

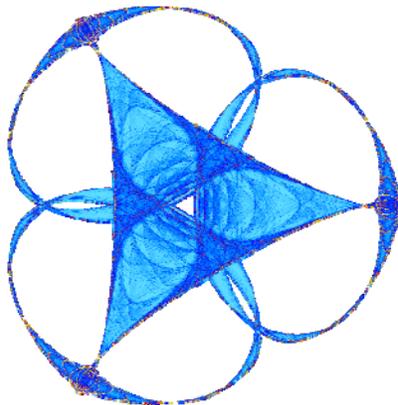
HELLY NUMBERS OF SUBSETS OF \mathbb{R}^D AND SAMPLING
TECHNIQUES IN OPTIMIZATION

By

J. A. De Loera, R. N. La Haye, D. Oliveros, E. Roldán-Pensado

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455-0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: <http://www.ima.umn.edu>

HELLY NUMBERS OF SUBSETS OF \mathbb{R}^d AND SAMPLING TECHNIQUES IN OPTIMIZATION

J. A. DE LOERA, R. N. LA HAYE, D. OLIVEROS, AND E. ROLDÁN-PENSADO

ABSTRACT. We present Helly-type theorems where the convex sets are required to intersect a subset S of \mathbb{R}^d . This is a continuation of prior work for $S = \mathbb{R}^d$, \mathbb{Z}^d , and $\mathbb{Z}^{d-k} \times \mathbb{R}^k$ (motivated by mixed-integer optimization). We are particularly interested in the case when S has some algebraic structure, in particular when S is a subgroup or the difference between a lattice and some sublattices. We give sharp bounds on the Helly numbers for S in several cases. By abstracting the ingredients of a general method we obtain colorful versions of many monochromatic Helly-type results, including several of our results. In the second part of the article we discuss a notion of S -optimization that generalizes continuous, integral, and mixed-integer optimization and show that two well-known randomized sampling algorithms, Clarkson's and Calafiore-Campi's algorithms, can be extended to work with more sophisticated variables over S as long as the S -Helly number $h(S)$ is finite.

1. INTRODUCTION

Eduard Helly stated his fundamental theorem slightly under a century years ago [23]. It says that a finite family of convex sets in \mathbb{R}^d intersect if every $d + 1$ of them intersect (see [28] for the basic introduction to this part of convexity). Since its discovery Helly's theorem has found many generalizations, extensions and applications in many areas of mathematics (see [13, 14, 18, 34] and references therein). Our paper presents new versions of Helly's theorem where the intersections in the hypotheses and conclusions of the theorems are restricted to exist in a proper subset S of \mathbb{R}^d . We also discuss new applications of the Helly numbers we obtain in the context of randomized algorithms for convex and chance-constrained optimization.

The 1970s and 1980s saw a growth of the research of *abstract convexity* where Helly-type theorems were explored in very abstract settings beyond Euclidean spaces. In this article, given a proper subset $S \subset \mathbb{R}^d$ we consider the *convexity space* whose convex sets are the intersections of the standard convex sets in \mathbb{R}^d with S (see [3, 17, 24, 25, 27, 33] and the many references therein for more on abstract convexity spaces). For example, there is a Helly-type theorem that talks about the existence of

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intersections over the *integer lattice* \mathbb{Z}^d , proved by Doignon [16] (later rediscovered in [9, 24, 30]); it states that a finite family of convex sets in \mathbb{R}^d intersect at a point of \mathbb{Z}^d if every 2^d of members of the family intersect at a point of \mathbb{Z}^d . A second example is the work of Averkov and Weismantel [5] who gave a *mixed* version of Helly's and Doignon's theorems which includes them both. This time the intersection of the convex sets is required to be in $\mathbb{Z}^{d-k} \times \mathbb{R}^k$ and this can be guaranteed if every $2^{d-k}(k+1)$ sets intersect in such a point (their formula had been previously stated by A.J. Hoffmann in [24]). Note that the size of subfamilies with intersection in S that guarantees an intersection of all members of the family in S depends on S . These are the *Helly numbers of S* from the title, which we now define precisely.

For a nonempty family \mathcal{K} of sets, the *Helly number* $h = h(\mathcal{K}) \in \mathbb{N}$ of \mathcal{K} is defined as the smallest number satisfying the following:

$$(1.1) \quad \forall i_1, \dots, i_h \in [m] : F_{i_1} \cap \dots \cap F_{i_h} \neq \emptyset \quad \implies \quad F_1 \cap \dots \cap F_m \neq \emptyset$$

for all $m \in \mathbb{N}$ and $F_1, \dots, F_m \in \mathcal{K}$. If no such h exists, then $h(\mathcal{K}) := \infty$. E.g., for the traditional Helly's theorem, \mathcal{K} is the family of all convex subsets of \mathbb{R}^d .

Finally for $S \subseteq \mathbb{R}^d$ we define

$$h(S) := h(\{S \cap K : K \subset \mathbb{R}^d \text{ is convex}\}).$$

Therefore $h(S)$ is the Helly number when the sets are required to intersect at points in S , we will call this the *S -Helly number*. We stress again that a set S and its convex sets $\mathcal{K}_S = \{S \cap K : K \subset \mathbb{R}^d \text{ is convex}\}$ form a convexity space. The key problem we study here is: given a set S , bound $h(S)$.

For instance, note that when S is finite then the bound $h(S) \leq \#(S)$ is trivial. The original Helly number is $h(\mathbb{R}^d) = d + 1$ and interestingly, if $S = \mathcal{F}$ is any subfield of \mathbb{R} (e.g., $\mathbb{Q}(\sqrt{2})$), then Radon's proof of Helly's theorem directly shows that the S -Helly number of \mathcal{F}^d is still $d + 1$. We know $h(\mathbb{Z}^{d-k} \times \mathbb{R}^k) = 2^{d-k}(k+1)$. However, if S is not necessarily a lattice but a general additive subgroup (e.g., $S = \{(\alpha\pi + \beta, \gamma) \in \mathbb{R}^2 : \alpha, \beta, \gamma \in \mathbb{Z}\}$), then the S -Helly number is not covered by prior results and in fact has a more complex answer. It is known that the Helly number may be infinite in some situations. In Section 2 we state some prior results which are useful in computing S -Helly numbers.

Now we are ready to state our contributions:

Bounds on Helly numbers for large classes of subsets of \mathbb{R}^d . We present new Helly-type theorems for large classes of sets $S \subset \mathbb{R}^d$. For simplicity we will always assume that the linear span of S is the whole space \mathbb{R}^d . For the case of dimension $d = 1$ it is easy to see that for any set S the Helly number $h(S)$ exists and is at most 2. In Section 3 we present the following two theorems:

Theorem 1.1. *Let S be a dense subset of \mathbb{R}^2 , then $h(S) \leq 4$. This result is sharp.*

Theorem 1.2. *Let L be a proper sublattice of \mathbb{Z}^2 . If $S = \mathbb{Z}^2 \setminus L$ then $h(S) \leq 6$.*

In Section 4 we explore the situation where S is a general additive subgroup of \mathbb{R}^d (not necessarily closed). This gives rise to an interesting open question. If S has additional structure we can obtain bounds for $h(S)$.

Theorem 1.3. *Let G be a dense subgroup of \mathbb{R} . If $S \subset \mathbb{R}^d$ is a G -module then $h(S) \leq 2d$.*

We then extend Doignon's theorem for lattices by proving that the difference of a lattice with the union of several sublattices has bounded Helly number. We prove the following theorem:

Theorem 1.4. *Let L_1, \dots, L_k be (translated) sublattices of \mathbb{Z}^d . Then the set $S = \mathbb{Z}^d \setminus (L_1 \cup \dots \cup L_k)$ has Helly number $h(S) \leq C_k 2^d$ for some constant C_k depending only on k .*

A quantitative version of Theorem 1.4, with different bounds, was obtained in [14].

Colorful versions. Many Helly-type theorems admit a colorful version. In this setting, each convex set is assigned one of N colors and every N convex sets of distinct colors are required to intersect. The conclusion is that there is a color for which all convex sets intersect. In Section 5 we analyze a popular method that yields colorful versions of several Helly-type theorems. As a result we find a few new colorful Helly-type theorems.

S -optimization and Randomized sampling algorithms. In section 6, we show new applications that Helly numbers have in optimization (this interaction has motivated a lot of papers about Helly numbers, e.g., [5, 9, 12, 20, 24, 30]). We introduce here the notion of S -optimization that naturally generalizes continuous, integral, and mixed-integer optimization:

Given $S \subset \mathbb{R}^d$, the optimization problem with equations and inequalities

$$\begin{aligned} & \max && f(\mathbf{x}) \\ & \text{subject to} && g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, n, \\ & && h_j(\mathbf{x}) = 0, \quad j = 1, 2, \dots, m, \\ & && \mathbf{x} \in S. \end{aligned}$$

will be called an S -optimization problem. When only linear constraints are present this is an S -linear program.

