A NOTE ON AN LP-BRUNN-MINKOWSKI INEQUALITY FOR CONVEX MEASURES IN THE UNCONDITIONAL CASE

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A note on an L^p-Brunn-Minkowski inequality for convex measures in the unconditional case

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Abstract

We consider a different L^p-Minkowski combination of compact sets in $\mathbb{R}^n$ than the one introduced by Firey and we prove an L^p-Brunn-Minkowski inequality, $p \in [0,1]$, for a general class of measures called convex measures that includes log-concave measures, under unconditional assumptions. As a consequence, we derive concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}} A)$, $p \in (0,1]$, for unconditional convex measures μ and unconditional convex body A in $\mathbb{R}^n$. We also prove that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures, completing recent works by Saroglou.

Keywords: Brunn-Minkowski-Firey theory, L^p-Minkowski combination, convex body, convex measure, (B)-conjecture.

1 Introduction

The Brunn-Minkowski inequality is a fundamental inequality in Mathematics, which states that for every convex subset $A, B \subset \mathbb{R}^n$ and for every

$\lambda \in [0,1]$, one has

$$|(1-\lambda)A+\lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}}+\lambda|B|^{\frac{1}{n}}, \tag{1}$$

where

$$A+B=\{a+b; a\in A, b\in B\}$$

denotes the *Minkowski sum* of A and B and where $|\cdot|$ denotes the Lebesgue measure. The book by Schneider [21] and the survey by Gardner [14] famously reference the Brunn-Minkowski inequality and its consequences.

Several extensions of the Brunn-Minkowski inequality have been developed during the last decades by establishing functional versions (see *e.g.* [16], [9], [10], [24]), by considering different measures (see *e.g.* [3], [4]), by generalizing the Minkowski sum (see *e.g.* [11], [12], [13], [18], [19]), among others.

In this paper, we will combine these extensions to prove an L^p-Brunn-Minkowski inequality for a large class of measures, including the log-concave measures.

Firstly, let us consider measures other than the Lebesgue measure. Following Borell [3], [4], we say that a Borel measure μ in $\mathbb{R}^n$ is *s-concave*, $s\in[-\infty,+\infty]$, if the inequality

$$\mu((1-\lambda)A+\lambda B) \geq M_s^{\lambda}(\mu(A),\mu(B))$$

holds for every $\lambda\in[0,1]$ and for every compact subset $A,B\subset\mathbb{R}^n$ such that $\mu(A)\mu(B)>0$, where $M_s^{\lambda}(a,b)$ denotes the s-mean of the non-negative real numbers a,b with weight λ, defined as

$$M_s^{\lambda}(a,b)=((1-\lambda)a^s+\lambda b^s)^{\frac{1}{s}} \quad \text{if } s\notin\{-\infty,0,+\infty\},$$

$M_{-\infty}^{\lambda}(a,b)=\min(a,b)$, $M_0^{\lambda}(a,b)=a^{1-\lambda}b^{\lambda}$, $M_{+\infty}^{\lambda}(a,b)=\max(a,b)$. Hence the Brunn-Minkowski inequality tells us that the Lebesgue measure in $\mathbb{R}^n$ is $\frac{1}{n}$-concave.

As a consequence of the Hölder inequality, one has $M_p^{\lambda}(a,b)\leq M_q^{\lambda}(a,b)$ for every $p\leq q$. Thus every s-concave measure is $-\infty$-concave. The $-\infty$-concave measures are also called *convex measures*.

For $s\leq\frac{1}{n}$, Borell showed that every measure μ, which is absolutely continuous with respect to the n-dimensional Lebesgue measure, is s-concave if and only if its density is an α-concave function, with

$$\alpha=\frac{s}{1-sn}\in[-\frac{1}{n},+\infty], \tag{2}$$

where a function $f:\mathbb{R}^n\to[0,+\infty)$ is said to be *α-concave*, with $\alpha\in[-\infty,+\infty]$, if the inequality

$$f((1-\lambda)x+\lambda y)\geq M_{\alpha}^{\lambda}(f(x),f(y))$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$.

Secondly, let us consider a generalization of the notion of the Minkowski sum introduced by Firey, which leads to an *L^p-Brunn-Minkowski theory.* For *convex bodies* A and B in $\mathbb{R}^n$ (*i.e.* compact convex sets containing the origin in the interior), the *L^p-Minkowski combination*, $p \in [-\infty, +\infty]$, of A and B with weight $\lambda \in [0, 1]$ is defined by

$$(1-\lambda) \cdot A \oplus_p \lambda \cdot B = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq M_p^\lambda(h_A(u), h_B(u)), \forall u \in S^{n-1}\},$$

where h_A denotes the *support function* of A defined by

$$h_A(u) = \max_{x \in A} \langle x, u\rangle, \quad u \in S^{n-1}.$$

Notice that for every $p \leq q$,

$$(1-\lambda) \cdot A \oplus_p \lambda \cdot B \subset (1-\lambda) \cdot A \oplus_q \lambda \cdot B.$$

