convex curve: the envelope.

On the other hand, let $C_{.,\bar{y}}$ be the convex envelope at the place ȳ and let be $C_{\bar{x}},.$ at the place in the other remain coordinate: clearly both of then are continues. Since the f is continuous then both, $C_{.,\bar{y}}$ and $C_{\bar{x}},.$ are continuous and convex,

The sets of the minimum are non-empty convex and compact

$$m_{\bar{x}},. = \left\{y \in [a_2, b_2]: C_{\bar{x}},.\,(y) = \min_z C_{\overline{x_{,.}}}(z)\right\}$$

$$m_{.,\bar{y}} = \left\{x \in [a_1, b_1]: C_{.,\bar{y}}(x) = \min_z C_{.,\bar{y}}(z)\right\}$$

Therefore the graph of both functions

G., = $\{(x, y): [x, y] \epsilon\, I\, y \epsilon\ m_{x},.\}$

G,. = $\{(x, y)$: $[x, y] \epsilon\ I\ x\ \epsilon\ m_{.,y}\}$

Are closed.

Now consider the multivalued function

φ: I => I

(x,y) => φ(x,y)

defined by

φ(x,y)=$m_{x.,}$ x $m_{.,\bar{y}}$

This is a multivalued function defined on I where the image is non-empty, convex and compact, then we apply Kakutani's fixed pewit theorem, which says:

Theorem 1 (Kakutani)

Led K c R^n be an non-empty and convex in compact and consider a multivalued function L: K=>K, such that the set L(x)≠ ∅□, is convex and compact and the graph is closed them there exists fixed pewit $\bar{x} \epsilon L(\bar{x})$.

Using this important result to our function under consideration, there exists a fixed point, namely:

($\bar{x}$,ȳ) ϵ φ ($\bar{x}$,ȳ)

From where, it means

$\bar{x}$ ϵ $m_{.,\bar{y}}$ and ȳ ϵ$m_{\bar{x},}$.

$C_{.,\bar{y}}(\bar{x})$ = $\min_{z}$ $C^{(z)}_{.,\bar{y}}$

$C_{\overline{x_{,.}}}(\bar{y})$ = $\min_{z}$ $C_{\bar{x}_{,.}}$; (z)

Now we will prove that at such point we reach the minimum off defined an the entire I. For this consider the convex envelope from below, now by the following there simple and important results proved by J. Martinez Legaz[5], that says that in any dimension the envelope from below reaches the same minimum as the function and to a same point. Moreover, there is a tangent separation hyperplane parallel to the plane of the orthant or of

constant value whose axis are x,y, reaching the minimum of the function f under consideration. The minimum is reached in a point part of the graph of f and the separation plane just mentioned.

Theorem 2 (Martin Legaz): let X be a vector topological space, C ⊆ X is a men- empty convex set, f: C →R is a teal function.

Which reaches the minimum in $\bar{x} \epsilon C$ and conv f is the convex envelope of: f: C → R, then $\bar{x}$ is also minimum of conv f.

Proof : Since the constant function of value f($\bar{x}$) is a convex minorant, then it holds $f(\bar{x})$ ≤ conv f, therefore

$f\ \bar{x})$ ≤ $\underset{x\epsilon C}{inf}$ (conv f) (x) ≤ (conv f) ($\bar{x}$) ≤ $f(\bar{x})$

From here we obtain that

$\underset{x\epsilon C}{inf}$ (conv f) (x) = (conv f) ($\bar{x}$)

q.e.d.

One has that there is a point ($\bar{x}$, ȳ) such that the hyperplane H minorant is parallel to the orthant x,y and moreover the value of H at the point ($\bar{x}$, ȳ) it is the name as$f$ $\bar{x}$, ȳ).
On the other hand H is generated by the two perpendicular straight lives $h_{\bar{x}}$ and $h_{\bar{y}}$ which intersect at point ($\bar{x}$, ȳ) and they generated H.
Thus we have proved in a general way that this new method of perpendicular contraction of the minimum: Clearly it is interesting to set what is the method with the best algorithm from a point of view of complexity and saving rime of real computation.
A new theory for linear programming is given by Marchi-Matons[6]
At such point, the function f reaches at least one minimum. According to the Martínez-Legaz theorem, the minimum of the function is reached at the same place as the convex envelope from below. Moreover since a minimum is interior, one of the plane of the convex envelope, it is parallel to the plane determinates orthant constant.

On the other hand the minimum it is clearly interesting to set what is the method with the best algorithm from a point of view of complexity and saving the time.