Clearly when $S = \mathbb{R}^d$ the S -optimization problem is the usual continuous optimization, when $S = \mathbb{Z}^d$, this is just integer optimization, and $S = \mathbb{Z}^k \times \mathbb{R}^{d-k}$ it is the case of mixed-integer optimization.

To demonstrate the natural power of S -optimization we show that two well-known methods of optimization that involve sampling can be extended to more sophisticated values of variables taking values over S as long as the S -Helly number $h(S)$ is finite. We present two theorems: one generalizes the scenario approximation method of Calafiore and Campi [10, 11] and the other generalizes Clarkson's randomized algorithm for convex and geometric optimization problems [12, 20].

Theorem 1.5. *Let $S \subseteq \mathbb{R}^d$ be a set with a finite Helly number $h(S)$. Let $0 < \epsilon \leq 1$, $0 < \delta < 1$; let $f(\mathbf{x}, w)$ be a convex function in the variables \mathbf{x} and measurable in w . Suppose there is an optimal value \mathbf{x}_* of the linear minimization chance-constrained problem*

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{subject to} \quad & \Pr[f(\mathbf{x}, w) \leq 0] \geq 1 - \epsilon, \\ & \mathbf{x} \in K \text{ convex set,} \\ & \mathbf{x} \in S. \end{aligned}$$

Then from a sufficiently large random sample of N different i.i.d values for w — w^1, w^2, \dots, w^N — \mathbf{x}_ can be δ -approximated by \mathbf{x}_N , the optimal solution of the convex optimization problem*

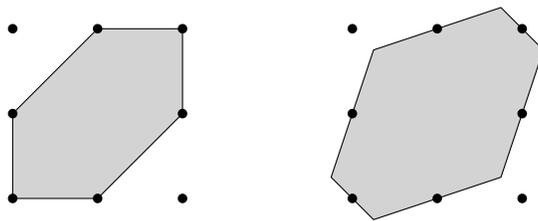
$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{subject to} \quad & f(\mathbf{x}, w^i) \leq 0, \quad i = 1, 2, \dots, N, \\ & \mathbf{x} \in K \text{ convex set,} \\ & \mathbf{x} \in S. \end{aligned}$$

More precisely, if $N \geq \frac{2(h(S)-1)}{\epsilon} \ln(1/\epsilon) + \frac{2}{\epsilon} \ln(1/\delta) + 2(h(S)-1)$, then the probability that \mathbf{x}_N does not satisfy $\Pr[f(\mathbf{x}_N, w) \leq 0] \geq 1 - \epsilon$ is less than δ .

Theorem 1.6. *Let $S \subseteq \mathbb{R}^d$ be a closed set with a finite Helly number $h(S)$. Using Clarkson's algorithm, one can find a solution of the S -convex optimization problem*

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{subject to} \quad & f_i(x) \leq 0, \quad f_i \text{ convex for all } i = 1, 2, \dots, m, \\ & \mathbf{x} \in S. \end{aligned}$$

in an expected $O(h(S)m + h(S)^{O(h(S))})$ calls to an oracle that solves smaller subsystems of the system above of size $O(h(S))$. Thus, when the oracle runs in polynomial time and $h(S)$ is small, Clarkson's algorithm runs in expected linear time in the number of constraints.

FIGURE 1. An S -vertex-polytope and an S -face-polytope.

To conclude we discuss the meaning of colorful and fractional Helly numbers for optimization problems.

2. USEFUL TOOLS

Now we discuss some useful methods used to bound the S -Helly number of a subset S of \mathbb{R}^d . They were studied first by Hoffman [24] in the setting of abstract convexity, and later Averkov and Weismantel in [5], and Averkov in [4].

We define an S -vertex-polytope as the convex hull of points $x_1, x_2, \dots, x_k \in S$ in convex position such that no other point of S is in $\text{conv}(x_1, \dots, x_k)$. Similarly, an S -face-polytope is defined as the intersection of semi-spaces H_1, H_2, \dots, H_k such that $\bigcap_i H_i$ has k faces and contains exactly k points of S , one contained in the interior of each face. Figure 1 shows an S -vertex-polytope with 6 vertices and an S -face-polytope with 6 sides in \mathbb{R}^2 .

Lemma 2.1 (Theorem 2.1 in [4] and Proposition 3 in [24]). *Assume $S \subset \mathbb{R}^d$ is discrete, then $h(S)$ is equal to the following two numbers:*

- (1) *The supremum of the number of faces of an S -face-polytope.*
- (2) *The supremum of the number of vertices of an S -vertex-polytope.*

The case when S is not discrete is a bit more delicate. However, there is a very general result due to Hoffman. Here we only state the case concerning subsets of \mathbb{R}^d .

Lemma 2.2 (Proposition 2 in [24]). *If $S \subset \mathbb{R}^d$, then $h(S)$ can be computed as the supremum of h such that the following holds: There exists a set $R = \{x_1, \dots, x_h\} \subset S$ such that $\bigcap_i \text{conv}(R \setminus \{x_i\})$ does not intersect S .*

Note that, in particular, the points of R must be in strictly convex position. As a direct application of Lemma 2.2 we have the following proposition.

Proposition 2.3. *If $S_1, S_2 \subseteq \mathbb{R}^d$, then $h(S_1 \cup S_2) \leq h(S_1) + h(S_2)$.*

Another way to compute Helly numbers is by using a theorem of Averkov and Weismantel.

Proposition 2.4 (Theorem 1.1 in [5]). *If M is a closed subset of \mathbb{R}^k then $h(\mathbb{R}^d \times M) \leq (d + 1)h(M)$.*

3. S -HELLY THEOREMS IN LOW DIMENSION

We start by looking at the one-dimensional case, here there is a very general result which immediately follows from Lemma 2.2.

Lemma 3.1. *If $S \subset \mathbb{R}$, then $h(S) = 2$.*

Unfortunately in \mathbb{R}^2 we no longer have such a nice theorem. Consider the example below.

Example 3.2. Let $S_n = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ be a set of n points in strict convex position. Then any subset of S_n can be expressed as $S_n \cap K$ where $K \subset \mathbb{R}^2$ is convex. Lemma 2.1 implies $h(S_n) = n$. If for each n we take a copy of S_n such that their convex hulls do not intersect, then their union S will have $h(S) = \infty$. A simpler, but non-discrete, example with $h(S) = \infty$ is a circumference.

In spite of Example 3.2, we still have a general result in dimension two.

Proof of Theorem 1.1. We begin by noting that the bound of four cannot be improved for arbitrary dense S . This can be done by taking $S = \mathbb{R}^2 \setminus \{0\}$ and convex sets $\{x \geq 0\}$, $\{x \leq 0\}$, $\{y \geq 0\}$, $\{y \leq 0\}$.

Now, assume that $h(S) \geq 5$. Lemma 2.2 provides a set $R = \{x_1, \dots, x_5\}$ such that $\bigcap_i (R \setminus \{x_i\})$ does not intersect S . But since the points of R are in strictly convex position, $\bigcap_i (R \setminus \{x_i\})$ has non-empty interior and must intersect S , a contradiction. \square

Theorem 1.1 does not hold in dimensions three and higher. In fact, we can construct a dense set S in \mathbb{R}^3 with $h(S) = \infty$.

Example 3.3. In \mathbb{R}^3 , consider a set of points S_0 on the plane $\{z = 0\}$ with $h(S_0) = \infty$. Let $S = (\mathbb{R}^3 \setminus \{z = 0\}) \cup S_0$. Then S is clearly dense and $h(S) = \infty$.

One very important family of subsets of \mathbb{R}^d are *lattices*, i.e., discrete subgroups of \mathbb{R}^d . As we mention before, Doignon in [16] made an interesting investigation of the Helly number of a lattice S of rank d and showed that $h(S) = 2^d$ (his work was independently rediscovered by D. Bell and H. Scarf [9, 30]). Recently this result was generalized to force not just a non-empty intersection with the lattice, but to control the number of lattice points in the intersection [1]. We state a particular case of this theorem below:

Lemma 3.4 (See Theorem 1 in [1]). *Let $k = 0, 1$ or 2 . Assume that \mathcal{F} is a family of convex sets in \mathbb{R}^2 such that their intersection contains exactly k points of \mathbb{Z}^2 . Then*

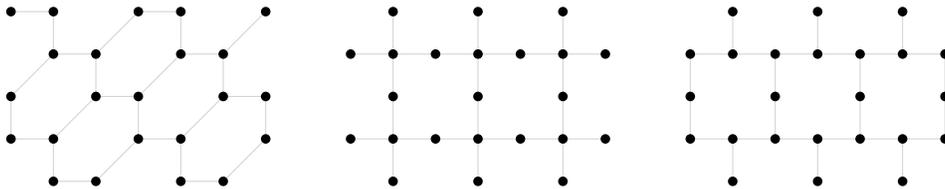


FIGURE 2. Three examples of lattice differences.

there is a subfamily of \mathcal{F} with at most 6 elements such that their intersection contains exactly k points of \mathbb{Z}^2 .