The support function is an important tool in Convex Geometry, having the property to determine a convex body and to be linear with respect to Minkowski sum and dilation:

$$A = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq h_A(u), \forall u \in S^{n-1}\}, \quad h_{A+B} = h_A + h_B, \quad h_{\mu A} = \mu h_A,$$

for every convex body A, B in $\mathbb{R}^n$ and every scalar $\mu \geq 0$. Thus,

$$(1-\lambda) \cdot A \oplus_1 \lambda \cdot B = (1-\lambda)A + \lambda B.$$

In this paper, we consider a different L^p-Minkowski combination. Before giving the definition, let us mention that the set of non-negative real numbers is denoted by $\mathbb{R}_+$. Let us recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *unconditional* if there exists a basis $(a_1, \cdots, a_n)$ of $\mathbb{R}^n$ (the canonical basis in the sequel) such that for every $x = \sum_{i=1}^n x_i a_i \in \mathbb{R}^n$ and for every $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{-1, 1\}^n$, one has $f(\sum_{i=1}^n \varepsilon_i x_i a_i) = f(x)$. A measure which is absolutely continuous with respect to the n-dimensional Lebesgue measure is *unconditional* if its density function is unconditional. For $\mathbf{p} = (p_1, \cdots, p_n) \in [-\infty, +\infty]^n$, $a = (a_1, \cdots, a_n) \in (\mathbb{R}_+)^n, b = (b_1, \cdots, b_n) \in (\mathbb{R}_+)^n$ and $\lambda \in [0, 1]$, let us denote

$$(1-\lambda)a +_{\mathbf{p}} \lambda b = (M_{p_1}^\lambda(a_1, b_1), \cdots, M_{p_n}^\lambda(a_n, b_n)) \in (\mathbb{R}_+)^n.$$

For subsets $A, B \subset \mathbb{R}^n$ such that $A \cap (\mathbb{R}_+)^n$ and $B \cap (\mathbb{R}_+)^n$ are non-empty, for $\mathbf{p} \in [-\infty, +\infty]^n$ and for $\lambda \in [0, 1]$, we define the L^p-Minkowski combination of A and B with weight λ, denoted by $(1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B$, to be the unconditional subset (*i.e.* the indicator function is unconditional) such that

$$((1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B) \cap (\mathbb{R}_+)^n = \{(1-\lambda)a +_{\mathbf{p}} \lambda b\, ;\ a \in A \cap (\mathbb{R}_+)^n, b \in B \cap (\mathbb{R}_+)^n\}.$$

This definition is consistent with the well known fact that an unconditional set (or function) is entirely determined on the positive octant $(\mathbb{R}_+)^n$. Moreover, this L^p-Minkowski combination coincides with the classical Minkowski sum when $\mathbf{p} = (1, \cdots, 1)$ and A, B are unconditional convex subsets of $\mathbb{R}^n$ (see Proposition 2.1 below).

Using an extension of the Brunn-Minkowski inequality discovered by Uhrin [24], we prove the following result:

Theorem 1.1. *Let $\mathbf{p} = (p_1, \cdots, p_n) \in [0,1]^n$ and let $\alpha \in [-\infty, +\infty]$ such that $\alpha \geq -\left(\sum_{i=1}^n p_i^{-1}\right)^{-1}$. Let μ be an unconditional measure in $\mathbb{R}^n$ that has an α-concave density function with respect to the Lebesgue measure. Then, for every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda \in [0,1]$,*

$$\mu((1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B) \geq M_\gamma^\lambda(\mu(A), \mu(B)), \tag{3}$$

where $\gamma = \left(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\right)^{-1}$.

In theorem 1.1, if α or one of the p_i is equal to 0, then $\left(\sum_{i=1}^n p_i^{-1}\right)^{-1}$ and γ are defined by continuity and are equal to 0.

The case of the Lebesgue measure and $\mathbf{p} = (0, \cdots, 0)$ is treated by Saroglou [21], answering a conjecture by Böröczky, Lutwak, Yang and Zhang [5] in the unconditional case.

Conjecture 1.2 (log-Brunn-Minkowski inequality [5])**.** *Let A, B be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in [0,1]$. Then,*

$$|(1-\lambda) \cdot A \oplus_0 \lambda \cdot B| \geq |A|^{1-\lambda}|B|^\lambda. \tag{4}$$

Useful links have been discovered by Saroglou [21], [22] between Conjecture 1.2 and the (B)-conjecture.

Conjecture 1.3 ((B)-conjecture [17], [8])**.** *Let μ be a symmetric log-concave measure in $\mathbb{R}^n$ and let A be a symmetric convex subset of $\mathbb{R}^n$. Then the function $t \mapsto \mu(\mathrm{e}^t A)$ is log-concave on $\mathbb{R}$.*

The (B)-conjecture was solved by Cordero-Erausquin, Fradelizi and Maurey [8] for the Gaussian measure and for the unconditional case. As a variant of the (B)-conjecture, one may study concavity properties of the function $t \mapsto \mu(V(t)A)$ where $V : \mathbb{R} \to \mathbb{R}_+$ is a convex function. As a consequence of Theorem 1.1, we deduce concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}} A)$, $p \in (0,1]$, for every unconditional s-concave measure μ and every unconditional convex body A in $\mathbb{R}^n$ (see Proposition 2.4 below).