THE CASE FOR n ARBITRARY DIMENSIONS

Now after remembering the case of one and two, we are going to study the case with an arbitrary number of dimensions. In the case of two dimensions, we have solved the problem by a rather constructing way namely: from the fact that we move or partial on "slice" plans that for it we approach by the convex envelope in one dimension and then to move it accordingly through the same coordinate. A further problem is the effective computational determination of the convex envelope. We remark that our method it works only with the explicit computation of an enveloping in one dimension, and then moving it. For this reason, intuitively we consider it the better one.
In the case of arbitrary number of dimensions we have a much more complex problem, however, applying the induction principle the proof turned it in rather simple. Consider the following continuous function

$$f : [\, a_{1,}\ b_1\,]\ x \ldots x\, [a_n, b_n] = I_n \ \rightarrow R$$
$$: x\ \in\ I_n \rightarrow f\ (x)$$

Having at least a global minimum interior
Take any point of the form

$x^1 = (x_1^1, \ldots, x_n^1) \in\ I_n$

And consider the "slight"
$f_i^{x_i} : \underset{j \neq i}{X}\ [a_j\,;\, b_j] = I_{m-1}^{x_i} \rightarrow R$

Defined as
$f_i^{x_i}(x_{-i}) = f\ (x_i\ \ , x_{-i})$
where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Note that $f_i^{x_i}$ is a continuous function on x_{-i} and $f_i^{\cdot}(x_{-i})$ is continuous in x_i , for any x_{-i} (Remind that the slight function $f_i^{\cdot}(x_{-i})$ is the defined as $f_i^{x_i}(x_{-i})$, which is the defined on the interval $I_i = [a_i, b_i]$)

Then the minima of $f_i^{\cdot}(x_{-i})$ are reached by the envelope and then all of minima of the envelope is a non-empty, convex closed and bounded set.

By the same argument the tangent at one minimum is parallel to the corresponding axis.

Now, if we take the multivalued function

$$\varphi : I_n \implies I_n$$

$$x \to m_1(x_{-1}) x\, m_2(x_{-2}) x \ldots x m_i(m_{-i}) x \ldots x m_n(x_{-n})$$

where $m_i(x_{-i}) = \min_{y_i} f^{yi}(x_{-i})\ C\ I_i$

by the same argument given in the case of two dimension applying the fixed point of Kakutani that asserts that for any multivalued function defined on a non-empty subset in any Euclidean space with non-empty, convex and compact images in the same set, with closed graph, it has a fixed point. This in our case since $\varphi(x)$ is convex, then we have the existence of a fixed point:

$$\bar{x} \in \varphi(\bar{x})$$

At this point we have the same situation as in the case of two dimension by Martínez-Legaz theorem the minimum of the function and envelope are the same, and there is a parallel plane to the orthant an separating the function. Trivially, this plane is generated by the half at straight lines of separating $f_i^{\circ}(\bar{x}_{-i})$

By the induction principle results true for any dimensions.

As a important consequence we have obtain that for any dimensions n it is possible to find the absolute minimum, step by step with only at each step having difficulty of one dimension.

This proof is inspired as analogous of non-cooperative game theory.

NOTE: We have used Kakutani's fixed point theorem. However for higher dimension we are going to use it again. But we indicate for the future study of the subject that another

strong result is the fixed point theorem by Eilenberg and Montgomery that instead of having convexity in the text of Kakutani's, is change for contractible.
We remaind that a set M is contractible if there exist a continuous function

$$\alpha: [0,1]x\ M\ \rightarrow M$$
$$\alpha\ (0,m) =\ m_0\ \in M$$
$$\alpha\ (1,m) = m$$

The theorem says that under general conditions there exist a fixed point.
It is my intuition that this theorem will be of great importance when one apply and study Morse theory, related to biotechnology.
More over a further generalization of Eilenberg Montgomery it is possible to find a fixed point theorem with more general topological conditions: Beagle's theorem.
Finally we would to say that might a mathematical tool to used in more sophisticate set up or related with this is the non-cooperative game theory and the Morse theory. For this reason we incorporate some specific bibliography.
As an addendum, we would lied to say that our method of face projection and the construction of the envelope, would be an important feature. Since in the future will probably be possible to construct the envelope then we include various papers regarding the envelope.

Final Remarks

In this paper we have solved proving the existence of a minimum point for the multiple minimum in an arbitrary dimension. We have solved it by constructing a solution by means of the different envelopes in each step in such a way that in each we have a solution of one dimension. Thus it is obtained by means of n - steps where in each step the envelop of is obtained after adding one more dimension.
Our method is clearly constructive.

Bibliography

[1]H. A. Scheraga: conformational energy calculation on polypeptides and proteins.(1991).Baker Laboratory of Chemistry. Cornell University. Ithaca, New York 14853-1301, U.S.A.