The family consisting of the difference between two lattices is a rather rich set of points in \mathbb{R}^d : they have interesting periodic patterns but contain complicated empty regions, and are closely related to tilings of space [22]. See Figure 2 for some examples. Next we present a tight result in dimension two, which we generalize in Theorem 1.4 to arbitrary dimensions.

Proof of Theorem 1.2. From Lemma 2.1 it is enough to bound the number of faces of an S -face-polygon. Assume $k > 6$ and there is an S -face-polygon K determined by the semiplanes H_1, H_2, \dots, H_k .

Note that if $|\text{int}(K) \cap \mathbb{Z}^2| \leq 2$, then by Lemma 3.4 applied to the interiors of the H_i , we can find H_{i_1}, \dots, H_{i_6} among our original semiplanes such that $\text{int}(\bigcap_j H_{i_j})$ contains no additional points of L . But the relative interior of at least one side of K is contained in $\text{int}(\bigcap_j H_{i_j})$, contradicting that there is a point of S in each side of K . Therefore $|K \cap L| \geq 3$. We look at two possible cases:

1. Suppose there is a triplet of non collinear points in $K \cap L$. Then we may choose $p, q, r \in K \cap L$ such that $T = \text{conv}(p, q, r)$ contains no other point from L . Consequently, the area of T is equal to $\frac{d(L)}{2}$, where $d(L)$ is the determinant of the lattice L . On the other hand, since $\text{int}(K) \cap S$ is empty, $\text{int}(K) \cap L = \text{int}(K) \cap \mathbb{Z}^2$ and therefore $T \cap \mathbb{Z}^2 = \{p, q, r\}$. Hence the area of T is also equal to $\frac{d(\mathbb{Z}^2)}{2}$. This contradicts the fact that L is a proper sublattice of \mathbb{Z}^2 .

2. Suppose that the points of $\text{int}(K) \cap L$ are collinear. After applying a suitable \mathbb{Z}^2 -preserving linear transformation we may assume that $\text{int}(K) \cap L$ is contained in the x -axis. Since K intersects the x -axis in an interval with at least 3 interior lattice points, its length is greater than 2. Suppose K contains a point with y -coordinate at least 2. Then by convexity $K \cap \{y = 1\}$ has width greater than 1 and contains an interior lattice point p . Because the elements of $\text{int}(K) \cap L$ are collinear, p cannot be in L , thus p must be in S . However, this is impossible as K is S -free. Therefore $K \subseteq \{|y| < 2\}$. Each edge of K contains a point from S in its relative interior and every lattice point in K has y -coordinate $-1, 0$, or 1 . But a convex body can

intersect each of the three lines $\{y = -1\}$, $\{y = 0\}$, $\{y = 1\}$ in at most 2 points. It follows that K has at most six sides, contradicting the assumption that K has at least seven sides. \square

As we see in Section 6, it is very useful for a set to have finite Helly number. At the same time, for some $S \subset \mathbb{R}^d$ there are very interesting mathematics lurking behind this and it is not trivial to find $h(S)$. We conclude with a conjecture that stresses these points.

Conjecture. *Let \mathbb{P} be the set of prime numbers. Then $h(\mathbb{P}^2) = \infty$.*

We have been able to show that $h(\mathbb{P}^2) \geq 12$ and $h((\mathbb{Z} \setminus \mathbb{P})^2)$ is finite (e.g., through a simple modification of argument used in Theorem 1.4 and Lemma 2.2). This problem appears to be related to the Gilbreath-Proth conjecture [29] on the behavior of differences of consecutive primes.

4. SUBSETS OF \mathbb{R}^d : SUBGROUPS AND DIFFERENCES OF LATTICES

Now we move to spaces of arbitrary dimension. As we will see, it is much harder to calculate Helly numbers in this general framework.

4.1. Subgroups of \mathbb{R}^d . We look at the case when S is an additive subgroup of \mathbb{R}^d . The most famous examples are lattices. We recall a lattice is defined as a discrete subgroup of \mathbb{R}^d ; it is well known that lattices in \mathbb{R}^d are generated by at most d elements (see [8]). This is precisely the context of Doignon's theorem, which says that $h(\mathbb{Z}^d) = 2^d$.

If a group $S \subset \mathbb{R}^d$ is finitely generated by m elements, then the natural epimorphism between \mathbb{Z}^m and S given by linear combinations yields a linear map from \mathbb{R}^m onto \mathbb{R}^d . From here Doignon's theorem implies that $h(S) \leq 2^m$. However, there is no dependency on d here, and m could be large compared to d .

On the other hand, it is known in the topological groups literature (see e.g. [15]) that every closed subgroup that linearly spans \mathbb{R}^d is of the form $\phi(\mathbb{R}^k \times \mathbb{Z}^{d-k})$, where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear bijection. In this case the mixed version of Helly's and Doignon's theorems by Averkov and Weismantel [5], guarantees that $h(\mathbb{Z}^{d-k} \times \mathbb{R}^k) = 2^{d-k}(k+1)$.

The closure \bar{S} of a group S is also a group; therefore we may assume that $\bar{S} = \mathbb{R}^k \times \mathbb{Z}^{d-k}$. This allows us to express S as a product of the form $\mathbb{Z}^{d-k} \times D$, where D is a dense subgroup of \mathbb{R}^k . In this case we might expect that $h(S) \leq (k+1)2^{d-k}$, but the following example shows that this is not the case.

Example 4.1. Let $k \leq d$ and $1, \alpha_1, \alpha_2, \dots, \alpha_k$ be linearly independent numbers when considered as a vector space over \mathbb{Q} . Let $\{e_1, e_2, \dots, e_d\}$ be the canonical basis of \mathbb{R}^d

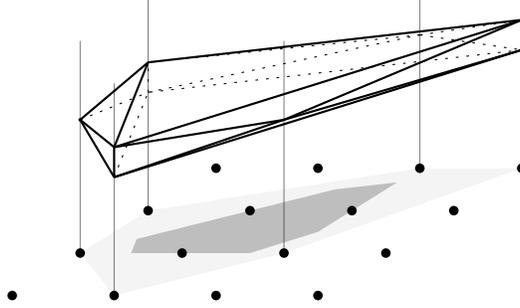


FIGURE 3. The convex hull of 9 points in $\mathbb{Z}^2 \times G$, in dark gray is the projection of the intersection described in Lemma 2.2

and consider the group

$$S = \langle e_1, \dots, e_{k-1}, \alpha_1(e_k + e_1), \dots, \alpha_{k-1}(e_k + e_{k-1}), \alpha_k e_k, e_{k+1}, \dots, e_d \rangle.$$

Note that $S = D \times \mathbb{Z}^{d-k}$, where

$$D = \langle e_1, \dots, e_{k-1}, \alpha_1(e_k + e_1), \dots, \alpha_{k-1}(e_k + e_{k-1}), \alpha_k e_k \rangle$$

is a dense set in \mathbb{R}^d , so the closure of S is $\mathbb{R}^k \times \mathbb{Z}^{d-k}$. Observe that if we intersect S with the space $\{x_k = 0\}$ we obtain a $(d - 1)$ -dimensional lattice. Since $h(S \cap \{x_k = 0\}) = 2^{d-1}$, we can construct a family \mathcal{F} with $2^{d-1} + 2$ elements containing $\{x_k \geq 0\}, \{x_k \leq 0\}$ to show that $h(S) \geq 2^{d-1} + 2$. If $k \geq 3$, then $h(S)$ is larger than $(k + 1)2^{d-k}$.

In spite of this, we state the following conjecture:

Conjecture. *For any subgroup $G \subset \mathbb{R}^d$, the Helly number $h(G)$ is finite.*

Note that this conjecture is true for all finitely generated groups by Doignon's theorem. One might be tempted to conjecture that $h(G) \leq 2^d$. As we see in the next example, this is not always case.