Saroglou [22] also proved that the log-Brunn-Minkowski inequality for the Lebesgue measure (inequality (4)) is equivalent to the log-Brunn-Minkowski inequality for all log-concave measures. We continue these kinds of equivalences by proving that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures (see Proposition 3.1 below).

We also investigate functional versions of the (B)-conjecture, which may be read as follows:

Conjecture 1.4 (Functional version of the (B)-conjecture)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be even log-concave functions. Then the function*

$$t \mapsto \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x)g(x)\,\mathrm{d}x$$

is log-concave on $\mathbb{R}$.

We prove that Conjecture 1.4 is equivalent to Conjecture 1.3 (see Proposition 3.2 below).

Let us note that other developments in the use of the earlier mentioned extensions of the Brunn-Minkowski inequality have been recently made as well (see *e.g.* [2], [6], [7], [15]).

The rest of the paper is organized as follows: In the next section, we prove Theorem 1.1 and we extend it to m sets, $m \geq 2$. We also compare our L^p-Minkowski combination to the Firey combination and derive an L^p-Brunn-Minkowski inequality for the Firey combination. We then discuss the consequences of a variant of the (B)-conjecture, namely we deduce concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}}A)$, $p \in (0, 1]$. In Section 3, we prove that the (B)-conjecture for all uniform measures is equivalent to the (B)-conjecture for all log-concave measures, and we also prove that the (B)-conjecture is equivalent to its functional version Conjecture 1.4.

2 Proof of Theorem 1.1 and consequences

2.1 Proof of Theorem 1.1

Before proving Theorem 1.1, let us show that our L^p-Minkowski combination coincides with the classical Minkowski sum when $\mathbf{p} = (1, \cdots, 1)$, for unconditional convex sets.

Proposition 2.1. *Let A, B be unconditional convex subsets of $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then,*

$$(1-\lambda)\cdot A +_{\mathbf{1}} \lambda \cdot B = (1-\lambda)A + \lambda B,$$

where $\mathbf{1} = (1, \cdots, 1)$.

Proof. Since the sets $(1-\lambda)\cdot A +_{\mathbf{1}} \lambda\cdot B$ and $(1-\lambda)A+\lambda B$ are unconditional, it is sufficient to prove that

$$((1-\lambda)\cdot A +_{\mathbf{1}} \lambda \cdot B)\cap(\mathbb{R}_+)^n = ((1-\lambda)A+\lambda B)\cap(\mathbb{R}_+)^n.$$

Let $x \in ((1-\lambda)A+\lambda B)\cap(\mathbb{R}_+)^n$. There exists $a=(a_1,\cdots,a_n)\in A$ and $b=(b_1,\cdots,b_n)\in B$ such that $x=(1-\lambda)a+\lambda b$ and for every $i\in\{1,\cdots,n\}$, $(1-\lambda)a_i+\lambda b_i \in \mathbb{R}_+$. Let $\varepsilon,\eta\in\{-1,1\}^n$ such that $(\varepsilon_1 a_1,\cdots,\varepsilon_n a_n)\in(\mathbb{R}_+)^n$ and $(\eta_1 b_1,\cdots,\eta_n b_n)\in(\mathbb{R}_+)^n$. Notice that for every $i\in\{1,\cdots,n\}$, $0\le (1-\lambda)a_i+\lambda b_i \le (1-\lambda)\varepsilon_i a_i+\lambda\eta_i b_i$. Since the sets A and B are convex and unconditional, it follows that $x\in(1-\lambda)(A\cap(\mathbb{R}_+)^n)+\lambda(B\cap(\mathbb{R}_+)^n) = ((1-\lambda)\cdot A +_{\mathbf{1}} \lambda\cdot B)\cap(\mathbb{R}_+)^n$.

The other inclusion is clear due to the definition of the set $(1-\lambda)\cdot A +_{\mathbf{1}} \lambda\cdot B$. □

Proof of Theorem 1.1. Let $\lambda\in[0,1]$ and let A,B be unconditional convex bodies in $\mathbb{R}^n$.

It has been shown by Uhrin [24] that if $f,g,h:(\mathbb{R}_+)^n\to\mathbb{R}_+$ are bounded measurable functions such that for every $x,y\in(\mathbb{R}_+)^n$, $h((1-\lambda)x +_{\mathbf{p}} \lambda y)\ge M_\alpha^\lambda(f(x),g(y))$, then

$$\int_{(\mathbb{R}_+)^n} h(x)\,\mathrm{d}x \ge M_\gamma^\lambda\left(\int_{(\mathbb{R}_+)^n} f(x)\,\mathrm{d}x, \int_{(\mathbb{R}_+)^n} g(x)\,\mathrm{d}x\right),$$

where $\gamma=\left(\sum_{i=1}^n p_i^{-1}+\alpha^{-1}\right)^{-1}$.