[2]E. O. Purisima and H. A. Scheraga: an approach to the multiple-minima problem by relaxing dimensionality. Proc. Natl. Acad. Sci. USA. Vol.83,pp.2782-2786,May 1986. Chemistry.
[3] L. Piela and J. Kostrowicki and H A. Scheraga: The Multiple-Minima Problem in the Conformational Analysis of Molecules. Deformation of the potential Energy Hypersurface by the Diffusion Equation Method. J.Phys. Chem.,1989, 93(8),pp 3339-3346
[4]E. Marchi. New Fundamental Result in two dimensional multiple minima.2015
[5] J.E.Martinez Legaz: Private Communication. 2015
[6] E.Marchi and Matons. A new theory for linear programming is given. Jama. 2012
[7]H. Carrillo and D. Lipman: The Multiple Sequence Alignment Problem in Biology. SIAM J. Appl. Math,45(5), 1073-1082.
[8] Zhenqin Li and H. A. Scheraga. Baker Laboratory of Chemistry, Cornell University, Itheca, NY 14853-1301.
[9] L. C. W. Dixon, & G.P. Szogo, (1978) in Towards Global Optimization, eds. Dixon, L.C.W & Szego,G.P. (North Holland, New York), Vol 2,pp. 1-15.3.
[10] Ezio Marchi and Martín Matons: The Face Projection Method in Linear Programming. J. Adv. Math. Appl.1 (2012)pp 234-238.
[11] G.B Dantzig: Linear Programming and Extension. Princeton University. Press. Princeton, NJ(1963).
[12] G. M.L. Lovász and A. Schirijver: The Ellipsoid Method and its Consequences in Combinational Optimization. Combinatoria I(1981) pp.169-197.
[13] L.V Kantorovich.(1939): Mathematics Methods of Organizing and Planning Production, Management Sciences, Vol6, No4(1960)pp. 366-395.
[14] N.A. Karmarkar,.: New Polynimial-time Algorithm for Linear Programming Combinatorica 4(1984)pp.373-395.
[15] E. Marchi: Foundations of non-cooperative games. Econometric Research Program Research Memorandum N° 97.1968.
[16] E. Marchi: E-Point of game. Proced. Nat. Acad. Sci. USA,5,878-882,1967.
[17] E. Buger: Introduction to The Theory of Games. Prentice Hall 1963.
[18] C. Berger: Topological spaces. Oliver and Boyd 1959
[19] F. Tardella: Existence and sum decomposition of vertex polyhedral convex envelope. Optimization letters. Junio 2008 vol2, Issue pp.363-375.
[20] F. Tardella: On the existence of polyhedral convex envelopes. Frontiers in global optimization and its applications. 2004 Vol 74. pp 563-573.

[21] S. Cafieri, J. Lee and L. Liberti: On convex relaxations of quadrilinear terms. 29 th April 2009.
[22] M. Locatelli: Convex envelopes for quadratic and polynomial functions over polytopes quadratic forms, Mathematical Programming 124-33-43(2010).
[23] Y. Crama, Recognition problems for polynomials in 0-1 variables, Mathematical Programming, 44:139-155 (1989).
[24] P.H. Diananda, non-negative forms in real variables some or all of which are non-negative, Proc. Cambridge Philos. Soc., 58:17-25(1967).
[25] M. Jach, D.Michaels, R. Wesmantel, The convex Envelope of (n-1)- Convex Functions, SIAM Journal on Optimization,19(3):1451-1466 (2008).
[26] J. Linderoth, A simplicial branch- and-bound algorithm for solving quadratically constrained quadratic programs, Mathematical Programming,103:251-282(2005).
[27] M. Locatelli, F. Schoen, On convex envelopes and underestimating for bivariate functions, submitted (2010).
[28] G.P. McCormick, Computability of global solutions to factorable nonconvex programs-Part I- Convex underestimating problems, Mathematical Programming,10:147-175(1976)
[29] C.A Mayer, C.A. Floudas, Convex envelopes for edge-concave functions. Mathematical Programming,103:207-224(2005).
[30] A. Rikun, A convex envelope formula for multilinear functions, Journal of Global Optimization, 10:425-437.
[31] H.D. Sherali, A. Alameddine, An explicit characterization of the convex envelope of a bivariate bilinear over special polytopes, Annals of Operations Reseatch,27:197-210.
[32] M. Tawarmalani, N.V. Sahinidis, Semidefinite relaxations of fractional programs vianovel convexification techniques, Journal of Global Optimization,20:137-158(2001).
[33] M.Tawarmalani, N.V. Sahinidis, Convex extensions and envelopes of lower semicontinuous functions, Mathematical Programming,93:247-263(2002).
[34] M. Tawarmalani, N.V. Sahinidis. a polyhedral branch-and-cut approach to global optimization, Mathematical Programming,103:225-249. (2005).
[35]F. Tardella. On the existence of polyhedral convex envelopes.
[36] C.A. Floudas, P.M. Pardalos (eds), Frontiers in Global Optimization, Dordrecht, Kluwer Academic Publishers,563-574(2003).
[37] J.M. Zamora, I.E. Grossmann, A Branch and Contract algorithm for problems with concave univariate, bilinear and fractional terms, Journal of Global Optimization,14,217-249(1999).

[38] R.T. Rockafellar . Convex analisis. Princeton University 1970.

[39] J. Milnor. Morse Theory. Princenton, New Jearsy. Princenton University Press. 1963