Example 4.2. Let $G_0 \subset \mathbb{R}$ be the group generated by 1, π and e , so that $G = \mathbb{Z}^2 \times G_0$ is a subgroup of \mathbb{R}^3 . Consider the points $x_1 = (0, 1, 3)$, $x_2 = (2, 1, e)$ and $x_3 = (2, 3, \pi)$ in \mathbb{R}^3 , the important property these points have is that the 3 midpoints they define are not in G . Let $H \subset \mathbb{R}^3$ be the plane through x_1, x_2, x_3 . Consider 3 additional points $x_4, x_5, x_6 \in G$ above $(0, 2), (1, 0), (3, 3) \in \mathbb{Z}^2$ and slightly above H and another 3 points x_7, x_8, x_9 above the same 3 points in \mathbb{Z}^2 but this time slightly below H . If $R = \{x_1, \dots, x_9\}$ then the hypothesis of Lemma 2.2 are satisfied (see Figure 3), and therefore $h(G) \geq 9$.

If the group G has some additional structure, we are able to bound its Helly number.

Proof of Theorem 1.3. We have already proved in Lemma 3.1 the case $d = 1$ and in Theorem 1.1 the case $d = 2$. For the general case we use induction over d .

Let $d > 1$; assume that the theorem is false and $h = h(S) > 2d$. This means that there is a family \mathcal{F} of convex sets such that $\bigcap \mathcal{F}$ does not intersect S but for every subfamily \mathcal{G} with $h - 1$ elements, $\bigcap \mathcal{G}$ intersects S .

Since S is dense, $\text{int}(\bigcap \mathcal{F}) = \emptyset$; therefore $m = \dim(\bigcap \mathcal{F}) < d$. Consequently, by Steinitz's theorem [32], there is a subfamily $\mathcal{H} \subset \mathcal{F}$ with $2(d - m)$ elements such that $\dim(\bigcap \mathcal{H}) = \dim(\bigcap \mathcal{F})$.

Since $2(d - m) < h$, there exists $s \in (\bigcap \mathcal{H}) \cap S$. Then, up to translation, $R = \text{span}(\bigcap \mathcal{H}) \cap S$ is a group contained in an m -dimensional subspace of \mathbb{R}^d . The induction hypothesis on R implies $h(R) \leq 2m$. By the definition of \mathcal{F} , every $(h - 1) - 2(d - m)$ elements of $\mathcal{F} \setminus \mathcal{H}$ intersect in R . However, $\bigcap (\mathcal{F} \setminus \mathcal{H})$ does not intersect R , as then $\bigcap \mathcal{F}$ would intersect $R \subset S$. Thus $(h - 1) - 2(d - m) < h(R)$ and therefore $h \leq h(R) + 2(d - m) \leq 2d$. This contradicts our choice of h . \square

To show that this is best possible we have the following example.

Example 4.3. Let $a_1, a_2, \dots, a_d, b_1, b_2, \dots, b_d, c_1, c_2, \dots, c_d$ be linearly independent numbers when considered as a vector space over \mathbb{Q} and satisfying $a_i < c_i < b_i$. Consider the \mathbb{Q} -module S generated by the $2d$ vectors of the form

$$\begin{aligned} A_i &= (c_1, \dots, c_{i-1}, a_i, c_{i+1}, \dots, c_d), \\ B_i &= (c_1, \dots, c_{i-1}, b_i, c_{i+1}, \dots, c_d). \end{aligned}$$

It is easy to see that $C = (c_1, \dots, c_d) \notin S$.

Let K_i^- and K_i^+ be the two semispaces with boundary through C orthogonal to e_i so that $A_i \in K_i^-$ and $B_i \in K_i^+$. If \mathcal{F} is the family consisting of these semispaces then $A_i \in \bigcap (\mathcal{F} \setminus \{K_i^+\})$ and $B_i \in \bigcap (\mathcal{F} \setminus \{K_i^-\})$ but $\bigcap \mathcal{F} = \{C\}$ does not intersect S ; therefore $h(S) \geq 2d$.

4.2. Difference of lattices in \mathbb{R}^d . Recall that the Ramsey number

$$R_k = R(\underbrace{3, 3, \dots, 3}_k)$$

is the minimum number needed to guarantee the existence of a monochromatic triangle in any edge-coloring with k colors of the complete graph with R_k vertices. We now prove Theorem 1.2 with constant $C_k = R_k - 1$.

Proof of Theorem 1.4. Assume that $h(S) > (R_k - 1)2^d$. Lemma 2.1 implies the existence of an S -vertex-polytope K with $h(S)$ vertices. We say that two elements of \mathbb{Z}^d have the same parity if their difference has only even entries, or equivalently, if their midpoint is in \mathbb{Z}^d . Since the vertex-set of K has more than $(R_k - 1)2^d$ elements, it contains a subset V consisting of R_k points with the same parity.

By definition of K , if $A, B \in V$ then their midpoint cannot be in S ; it must be contained in some L_i . Consider V as the vertex-set of the complete graph G and assign to each edge AB a color i so that the midpoint of A and B is in L_i .

Since G has R_k vertices, by Ramsey's theorem it contains a monochromatic triangle. This means that there exist $A_1, A_2, A_3 \in V \subset S$ such that the three midpoints $M_j = \frac{1}{2}(A_j + A_{j+1})$ are in some L_i . But then $A_1 = M_1 - M_2 + M_3 \in L_i$ which contradicts the fact that $A_1 \in S$. \square

5. ANALYSIS OF A POPULAR METHOD TO COLOR HELLY-TYPE THEOREMS

The main purpose of this section is to discuss the essential features of a general method to obtain colorful versions of some Helly-type theorems in the sense of Bárány and Lovász. This method has been used in a number of occasions [5, 7] and can be applied to the usual Helly's theorem, Doignon's theorem, Theorem 1 in [1] and our Theorems 1.2 and 1.4 presented in this paper. This idea has been around for some time and it is based on Lovász's proof of Helly's theorem, but as far as we know it has never been abstracted to its essential features.

Let $\mathcal{P}(K)$ be a property of a convex set $K \subset \mathbb{R}^d$. By a property we mean a function from the set of convex sets in \mathbb{R}^d that takes values in $\{\text{true}, \text{false}\}$. As examples of properties $\mathcal{P}(K)$ we have:

- (i) K intersects a fixed set S ,
- (ii) K contains at least k integer lattice points,
- (iii) K is at least k -dimensional,
- (iv) K has volume at least 1.

The following generalizes the definition of Helly number we gave in the introduction to properties shared by sets.

Definition 5.1. For a property \mathcal{P} as above, the *Helly number* $h = h(\mathcal{P}) \in \mathbb{N}$ is defined as the smallest number satisfying the following.

$$(5.1) \quad \forall i_1, \dots, i_h \in [m] : \mathcal{P}(F_{i_1} \cap \dots \cap F_{i_h}) \implies \mathcal{P}(F_1 \cap \dots \cap F_m)$$

for all $m \in \mathbb{N}$ and convex sets F_1, \dots, F_m . If no such h exists, then $h(\mathcal{P}) := \infty$.

In particular, if \mathcal{P} is the property (i) then $h(\mathcal{P}) = h(S)$. For property (ii), the quantitative Doignon theorem in [1] gives bounds for $h(\mathcal{P})$. A result from Grünbaum [21] bounds and determines $h(\mathcal{P})$ for property (iii) with $0 \leq k \leq d$. In the case of property (iv), it is easy to show that $h(\mathcal{P}) = \infty$, despite the existence of a quantitative Helly theorem [6].

Definition 5.2. We say that property \mathcal{P} is:

1. *Helly* if the corresponding Helly number $h(\mathcal{P})$ is finite.
2. *Monotone* if $K \subset K'$ and $\mathcal{P}(K)$ is true implies that $\mathcal{P}(K')$ is true too.

3. *Orderable* if for any finite family \mathcal{F} of convex sets there is a direction v such that:
 - (a) For every $K \in \mathcal{F}$ where $\mathcal{P}(K)$ is true, there is a containment-minimal v -semispace (i.e., a half-space of the form $\{x : v^T x \geq 0\}$) H such that $\mathcal{P}(K \cap H)$ is true.
 - (b) There is a unique minimal $K' \subset K \cap H$ with $\mathcal{P}(K')$ true.
4. *N -colorable* if for any given finite families $\mathcal{F}_1, \dots, \mathcal{F}_N$ of closed convex sets in \mathbb{R}^d (each a colored family), such that $\mathcal{P}(\bigcap \mathcal{G})$ is true for every rainbow subfamily \mathcal{G} (i.e. a family with $\mathcal{G} \cap \mathcal{F}_i = 1$ for every i), then $\mathcal{P}(\bigcap \mathcal{F}_i)$ is true for some family \mathcal{F}_i .