Let us denote by ϕ the density function of μ and let us set $h=1_{(1-\lambda)\cdot A+_{\mathbf{p}}\lambda\cdot B}\phi$, $f=1_A\phi$ and $g=1_B\phi$. By assumption, the function ϕ is unconditional and α-concave, hence ϕ is non-increasing in each coordinate on the octant $(\mathbb{R}_+)^n$. Then for every $x,y\in(\mathbb{R}_+)^n$, one has

$$\phi((1-\lambda)x +_{\mathbf{p}} \lambda y)\ge\phi((1-\lambda)x+\lambda y)\ge M_\alpha^\lambda(\phi(x),\phi(y)).$$

Hence,

$$h((1-\lambda)x +_{\mathbf{p}} \lambda y)\ge M_\alpha^\lambda(f(x),g(y)).$$

Thus we may apply the result mentioned at the beginning of the proof to obtain that

$$\int_{(\mathbb{R}_+)^n} h(x)\,\mathrm{d}x \ge M_\gamma^\lambda\left(\int_{(\mathbb{R}_+)^n} f(x)\,\mathrm{d}x, \int_{(\mathbb{R}_+)^n} g(x)\,\mathrm{d}x\right),$$

where $\gamma=\left(\sum_{i=1}^n p_i^{-1}+\alpha^{-1}\right)^{-1}$. In other words, one has

$$\mu(((1-\lambda)\cdot A +_{\mathbf{p}} \lambda\cdot B)\cap(\mathbb{R}_+)^n)\ge M_\gamma^\lambda(\mu(A\cap(\mathbb{R}_+)^n),\mu(B\cap(\mathbb{R}_+)^n)).$$

Since the sets $(1-\lambda)\cdot A +_{\mathbf{p}} \lambda\cdot B$, A and B are unconditional, it follows that

$$\mu((1-\lambda)\cdot A +_{\mathbf{p}} \lambda\cdot B)\ge M_\gamma^\lambda(\mu(A),\mu(B)).$$

□

Remark. One may similarly define the L^p-Minkowski combination

$$\lambda_1 \cdot A_1 +_{\mathbf{p}} \cdots +_{\mathbf{p}} \lambda_m \cdot A_m,$$

for m convex bodies $A_1, \ldots, A_m \subset \mathbb{R}^n$, $m \geq 2$, where $\lambda_1, \ldots, \lambda_m \in [0,1]$ are such that $\sum_{i=1}^m \lambda_i = 1$, by extending the definition of the p-mean M_p^λ to m non-negative numbers. By induction, one has under the same assumptions of Theorem 1.1 that

$$\mu(\lambda_1 \cdot A_1 +_{\mathbf{p}} \cdots +_{\mathbf{p}} \lambda_m \cdot A_m) \geq M_\gamma^\lambda(\mu(A_1), \cdots, \mu(A_m)), \tag{5}$$

where $\gamma = \left(\sum_{i=1}^n p_i^{-1} + \alpha^{-1}\right)^{-1}$. Indeed, let $m \geq 2$ and let us assume that inequality (5) holds. Notice that

$$\lambda_1 \cdot A_1 +_{\mathbf{p}} \cdots +_{\mathbf{p}} \lambda_m \cdot A_m +_{\mathbf{p}} \lambda_{m+1} \cdot A_{m+1} = \left(\sum_{i=1}^m \lambda_i\right) \cdot \widetilde{A} +_{\mathbf{p}} \lambda_{m+1} \cdot A_{m+1},$$

where

$$\widetilde{A} := \left(\frac{\lambda_1}{\sum_{i=1}^m \lambda_i} \cdot A_1 +_{\mathbf{p}} \cdots +_{\mathbf{p}} \frac{\lambda_m}{\sum_{i=1}^m \lambda_i} \cdot A_m\right).$$

Thus,

$$\begin{aligned} \mu\left(\left(\sum_{i=1}^m \lambda_i\right) \cdot \widetilde{A} +_{\mathbf{p}} \lambda_{m+1} \cdot A_{m+1}\right) &\geq \left(\left(\sum_{i=1}^m \lambda_i\right) \mu(\widetilde{A})^\gamma + \lambda_{m+1}\mu(A_{m+1})^\gamma\right)^{\frac{1}{\gamma}} \\ &\geq \left(\sum_{i=1}^{m+1} \lambda_i \mu(A_i)^\gamma\right)^{\frac{1}{\gamma}}. \end{aligned}$$

2.2 Consequences

The following result compares the L^p-Minkowski combinations $\oplus_p$ and $+_{\mathbf{p}}$.

Lemma 2.2. *Let $p \in [0,1]$ and set $\mathbf{p} = (p, \cdots, p) \in [0,1]^n$. For every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda \in [0,1]$, one has*

$$(1-\lambda) \cdot A \oplus_p \lambda \cdot B \supset (1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B.$$

Proof. The case $p = 0$ is proved in [21]. Let $p \neq 0$. Since the sets $(1-\lambda) \cdot A \oplus_p \lambda \cdot B$ and $(1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B$ are unconditional, it is sufficient to prove that

$$((1-\lambda) \cdot A \oplus_p \lambda \cdot B) \cap (\mathbb{R}_+)^n \supset ((1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B) \cap (\mathbb{R}_+)^n.$$

Let $u \in S^{n-1} \cap (\mathbb{R}_+)^n$ and let $x \in ((1-\lambda) \cdot A +_{\mathbf{p}} \lambda \cdot B) \cap (\mathbb{R}_+)^n$. One has,

$$\begin{aligned}
\langle x, u \rangle &= \sum_{i=1}^{n} ((1-\lambda)a_i^p + \lambda b_i^p)^{\frac{1}{p}} u_i \\
&= \sum_{i=1}^{n} ((1-\lambda)(a_i u_i)^p + \lambda (b_i u_i)^p)^{\frac{1}{p}} \\
&= \|(1-\lambda)X + \lambda Y\|_{\frac{1}{p}}^{\frac{1}{p}},
\end{aligned}$$

where $X = ((a_1 u_1)^p, \cdots, (a_n u_n)^p)$ and $Y = ((b_1 u_1)^p, \cdots, (b_n u_n)^p)$. Notice that $\|X\|_{\frac{1}{p}} \leq h_A(u)^p$, $\|Y\|_{\frac{1}{p}} \leq h_B(u)^p$ and that $\|\cdot\|_{\frac{1}{p}}$ is a norm. It follows that

$$\langle x, u \rangle \leq \left((1-\lambda)\|X\|_{\frac{1}{p}} + \lambda \|Y\|_{\frac{1}{p}}\right)^{\frac{1}{p}} \leq ((1-\lambda) h_A(u)^p + \lambda h_B(u)^p)^{\frac{1}{p}}.$$

Hence, $x \in ((1-\lambda) \cdot A \oplus_p \lambda \cdot B) \cap (\mathbb{R}_+)^n$. □

From Lemma 2.2 and Theorem 1.1, one obtains the following result:

Corollary 2.3. *Let $p \in [0,1]$. Let μ be an unconditional measure in $\mathbb{R}^n$ that has an α-concave density function, with $\alpha \geq -\frac{p}{n}$. Then for every unconditional convex body A, B in $\mathbb{R}^n$ and for every $\lambda \in [0,1]$,*

$$\mu((1-\lambda) \cdot A \oplus_p \lambda \cdot B) \geq M_\gamma^\lambda(\mu(A), \mu(B)), \tag{6}$$

where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$.

In Corollary 2.3, if α or p is equal to 0, then γ is defined by continuity and is equal to 0.

Remarks.

1. By taking $\alpha = 0$ in Corollary 2.3 (corresponding to log-concave measures), one obtains

$$\mu((1-\lambda) \cdot A \oplus_0 \lambda \cdot B) \geq \mu(A)^{1-\lambda} \mu(B)^\lambda.$$

2. By taking $\alpha = +\infty$ in Corollary 2.3 (corresponding to $\frac{1}{n}$-concave measures), one obtains that for every $p \in [0,1]$,

$$\mu((1-\lambda) \cdot A \oplus_p \lambda \cdot B)^{\frac{p}{n}} \geq (1-\lambda)\mu(A)^{\frac{p}{n}} + \lambda \mu(B)^{\frac{p}{n}}.$$

Equivalently, for every $p \in [0,1]$, for every unconditional convex body A, B in $\mathbb{R}^n$ and for every unconditional convex set $K \subset \mathbb{R}^n$,

$$|((1-\lambda) \cdot A \oplus_p \lambda \cdot B) \cap K|^{\frac{p}{n}} \geq (1-\lambda)|A \cap K|^{\frac{p}{n}} + \lambda |B \cap K|^{\frac{p}{n}}.$$

Let us recall that the function $t \mapsto \mu(\mathrm{e}^t A)$ is log-concave on $\mathbb{R}$ for every unconditional log-concave measure μ and every unconditional convex body A in $\mathbb{R}^n$ (see [8]). By adapting the argument of [20], Proof of Proposition 3.1 (see Proof of Corollary 2.5 below), it follows that the function $t \mapsto \mu(t^{\frac{1}{p}}A)$ is $\frac{p}{n}$-concave on $\mathbb{R}_+$, for every $p \in (0,1]$, for every unconditional s-concave measure μ, with $s \geq 0$, and for every unconditional convex body A in $\mathbb{R}^n$. However, no concavity properties are known for the function $t \mapsto \mu(\mathrm{e}^t A)$ when μ is an s-concave measure with $s < 0$. Instead, for these measures we prove concavity properties of the function $t \mapsto \mu(t^{\frac{1}{p}}A)$.