It is not too difficult to see that property (i) is Helly, monotone and orderable when S is discrete or the whole space. Property (ii) is also Helly, monotone and orderable, whereas property (iii) fails to be orderable and property (iv) is neither Helly nor orderable. Under this definition we can state the key result:

Theorem 5.3 (Generic Colorful Helly). *A Helly, monotone, orderable property \mathcal{P} is always $h(\mathcal{P})$ -colorable.*

Proof. Let \mathcal{F} be the family consisting of all the convex sets $\bigcap \mathcal{G}$ where \mathcal{G} is a family of convex sets such that, for some j , $\mathcal{G} \cap \mathcal{F}_j = 0$ and $\mathcal{G} \cap \mathcal{F}_i = 1$ for every $i \neq j$. Let v be the direction provided by the hypothesis that the property is orderable.

By hypothesis and monotonicity, $\mathcal{P}(K)$ is true for every $K \in \mathcal{F}$ and therefore there is a semispace H_K as in condition (a) of orderability. Among these semispaces, set $H = H_K$ as the containment-maximal one and let K' be as in condition (b) of orderability. We may assume that $K = K_1 \cap \dots \cap K_{N-1}$ with $K_i \in \mathcal{F}_i$ for $i = 1, \dots, N - 1$.

Take K_N to be an arbitrary element of \mathcal{F}_N , we now show that $K' \subset K_N$. Consider the family $\mathcal{F}' = \{K_1, \dots, K_{N-1}, K_N, H_K\}$, it is not difficult to see that it satisfies (5.1) with $m = N$. Condition 1 of Definition 5.2 implies that $\mathcal{P}(\bigcap \mathcal{F}')$ is true, but the uniqueness and minimality of K' imply that $K' \subset K_N$. Since K_N is arbitrary, we may conclude that $\mathcal{P}(\bigcap \mathcal{F}_N)$ is true. \square

As a corollary we can obtain colored versions of three new Helly-type theorems:

Corollary 5.4. *Let \mathcal{P} be one of the following properties, all of which have an associated Helly-type theorem:*

- (1.) K intersects a fixed set $S \subset \mathbb{R}^d$, with $h(S) < \infty$, S discrete.
- (2.) K contains at least k integer lattice points (see [1]),
- (3.) K contains at least k points from a difference of lattices (see [14]),

Then \mathcal{P} is $h(\mathcal{P})$ -colorable.

6. HELLY NUMBERS IN OPTIMIZATION: RANDOMIZED ALGORITHMS OVER SUBSETS OF \mathbb{R}^d

In this section we present many applications to optimization of the S -Helly numbers calculated in the first half of the paper. In fact, it is fair to say that applications in optimization have prompted already many papers about Helly numbers [5, 9, 12, 20, 24, 30]. We presented in the introduction the notion of S -optimization that generalizes continuous, integral, and mixed-integer optimization

Clearly when $S = \mathbb{R}^d$ the problem is usual continuous optimization, when $S = \mathbb{Z}^d$, this is integer optimization, and $S = \mathbb{Z}^k \times \mathbb{R}^{d-k}$ is the case of mixed-integer optimization. Here are some more unfamiliar examples: we show S -optimization problems can model succinctly with more sophisticated $S \subset \mathbb{R}^d$ (typically discrete sets). The alert reader will recognize S before we announce it!

Example 6.1. Given a graph $G = (V, E)$, we formulate the classic graph K -coloring query as the solvability of the following linear system of modular inequations: For all (i, j) in $E(G)$ consider the inequations $c_i \not\equiv c_j \pmod K$. This is a system on $|V|$ variables and it has a solution if and only if the graph is K -colorable. Note that the set of points $\mathbf{c} = (c_1, \dots, c_{|V|})$ with $c_i \equiv c_j \pmod K$ is a lattice, which we call $L_{i,j}$. Therefore, solving our system of inequalities is equivalent to finding a $\mathbf{c} \in \mathbb{Z}^{|V|} \setminus (\bigcup_{i,j} L_{i,j})$.

Example 6.2. We are interested in the geometry of the solutions of the following modular mixed-integer optimization problem:

$$\begin{aligned}
 \min \quad & 3x_1 + 7x_2 + 4x_3 + \sum_{i \geq 4}^N (100 - i)x_i \\
 \text{subject to} \quad & 8x_1 + 3x_2 + 5x_3 \equiv 6 \pmod{11}, \\
 & 6x_1 + 4x_2 - 3x_3 \equiv 1 \pmod{2}, \\
 & x_1 \not\equiv x_3 \pmod{5}, \\
 & x_1 + x_2 + x_3 + \sum_{i \geq 6}^N x_i \leq 1000, \\
 & x_1 \not\equiv 2, 4, 16 \pmod{23}, \quad x_2 \equiv 0 \pmod{2}, \quad x_3 \equiv 2 \pmod{3}, \\
 & x_1, x_2, x_3 \geq 0 \text{ and integral,} \\
 & x_i, i = 4, \dots, 50 \text{ continuous.}
 \end{aligned}$$

Note that only the integral variables have modular restrictions. We can reformulate this problem adding integer slacks as problem with only six integral variables with modular restrictions and N continuous variables.

$$\begin{aligned}
\min \quad & 3x_1 + 7x_2 + 4x_3 + \sum_{i \geq 4}^N (100 - i)x_i \\
\text{subject to} \quad & 8x_1 + 3x_2 + 5x_3 = y_1 \\
& 6x_1 + 4x_2 - 3x_3 = y_2, \\
& x_1 - x_3 = y_3, \\
& x_1 + x_2 + x_3 + \sum_{i \geq 6}^N x_i \leq 1000, \\
& x_1 \not\equiv 2, 4, 16 \pmod{23}, \quad x_2 \equiv 0 \pmod{2}, \quad x_3 \equiv 2 \pmod{3}, \\
& y_1 \equiv 6 \pmod{11}, \quad y_2 \equiv 1 \pmod{2}, \quad y_3 \not\equiv 0 \pmod{5}, \\
& x_1, x_2, x_3 \geq 0 \text{ and integral,} \\
& x_i, i = 4, \dots, 50 \text{ continuous.}
\end{aligned}$$

Where do the solutions exist? What is the set $S \subset \mathbb{R}^6$ where the variables take on values for this situation? The answer can be described first as $S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6 \times \mathbb{R}^{N-3}$, where S_i can be described as the difference between \mathbb{Z} and the subtraction of cosets (or translated sublattices) with respect to lattices of multiples of an integer q . Thus at the the end $S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6$ can be written in the format of Theorem 1.4, i.e. the lattice \mathbb{Z}^6 from which we subtract the union of several translated sublattices.

In the following subsections we demonstrate how two well-known methods of optimization that involve a sampling scheme can be extended to more restrict the the variables to more sophisticated sets S as long as the S -Helly number $h(S)$ is finite. We present two theorems, one generalizing the scenario approximation method of Calafiore and Campi [11] and the other generalizing Clarkson's randomized algorithm for convex optimization problems [12, 20].

6.1. Chance-constrained S -optimization problems. Here we extend the scenario approximation scheme from [10, 11]. Recall we are concerned with the linear minimization chance-constrained problem

$$\begin{aligned}
(6.1) \quad & \min \quad c^T \mathbf{x} \\
& \text{subject to} \quad Pr[f(\mathbf{x}, w) \leq 0] \geq 1 - \epsilon, \quad i = 1, 2, \dots, k, \\
& \quad \mathbf{x} \in K \text{ a convex set,} \\
& \quad \mathbf{x} \in S.
\end{aligned}$$

The assumptions are that $0 < \epsilon \leq 1$, the function f is convex in the variables \mathbf{x} and measurable in w (chance variables that represent stochasticity). We also assume that the problem has an optimal solution. The idea to consider a random sample of the values of N different *i.i.d* values w^i for w , and create a fixed convex optimization problem from this sample:

$$(6.2) \quad \begin{aligned} & \min \quad c^T \mathbf{x} \\ & \text{subject to} \quad f(\mathbf{x}, w^i) \leq 0, \quad i = 1, 2, \dots, k, \\ & \quad \quad \quad \mathbf{x} \in K \text{ convex set,} \\ & \quad \quad \quad \mathbf{x} \in S. \end{aligned}$$

Denote by \mathbf{x}_N the (uniquely selected) optimal solution of the problem (6.2). We can state now our main result in this portion:

Before we start the proof of Theorem 1.5 we require a purely technical estimation:

Lemma 6.3. *If*

$$(6.3) \quad N \geq \frac{1}{1-r} \left(\left(\frac{1}{\epsilon} \right) \ln \left(\frac{1}{\delta} \right) + h + \left(\frac{h}{\epsilon} \right) \ln \left(\frac{1}{r\epsilon} \right) + \frac{1}{\epsilon} \ln \left(\left(\frac{h}{\epsilon} \right)^h \left(\frac{1}{h!} \right) \right) \right)$$

then $\binom{N}{h} (1-\epsilon)^{N-h} \leq \delta$.