Proposition 2.4. *Let $p \in (0,1]$, let μ be an unconditional measure that has an α-concave density function, with $\alpha \in [-\frac{p}{n}, 0)$ and let A be an unconditional convex body in $\mathbb{R}^n$. Then the function $t \mapsto \mu(t^{\frac{1}{p}}A)$ is $\left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$-concave on $\mathbb{R}_+$.*

Proof. Let $t_1, t_2 \in \mathbb{R}_+$. By applying Corollary 2.3 to the sets $t_1^{\frac{1}{p}}A$ and $t_2^{\frac{1}{p}}A$, one obtains

$$\mu(((1-\lambda)t_1 + \lambda t_2)^{\frac{1}{p}}A) = \mu((1-\lambda)\cdot t_1^{\frac{1}{p}}A \oplus_p \lambda \cdot t_2^{\frac{1}{p}}A) \geq M_\gamma^\lambda(\mu(t_1^{\frac{1}{p}}A), \mu(t_2^{\frac{1}{p}}A)),$$

where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$. Hence the function $t \mapsto \mu(t^{\frac{1}{p}}A)$ is γ-concave on $\mathbb{R}_+$. $\square$

As a consequence, we derive concavity properties for the function $t \mapsto \mu(tA)$.

Corollary 2.5. *Let $p \in (0,1]$, let μ be an unconditional measure that has an α-concave density function, with $\alpha \in [-\frac{p}{n}, 0)$, and let A be an unconditional convex body in $\mathbb{R}^n$. Then the function $t \mapsto \mu(tA)$ is $\left(\frac{1-p}{n} + \gamma\right)$-concave on $\mathbb{R}_+$, where $\gamma = \left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}$.*

Proof. We adapt [20], Proof of Proposition 3.1. Let us denote by ϕ the density function of the measure μ and let us denote by F the function $t \mapsto \mu(tA)$. From Proposition 2.4, the function $t \mapsto F(t^{\frac{1}{p}})$ is γ-concave, hence the right derivative of F, denoted by F'_+, exists everywhere and the function $t \mapsto \frac{1}{p}t^{\frac{1}{p}-1}F'_+(t^{\frac{1}{p}})F(t^{\frac{1}{p}})^{\gamma-1}$ is non-increasing. Notice that

$$F(t) = t^n \int_A \phi(tx)\,\mathrm{d}x,$$

and that $t \mapsto \phi(tx)$ is non-increasing, thus the function $t \mapsto \frac{1}{t^{1-p}}F(t)^{\frac{1-p}{n}}$ is non-increasing. Since

$$F'_+(t)F(t)^{\frac{1-p}{n}+\gamma-1} = t^{1-p}F'_+(t)F(t)^{\gamma-1} \cdot \frac{1}{t^{1-p}}F(t)^{\frac{1-p}{n}},$$

it follows that $F'_+(t)F(t)^{\frac{1-p}{n}+\gamma-1}$ is non-increasing as the product of two non-negative non-increasing functions. Hence F is $\left(\frac{1-p}{n}+\gamma\right)$-concave. $\square$

Remark. For every s-concave measure μ and every convex subset $A \subset \mathbb{R}^n$, the function $t \mapsto \mu(tA)$ is s-concave. Hence Corollary 2.5 is of value only if $\frac{1-p}{n}+\gamma \geq \frac{\alpha}{1+\alpha n}$ (see relation (2)). Notice that this condition is satisfied if $\alpha \geq -\frac{p}{n(1+p)}$. We thus obtain the following corollary:

Corollary 2.6. *Let $p \in (0,1]$, let μ be an unconditional measure that has an α-concave density function, with $-\frac{p}{n(1+p)} \leq \alpha < 0$ and let K be an unconditional convex body in $\mathbb{R}^n$. Then, for every subsets $A, B \in \{\mu K; \mu > 0\}$ and every $\lambda \in [0,1]$, one has*

$$\mu((1-\lambda)A+\lambda B) \geq M^{\lambda}_{\frac{1-p}{n}+\gamma}(\mu(A),\mu(B)),$$

where $\gamma = \left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}$.

In [20], the author investigated improvements of concavity properties of convex measures under additional assumptions, such as symmetries. Notice that Corollary 2.6 follows the same path and completes the results that can be found in [20]. Let us conclude this section by the following remark, which concerns the question of the improvement of concavity properties of convex measures.

Remark. Let μ be a Borel measure that has a density function with respect to the Lebesgue measure in $\mathbb{R}^n$. One may write the density function of μ in the form e^{-V}, where $V : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Let us assume that V is C^2. Let $\gamma > 0$. The function e^{-V} is γ-concave if $\mathrm{Hess}(\gamma \mathrm{e}^{-\gamma V})$, the Hessian of $\gamma \mathrm{e}^{-\gamma V}$, is non-positive (in the sense of symmetric matrices). One has

$$\mathrm{Hess}(\gamma \mathrm{e}^{-\gamma V}) = -\gamma^2 \nabla \cdot (\nabla V \mathrm{e}^{-\gamma V}) = \gamma^2 \mathrm{e}^{-V}(\gamma \nabla V \otimes \nabla V - \mathrm{Hess}(V)),$$

where $\nabla V \otimes \nabla V = \left(\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j}\right)_{1 \leq i,j \leq n}$. Hence the matrix $\mathrm{Hess}(\gamma \mathrm{e}^{-\gamma V})$ is non-positive if and only if the matrix $\gamma \nabla V \otimes \nabla V - \mathrm{Hess}(V)$ is non-positive.

Let us apply this remark to the Gaussian measure

$$\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x, \quad x \in \mathbb{R}^n.$$

Here $V(x) = \frac{|x|^2}{2} + c_n$, where $c_n = \frac{n}{2}\log(2\pi)$. Thus, $\nabla V \otimes \nabla V = (x_i x_j)_{1 \leq i,j \leq n}$ and $\mathrm{Hess}(V) = Id$ the Identity matrix. Notice that the eigenvalues of $\gamma \nabla V \otimes$

$\nabla V - \mathrm{Hess}(V)$ are -1 (with multiplicity $(n-1)$) and $\gamma|x|^2 - 1$. Hence if $\gamma|x|^2 - 1 \leq 0$, then $\gamma\nabla V \otimes \nabla V - \mathrm{Hess}(V)$ is non-positive. One deduces that for every $\gamma > 0$, for every compact sets $A, B \subset \frac{1}{\sqrt{\gamma}} B_2^n$ and for every $\lambda \in [0,1]$, one has

$$\gamma_n((1-\lambda)A + \lambda B) \geq M^{\lambda}_{\frac{\gamma}{1+\gamma n}}(\gamma_n(A), \gamma_n(B)), \tag{7}$$

where B_2^n denotes the Euclidean closed unit ball in $\mathbb{R}^n$.

Since the Gaussian measure is a log-concave measure, inequality (7) is an improvement of the concavity of the Gaussian measure when restricted to compact sets $A, B \subset \frac{1}{\sqrt{\gamma}} B_2^n$.

3 Equivalence between (B)-conjecture-type problems

In the following proposition, we demonstrate that it is sufficient to prove the (B)-conjecture for all uniform measures in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, to obtain the (B)-conjecture for all symmetric log-concave measures in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$. This completes recent works by Saroglou [21], [22].

In the following, we say that a measure μ satisfies the (B)-property if the function $t \mapsto \mu(\mathrm{e}^t A)$ is log-concave on $\mathbb{R}$ for every symmetric convex set $A \subset \mathbb{R}^n$.

Proposition 3.1. *If every symmetric uniform measure in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, satisfies the (B)-property, then every symmetric log-concave measure in $\mathbb{R}^n$, for every $n \in \mathbb{N}^*$, satisfies the (B)-property.*

Proof. The proof is inspired by [1], beginning of Section 3.

Step 1: Stability under orthogonal projection
Let us show that the (B)-property is stable under orthogonal projection onto an arbitrary subspace.

Let F be a k-dimensional subspace of $\mathbb{R}^n$. Let us define for every compactly supported measure μ in $\mathbb{R}^n$ and every measurable subset $A \subset F$,

$$\Pi_F\mu(A) := \mu(\Pi_F^{-1}(A)),$$

where Π_F denotes the orthogonal projection onto F and $\Pi_F^{-1}(A) := \{x \in \mathbb{R}^n; \Pi_F(x) \in A\}$.

Notice that $\Pi_F^{-1}(\mathrm{e}^t A) = \mathrm{e}^t(A \times F^{\perp})$, where $F^{\perp}$ denotes the orthogonal complement of F. Hence if μ satisfies the (B)-property, then $\Pi_F\mu$ satisfies the (B)-property.

Step 2: Approximation of log-concave measures
Let us show that for every compactly supported log-concave measure μ in

$\mathbb{R}^n$ there exists a sequence $(K_p)_{p\in\mathbb{N}^*}$ of convex subsets of $\mathbb{R}^{n+p}$ such that $\lim_{p\to+\infty}\Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$ in the sense that the density function of μ is the pointwise limit of the density functions of $(\mu_{K_p})_{p\in\mathbb{N}^*}$, where μ_{K_p} denotes the uniform measure on K_p (up to a constant).

Let μ be a compactly supported log-concave measure in $\mathbb{R}^n$ with density function $f = \mathrm{e}^{-V}$, where $V : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a convex function. Notice that for every $x \in \mathbb{R}^n$, $\mathrm{e}^{-V(x)} = \lim_{p\to+\infty}(1-\frac{V(x)}{p})_+^p$, where for every $a \in \mathbb{R}$, $a_+ = \max(a, 0)$. Let us define for every $p \in \mathbb{N}^*$,

$$K_p = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p; |y| \leq \left(1-\frac{V(x)}{p}\right)_+\}.$$

One has for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\left(1-\frac{V(x)}{p}\right)_+^p &= \int_0^{\left(1-\frac{V(x)}{p}\right)_+} pr^{p-1}\,\mathrm{d}r \\
&= p\int_0^{+\infty} 1_{[0,\left(1-\frac{V(x)}{p}\right)_+]}(r)r^{p-1}\,\mathrm{d}r \\
&= \frac{1}{v_p}\int_{\mathbb{R}^p} 1_{K_p}(x,y)\,\mathrm{d}y,
\end{aligned}$$

the last equality follows from an integration in polar coordinates, where v_p denotes the volume of the Euclidean closed unit ball in $\mathbb{R}^p$. By denoting μ_{K_p} the measure in $\mathbb{R}^{n+p}$ with density function

$$\frac{1}{v_p}1_{K_p}(x,y), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^p,$$

it follows that for every $p \in \mathbb{N}^*$, the measure $\Pi_{\mathbb{R}^n}\mu_{K_p}$ has density function

$$\left(1-\frac{V(x)}{p}\right)_+^p, \quad x \in \mathbb{R}^n.$$

We conclude that $\lim_{p\to+\infty}\Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$.