Proof.

$$N \geq \frac{1}{1-r} \left(\left(\frac{1}{\epsilon} \right) \ln \left(\frac{1}{\delta} \right) + h + \left(\frac{h}{\epsilon} \right) \ln \left(\frac{1}{r\epsilon} \right) + \frac{1}{\epsilon} \ln \left(\left(\frac{h}{\epsilon} \right)^h \left(\frac{1}{h!} \right) \right) \right)$$

is the same as

$$(1-r)N \geq \left(\frac{1}{\epsilon} \right) \ln \left(\frac{1}{\delta} \right) + h + \left(\frac{h}{\epsilon} \right) \left(\ln \left(\frac{h}{r\epsilon} \right) - 1 \right) - \frac{1}{\epsilon} \ln(h!),$$

so

$$N \geq \left(\frac{1}{\epsilon} \right) \ln \left(\frac{1}{\delta} \right) + h + \left(\frac{h}{\epsilon} \right) \left(\ln \frac{h}{r\epsilon} - 1 + \frac{rN\epsilon}{h} \right) - \frac{1}{\epsilon} \ln(h!).$$

But then, using the fact that $\ln(x) \geq 1 - \frac{1}{x}$ for positive values of x and applying it to $x = \frac{h}{rN\epsilon}$, we obtain

$$N \geq \left(\frac{1}{\epsilon} \right) \ln \left(\frac{1}{\delta} \right) + h + \left(\frac{h}{\epsilon} \right) \ln(N) - \frac{1}{\epsilon} \ln(h!).$$

From this last equation one can bound the logarithm of δ , in such a way that $\ln(\delta) \geq -\epsilon N + \epsilon h + h \ln(N) - \ln(h!)$. Therefore, using the fact that $e^{-\epsilon(N-h)} \geq$

$(1 - \epsilon)^{N-h}$ (because $-\epsilon \geq \ln(1 - \epsilon)$),

$$\delta \geq \frac{N^h}{h!} e^{-\epsilon(N-h)} \geq \frac{N(N-1)\dots(N-h+1)}{h!} (1 - \epsilon)^{N-h}.$$

This last inequality can be rewritten as $\delta \geq \binom{N}{h} (1 - \epsilon)^{N-h}$, finishing the proof of the statement. \square

Proof of Theorem 1.5. Suppose we have the sampling set $\{w^1, \dots, w^N\}$. Denote by x_N the optimal solution for the auxiliary problem (6.2) obtained from the sampling. Note that because f is convex, each choice w^i gives an S -convex set $K_i = \{x \in K : f(x, w^i) \leq 0\}$. The proof will require the use of the S -Helly number. The most important fact to do the estimations is that if we have the optimum value of (6.2), the optimal solution is defined by no more than $(h(S) - 1)$ of the K_i . This is because $K_i, i = 1 \dots N$ together with $c^T x < c^T x_N$ is a convex set which has no solutions in S , thus by the definition of the S -Helly number there are no more than $h(S)$ infeasible subfamily but this mean that from the original K_i only $h(S) - 1$ participate. We call these $h(S) - 1$ subsets the witness constraints of the problem (6.2).

Let Γ_N be the set of all possible values N i.i.d samples can take w^1, w^2, \dots, w^N . Now consider all possible index sets $I \subset [N] = \{1, \dots, N\}$ of cardinality $(h(S) - 1)$ and define

$$\Gamma_N^I = \{(w^1, \dots, w^N) \in \Gamma_N : (w^i)_{i \in I} \text{ define the witness constraints of (6.2)}\}.$$

Therefore, Γ_N can be written as the union of the Γ_N^I for all possible choices of I . Using this we will bound the probability that x_N is not in the solution set of (6.1). For simplicity, let

$$R_\epsilon = \{\mathbf{x} \in K \cap S : Pr[f(\mathbf{x}, w) \leq 0] \geq 1 - \epsilon\} \quad \text{and} \quad G_\epsilon = (K \cap S) \setminus R_\epsilon.$$

$$\begin{aligned}
& Pr \left[\{(w^1, \dots, w^N) \in \Gamma_N : x_N \notin G_\epsilon\} \right] \leq \\
& \sum_{I \subset [N], |I|=h(S)-1} Pr \left[\{(w^1, \dots, w^N) \in \Gamma_N^I : x_I \notin G_\epsilon\} \right] = \\
& \sum_{I \subset [N], |I|=h(S)-1} Pr \left[\left\{ (w^i)_{i \in I} : x_I \notin G_\epsilon \right\} \cap \left\{ (w^i)_{i \in I} : f(x_I, w^i) \leq 0, i \notin I \right\} \right] = \\
& \sum_{I \subset [N], |I|=h(S)-1} Pr \left[\left\{ (w^i)_{i \in I} : x_I \notin G_\epsilon \right\} \right] \cdot \\
& Pr \left[\left\{ (w^i)_{i \in I} : f(x_I, w^i) \leq 0, i \notin I \right\} \mid \left\{ (w^i)_{i \in I} : x_I \notin G_\epsilon \right\} \right] = \\
& \sum_{I \subset [N], |I|=h(S)-1} Pr \left[\left\{ (w^i)_{i \in I} : x_I \notin G_\epsilon \right\} \right] \cdot \\
& \prod_{i \notin I} Pr \left[\left\{ (w^i)_{i \in I} : f(x_I, w^i) \leq 0, i \notin I \right\} \mid \left\{ (w^i)_{i \in I} : x_I \notin G_\epsilon \right\} \right] \leq \\
& \binom{N}{h(S)-1} (1-\epsilon)^{N-(h(S)-1)}.
\end{aligned}$$

The last inequality is true because each of the factors in the product has probability no more than $1 - \epsilon$. Therefore we wish to choose N in such a way that we get that $\binom{N}{h(S)-1} (1-\epsilon)^{N-(h(S)-1)} \leq \delta$.

Finally, from Lemma 6.3 one can derive the bound stated in the theorem by two simple observations: First simply set $h = (h(S) - 1)$, second the last term can be dropped because it is not positive (this is the case since $n! \geq (n/e)^n$). In addition one can take r to be any value between 0 and 1. Thus, taking $r = 1/2$ one gets the statement of the theorem. \square

6.2. A Clarkson-type sampling algorithm for S -convex optimization. Here is another application of S -Helly numbers. We use them, together with the theory of violator spaces, to give a Clarkson-type algorithm, to compute the optimal solutions to a given S -convex optimization problem. We consider now the solution of S -optimization problem with linear objective function and convex constraints

$$\begin{aligned}
& \min \quad c^T \mathbf{x} \\
& \text{subject to} \quad f_i(x) \leq 0, \quad f_i \text{ convex for all } i = 1, 2, \dots, m, \\
& \quad \mathbf{x} \in S.
\end{aligned}$$

Our key contribution here is to demonstrate that a well-known algorithm due to Clarkson can be extended to S -optimization as long as S is closed and has a finite

Helly number. The method devised by Clarkson [12] works particularly well for geometric optimization problems in few variables. Examples of applications include convex and linear programming, integer linear programming, the problem of computing the minimum-volume ball or ellipsoid enclosing a given point set in \mathbb{R}^n , and the problem of finding the distance of two convex polytopes in \mathbb{R}^n . E.g., Clarkson stated the following result about linear programs and integer linear programs (ILPs), which gives:

Theorem 6.4 (Clarkson). *Given A an $m \times n$ matrix, vector $\mathbf{b} \in \mathbb{R}^m$ and the integer program $\min\{c^T \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n, \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}\}$, one can find a solution to this problem in a expected number of steps of order $O(n^2 m \log(m)) + n \log(m) O(n^{n/2})$. While the algorithm is exponential it gives the best complexity for solving ILPs when the number of variables n is fixed.*

Clarkson's algorithm requires that many small-size subsystems of the original problem are solved. This requires the call to an oracle to solve the small systems. The oracle originally provided by Clarkson in the case of regular integer programming was Lenstra's IP algorithm in fixed dimension. As a consequence, when the number of variables is constant, Clarkson's algorithm gives a remarkable linear bound on the complexity (see recent work by Eisenbrand [19]). Here we prove Theorem 1.6 which is a direct generalization of Clarkson's theorem for convex continuous and integral optimization.