Step 3: Conclusion

Let $n \in \mathbb{N}^*$ and let μ be a symmetric log-concave measure in $\mathbb{R}^n$. By approximation, one can assume that μ is compactly supported. Since μ is symmetric, the sequence $(K_p)_{p\in\mathbb{N}^*}$ defined in Step 2 is a sequence of symmetric convex subsets of $\mathbb{R}^{n+p}$. If we assume that the (B)-property holds for all uniform measures in $\mathbb{R}^m$, for every $m \in \mathbb{N}^*$, then for every $p \in \mathbb{N}^*$, μ_{K_p} satisfies the (B)-property. It follows from Step 1 that for every $p \in \mathbb{N}^*$, $\Pi_{\mathbb{R}^n}\mu_{K_p}$ satisfies the (B)-property. Since $\lim_{p\to+\infty}\Pi_{\mathbb{R}^n}\mu_{K_p} = \mu$ (*c.f.* Step 2) and since a pointwise limit of log-concave functions is log-concave, we conclude that μ satisfies the (B)-property. $\square$

Similarly, let us prove that the functional form of the (B)-conjecture (Conjecture 1.4) is equivalent to the classical (B)-conjecture (Conjecture 1.3).

Proposition 3.2. *One has equivalence between the following properties:*

1. *For every $n \in \mathbb{N}^*$, for every symmetric log-concave measure μ in $\mathbb{R}^n$ and for every symmetric convex subset A of $\mathbb{R}^n$, the function $t \mapsto \mu(\mathrm{e}^t A)$ is log-concave on $\mathbb{R}$.*

2. *For every $n \in \mathbb{N}^*$, for every even log-concave functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, the function $t \mapsto \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x)g(x)\,\mathrm{d}x$ is log-concave on $\mathbb{R}$.*

Proof. 2. $\Longrightarrow$ 1. This is clear by taking $f = 1_A$, the indicator function of a symmetric convex set A, and by taking g to be the density function of a log-concave measure μ.
1. $\Longrightarrow$ 2. Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be even log-concave functions. By approximation, one may assume that f and g are compactly supported. Let us write $g = \mathrm{e}^{-V}$, where $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is an even convex function. One has

$$G(t) := \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x)\mathrm{e}^{-V(x)}\,\mathrm{d}x = \lim_{p \to +\infty} \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x)\left(1 - \frac{V(x)}{p}\right)_+^p \mathrm{d}x,$$

where for every $a \in \mathbb{R}$, $a_+ = \max(a, 0)$. Let us denote for $t \in \mathbb{R}$,

$$G_p(t) = \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x)\left(1 - \frac{V(x)}{p}\right)_+^p \mathrm{d}x.$$

We have seen in the proof of Proposition 3.1 that

$$\left(1 - \frac{V(x)}{p}\right)_+^p = \frac{1}{v_p}\int_{\mathbb{R}^p} 1_{K_p}(x, y)\,\mathrm{d}y,$$

where $K_p := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p; |y| \leq \left(1 - \frac{V(x)}{p}\right)_+\}$ and where v_p denotes the volume of the Euclidean closed unit ball in $\mathbb{R}^p$. Hence,

$$G_p(t) = \frac{1}{v_p}\int_{K_p} f(\mathrm{e}^{-t}x)1_{\mathbb{R}^p}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Notice that K_p is a symmetric convex subset of $\mathbb{R}^{n+p}$. The change of variable $\widetilde{x} = \mathrm{e}^{-t}x$ and $\widetilde{y} = \mathrm{e}^{-t}y$ leads to

$$G_p(t) = \frac{\mathrm{e}^{t(n+p)}}{v_p}\mu_p(\mathrm{e}^{-t}K_p),$$

where μ_p is the measure with density function

$$h(x, y) = f(x)1_{\mathbb{R}^p}(y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^p.$$

Since a pointwise limit of log-concave functions is log-concave, we conclude that the function G is log-concave on $\mathbb{R}$ as the pointwise limit of the log-concave functions G_p, $p \in \mathbb{N}^*$. $\square$

Recall that the (B)-conjecture holds true for the Gaussian measure and for the unconditional case (see [8]). From the techniques of the proof of Proposition 3.2, it follows that Conjecture 1.4 holds true if one function is the density function of the Gaussian measure or if both functions are unconditional.

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