The key idea is to use the theory of *violator space* introduced by Gärtner, Matoušek, Rüst and Škovroň [20]. They showed it can be used as a general framework to work with convex optimization problems. Essentially, a violator space is an abstract optimization problem in which we have a finite set of constraints or elements H and a function that given any subset of constraints G , indicates which other constraints in $H \setminus G$ violate the feasible solutions to G . If one has a violator space structure, the optimal solution of the problem can be computed via a randomized method whose running time is *linear* in the number of constraints defining the problem, and *subexponential* in the dimension of the problem. Violator spaces include all prior abstractions such as [2, 31]. The key definition from [20] is the following:

Definition 6.5. A *violator space* is a pair (H, \mathbf{V}) , where H is a finite set and \mathbf{V} a mapping $2^H \rightarrow 2^H$, such that the following two axioms hold:

Consistency: $G \cap \mathbf{V}(G) = \emptyset$ holds for all $G \subseteq H$, and

Locality: $\mathbf{V}(G) = \mathbf{V}(F)$ holds for all $F \subseteq G \subseteq H$ such that $G \cap \mathbf{V}(F) = \emptyset$.

There are three important ingredients of every violator space: a basis, combinatorial dimension, and a primitive test (which will be answered by an oracle). First, just like in the simplex method for linear programming the problem will be defined by bases, we need to have a notion of basis for our optimal solutions.

Definition 6.6 ([20]). A *basis* of a violator space is defined in analogy to a basis of a linear programming problem: a minimal set of constraints that defines a solution space. Specifically, [20, Definition 7] defines $B \subseteq H$ to be a *basis* if $B \cap V(F) \neq \emptyset$ holds for all proper subsets $F \subset B$. For $G \subseteq H$, a basis of G is a minimal subset B of G with $V(B) = V(G)$.

Moreover, violator space bases come with a natural combinatorial invariant, which is strongly related to Helly numbers we discussed earlier. The size of a largest basis of a violator space (H, V) is called the *combinatorial dimension* of the violator space and denoted by $\delta = \delta(H, V)$.

[20] proved a crucial property: knowing the violations $V(G)$ for all $G \subseteq H$ is enough to compute a largest basis. To do so, one can utilize Clarkson's randomized algorithm to compute a basis of some violator space (H, V) with $m = |H|$.

The results about the runtime and the size of the sets involved are summarized below. The primitive operation, used as black box in all stages of the algorithm, is the so-called *violation tests primitive*. Given a violator space (H, V) , some set $G \subset H$, and some element $h \in H \setminus G$, the *primitive* test decides whether $h \in V(G)$.

The main idea to improve over a brute-force search is due to Clarkson [12]. He presented his method in two stages, referred to as Clarkson's first and Clarkson's second algorithm, which we outline below. In the first round, we draw a random sample $R \subset H$ of size $r = |R|$ and then compute a basis of R using some other algorithm. The crucial point here is that r is much much smaller than $|H|$, a Helly number of sorts. Obviously, this first round may fail to find a basis of H , and we have to do more rounds. The second stage, Clarkson's second algorithm, is only called with sets of size at most $3\delta\sqrt{|G|}$ or $9\delta^2$.

Clarkson's first algorithm: Basis1

Input: $G \subseteq H$, $\delta =$ the combinatorial complexity.

Output: B , a basis for G of size δ .

Outline: If $|G| \leq 9\delta^2$: return Basis2(G).

Else: $W := \emptyset$.

Do: Choose a random subset $R \subset G \setminus W$ with $\lfloor \delta\sqrt{|G|} \rfloor$ elements.

Set $C := \text{Basis2}(W \cup R)$ and $V := \{h \in G \setminus C \text{ s.t. } h \in V(C)\}$.

If $|V| \leq 2\sqrt{|G|}$ then $W := W \cup V$.

Until $V = \emptyset$

return C .

Clarkson's second algorithm (Basis2 below) iteratively picks a random small ($6\delta^2$ elements) subset R of G , finds a basis C for R by exhaustively testing each possible subset (BruteForce) (where we take advantage on the fact the sample is small), and then calculates the violators of $G \setminus C$. At each iteration, elements that appear in bases with small violator sets get a higher probability of being selected. This idea is

very important: we are biasing the sampling process, so that some constraints will be more likely to be chosen. This is accomplished by considering every element h of the set G as having a multiplicity $\mathbf{m}(h)$; the multiplicity of a set is the sum of the multiplicities of its elements. The process is repeated until a basis of G is found, i.e., until $\mathbf{V}(G \setminus C)$ is empty.

Clarkson's second algorithm: Basis2

Input: $G \subseteq H$; δ : combinatorial complexity.

Output: B : a basis.

Outline: If $|G| \leq 6\delta^2$: return BruteForce(G).

Else: Choose a random subset $R \subset G$ with $6\delta^2$ elements.

$C := \text{BruteForce}(R)$.

$V := \{h \in G \setminus C \text{ s.t. } h \in \mathbf{V}(C)\}$.

If $\mathbf{m}(V) \leq \mathbf{m}(G)/3\delta$ then

$\mathbf{m}(h) := 2\mathbf{m}(h), h \in V$.

Until $V = \emptyset$.

return C .

As described above, all one needs is to be able to answer the *Primitive query*: Given $G \subset H$ and $h \in H \setminus G$, decide whether $h \in \mathbf{V}(G)$. Second, the runtime is given in terms of the combinatorial dimension $\delta(H, V)$ and the size of the input set of constraints H . The key result we will use in the rest of the paper is about the complexity of finding a basis:

Theorem 6.7. [20, Theorem 27] *Using Clarkson's algorithms, a basis of H in a violator space (H, \mathbf{V}) can be found by answering the primitive query an expected $O(\delta|H| + \delta^{O(\delta)})$ times.*

It is very important to note that, in both stages of Clarkson's method, the query $h \in \mathbf{V}(C)$ is answered via calls to the primitive as a black box. In our algebraic applications, the primitive computation requires solving a small-size subsystem. On the other hand, the combinatorial dimension relates to the Helly number of the problem which is usually a number that is problem-dependent and requires non-trivial mathematical results.

The algorithms we will derive are randomized but run in expected polynomial time complexity when the number of discrete variables is fixed. More over the algorithmic complexity is in fact linear in the number of constraints, and it depends on calls to an oracle that solves small size subproblems. The size of these problems is precisely the S -Helly number. A concrete situation when S -problems are efficiently solvable is when the number of variables is constant.

Proof of Theorem 1.6. Let $H = \{f_1, f_2, \dots, f_m\}$ be the constraints of the S -convex optimization problem of the statement of Theorem 1.6. We define a the violator set

operator $V(G)$ for a subset of inequalities $G \subset H$ as follows: We provide each S -program with a universal tie-breaking rule, for instance, using lexicographic ordering. A constraint $h \in H$ is in $V(G)$ if the optimal solution value of the subsystem G with respect to the objective function, denoted \vec{x}_G , is not equal to the unique optimal solution of $G \cup \{h\}$, denoted $\vec{x}_{G \cup \{h\}}$. Note that we need to have a total ordering on the possible feasible solutions of G and the fact that S is closed to have a unique optimum.

For our proof we define the violator map as follows: a constraint $h \in H$ is in $V(G)$ if the optimal solutions satisfy $\vec{x}_G > \vec{x}_{G \cup \{h\}}$. If we assume that G has no feasible solutions, we define $V(G)$ as being the empty set. Indeed any new constraint added to the integer program can only decrease the number of feasible solutions. We need to check that the two conditions presented in the definition of violator spaces. are satisfied. The consistency condition is clearly satisfied.

Assume now that $F \subseteq G \subseteq H$ and $G \cap V(F) = \emptyset$. To show locality we want to show that $V(F) = V(G)$. Note that by the hypothesis $G \cap V(F)$ it means that $\vec{x}_G = \vec{x}_F$ because otherwise at least one element in G must be in $V(F)$.

Now we verify first the containment $V(F) \subseteq V(G)$. Take $h \in V(F)$; if $h \notin V(G)$ then $\vec{x}_{G \cup \{h\}} = \vec{x}_G = \vec{x}_F > \vec{x}_{F \cup \{h\}}$. However, $F \cup \{h\} \subset G \cup \{h\}$. It follows that $\vec{x}_{F \cup \{h\}} \geq \vec{x}_{G \cup \{h\}}$ too—a contradiction. Now we check $V(G) \subset V(F)$. Take $h \in V(G)$, if $\vec{x}_{F \cup \{h\}} = \vec{x}_F = \vec{x}_G > \vec{x}_{G \cup \{h\}}$ But then there exist $g \in G$ such that $g \in V(F \cup \{h\}) = V(F)$ a contradiction.

Since the two conditions of a violator space are satisfied, all that is left to apply Theorem 6.7 is to outline what the combinatorial dimension and the primitive test are. First, a basis for G , using this violator space, represents an optimal solution of the S -subproblem. But if we have the optimum value \vec{x}_G , then the optimal solution is defined by no more than $(h(S) - 1)$ of the f_i . This is because $f_i, i = 1 \dots N$ together with $c^T x < c^T x_N$ is an S -convex set which has no solutions in S . Thus by the definition of the S -Helly number, there are no more than $h(S)$ infeasible subfamily, but this means that from the original $f_i \in G$ only $h(S) - 1$ participate. Therefore the combinatorial dimension of this violator space is $h(S) - 1$. The primitive test is provided by an oracle that solves smaller problems of size $O(h(S))$. Therefore, the conclusion of Theorem 1.6 follows by applying Theorem 6.7. \square

6.3. S -feasibility problems. In general, when one has a system of convex constraints one is interested on the feasibility of the system. When the system is infeasible the next best thing would be to guarantee that a large subsystem is sure to be feasible. For this we would like to stress that Helly's theorem can hold with some tolerances too.

First, we have seen the colorful version of Helly's theorem. It can clearly be interpreted as a situation when the constraints are divided into colors, where the

colors indicate constraint types or categories. What the theorem guarantees then is a situation in which at least one entire color class has a common solution. Second, there is the *fractional Helly theorem* for $S \subseteq \mathbb{R}^d$. In its general form, it says that there is a number γ with the property that for any family of n convex sets C_1, C_2, \dots, C_n , if $\bigcap_{i \in I} C_i \neq \emptyset$ for at least a fraction α of all possible $\binom{n}{\gamma}$ index sets $I \subseteq \{1, \dots, n\}$ of cardinality γ , then there is a fraction β for which at least βn of the convex sets C_1, \dots, C_n have a common point in S . It is known that when $\gamma(\mathbb{R}^d) = \gamma(\mathbb{Z}^d) = d+1$, (see Katchalski and Liu [26] and Bárány and Matoušek [7]). Recently Averkov and Weismantel have proved in [5] that if $S \subseteq \mathbb{R}^d$ is non-empty and closed and has $h(S) < \infty$ then $\gamma(S) \leq d+1$ too. This new theorem applies to many more situations, including $S = \mathbb{Z}^k \times \mathbb{R}^{d-k}$, and also the S in our Theorem 1.4. Note that here we are only interested on checking that a positive fraction of all the constraints are feasible; to do this one only needs to verify a fraction of the $d+1$ -subsets of constraints.

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REFERENCES

1. I. Aliev, R. Bassett, J. A. De Loera, and Q. Louveaux, *A quantitative Doignon-Bell-Scarfi theorem*, to appear in *Combinatorica*, available in arXiv:1405.2480 (2014).
2. N. Amenta, *Helly theorems and generalized linear programming*, *Discrete and Computational Geometry* **12** (1994), 241–261.
3. J. L. Arocha, J. Bracho, and L. Montejano, *Flat transversals to flats and convex sets of a fixed dimension*, *Adv. Math.* **213** (2007), no. 2, 902–918. MR 2332615 (2009e:52013)
4. G. Averkov, *On maximal S -free sets and the Helly number for the family of S -convex sets*, *SIAM J. Discrete Math.* **27** (2013), no. 3, 1610–1624.
5. G. Averkov and R. Weismantel, *Transversal numbers over subsets of linear spaces*, *Adv. Geom.* **12** (2012), no. 1, 19–28.
6. I. Bárány, *A generalization of Carathéodory’s theorem*, *Discrete Math.* **40** (1982), no. 2-3, 141–152.
7. I. Bárány and J. Matoušek, *A fractional Helly theorem for convex lattice sets*, *Adv. Math.* **174** (2003), no. 2, 227–235. MR 1963693 (2003m:52006)
8. A. Barvinok, *A course in convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, RI, 2002.
9. D. E. Bell, *A theorem concerning the integer lattice*, *Studies in Appl. Math.* **56** (1976), no. 2, 187–188.

10. G. Calafiore and M. C. Campi, *Uncertain convex programs: randomized solutions and confidence levels*, Math. Program. **102** (2005), no. 1, Ser. A, 25–46. MR 2115479 (2005k:90104)
11. G. C. Calafiore and M. C. Campi, *The scenario approach to robust control design*, IEEE Trans. Automat. Control **51** (2006), no. 5, 742–753. MR 2232597 (2007a:93075)
12. K. L. Clarkson, *Las Vegas algorithms for linear and integer programming*, Journal of the ACM **42** (1995), no. 2, 488–499.
13. L. Danzer, B. Grünbaum, and V. Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180. MR 0157289 (28 #524)
14. J.A. De Loera, R.N. La Haye, D. Rolnick, and P. Soberón, *Quantitative Tverberg, Helly, & Carathéodory theorems*, arXiv preprint arXiv:1503.06116 (2015).
15. D. Dikranjan, *Introduction to topological groups*, <http://users.dimi.uniud.it/~{}dikran.dikranjan/ITG.pdf> (2013).
16. J-P. Doignon, *Convexity in crystallographical lattices*, J. Geometry **3** (1973), 71–85.
17. J-P. Doignon, J. R. Reay, and G. Sierksma, *A Tverberg-type generalization of the Helly number of a convexity space*, J Geom **16** (1981), no. 1, 117–125.
18. J. Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 389–448. MR 1242986 (94k:52010)
19. F. Eisenbrand and G. Shmonin, *Parametric integer programming in fixed dimension*, Math. Oper. Res. **33** (2008), no. 4, 839–850. MR 2464645 (2009j:90078)
20. B. Gärtner, J. Matoušek, L. Rüst, and P. Škovroň, *Violator spaces: structure and algorithms*, Discrete Appl. Math. **156** (2008), no. 11, 2124–2141. MR 2437006 (2009i:90081)
21. B. Grünbaum, *The dimension of intersections of convex sets*, Pacific J. Math. **12** (1962), 197–202.
22. B. Grünbaum and G. C. Shephard, *Tilings and patterns*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Company, New York, 1989, An introduction. MR 992195 (90a:52027)
23. E. Helly, *über mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresberichte der Deutschen Math.-Vereinigung. **32** (1923), 175–176.
24. A. J. Hoffman, *Binding constraints and Helly numbers*, Second International Conference on Combinatorial Mathematics (New York, 1978), Ann. New York Acad. Sci., vol. 319, New York Acad. Sci., New York, 1979, pp. 284–288. MR 556036 (82d:90081)
25. R. Jamison, *Partition numbers for trees and ordered sets*, Pacific Journal of Mathematics **96** (1981), no. 1, 115–140.
26. M. Katchalski and A. Liu, *A problem of geometry in \mathbf{R}^n* , Proc. Amer. Math. Soc. **75** (1979), no. 2, 284–288. MR 532152 (80h:52010)
27. D. Kay and E. W. Womble, *Axiomatic convexity theory and relationships between the Carathéodory, Helly, and Radon numbers*, Pacific Journal of Mathematics **38** (1971), no. 2, 471–485.
28. J. Matoušek, *Lectures on discrete geometry*, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002. MR 1899299 (2003f:52011)
29. A. M. Odlyzko, *Iterated absolute values of differences of consecutive primes*, Math. Comp. **61** (1993), no. 203, 373–380. MR 1182247 (93k:11119)
30. H. E. Scarf, *An observation on the structure of production sets with indivisibilities*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 9, 3637–3641.

31. M. Sharir and E. Welzl, *A combinatorial bound for linear programming and related problems*, Proc. 9th Symposium on Theoretical Aspects of Computer Science (STACS), Lecture Notes in Computer Science, vol. 577, Springer-Verlag, 1992, pp. 569–579.
32. E. Steinitz, *Bedingt konvergente reihen und konvexe systeme.*, Journal für die reine und angewandte Mathematik **143** (1913), 128–176.
33. M. L. J. van de Vel, *Theory of convex structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993. MR 1234493 (95a:52002)
34. R. Wenger, *Helly-type theorems and geometric transversals*, Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997, pp. 63–82.

(J. A. De Loera, R. N. La Haye) DEPARTMENT OF MATHEMATICS, UC DAVIS
E-mail address: deloera@math.ucdavis.edu, rlahaye@math.ucdavis.edu

(D. Oliveros, E. Roldán-Pensado) INSTITUTO DE MATEMÁTICAS, UNAM CAMPUS JURQUILLA
E-mail address: dolivero@matem.unam.mx, e.rolدان@im.unam.mx