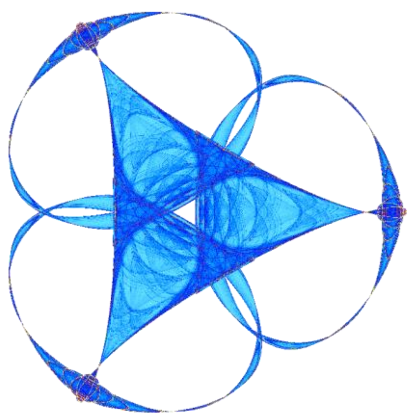
THE EFFECTIVE LVMM METHOD IN LOTKA-VOLTERRA SYSTEMS

By

Ezio Marchi

IMA Preprint Series #2441

(October 2014)



INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436
Phone: 612-624-6066 Fax: 612-626-7370
URL: http://www.ima.umn.edu

The effective LVMM method in Lotka-Volterra Systems

By

Ezio Marchi
Emeritus Professor UNSL
Founder and First Director of the IMASL
San Luis, Argentina
emarchi@speedy.com.ar

Abstract: *The Lotka – Volterra equation for three species are integrated in an original way obtaining the existence of cycle. No studies of the cycle stability are held. These would be very interesting kind as in two species. However our results are of value.*

Key words: cycles, three species, existence

Introduction

In life science research, the Lotka–Volterra model is considered a classical dynamic model which has been intensively used to explain dynamic phenomena in population ecology and other life science fields. In the past two decades, an increasing amount of attention has been addressed to the study of dynamical system models arising from biological systems. This is part of the development of a rich and diversified field that lies somewhere between mathematics and the sciences.

In this paper we present and study a general case of ecological competition among three species using the Lotka-Volterra equations. The used technique is new which combine Volterra´s [1] idea for the problem of two species and the approach due to Montroll [2]. The essential idea is to eliminate one variable by the knowledge of a system integral. Such elimination can be obtained from the selection of a very complicate partial differential equation. The result is that the third variable is a function of the remaining two. Since we are interested not only the existence of cycles, which may be limit (see ZHENGYI Lu and YONG Luo [5]-[6]), but also in the real computation of then. Therefore we mention the fact that assuming the existence of a cycle which is a one-dimentional manifold therefore one of the three variables is function of the previous two. This is essentially our approach.

There are generalizations of the Bendixon-Poincare theorem for three or higher dimensional non-linear systems, but these generalizations, however, are inapplicable to most systems of interest. At this point we would like to mention that our model has essentially the power of the theorem of Bendixon-Poincare.

Three species model

Consider the system of differential equations following Lotka-Volterra by the problem of three interacting species:

$$\frac{dx}{dt} = x(e_1 - a_{12}y - a_{13}z)$$
$$\frac{dy}{dt} = y(-e_2 + a_{21}x - a_{23}z) \qquad (1)$$
$$\frac{dz}{dt} = z(-e_3 + a_{31}x + a_{32}y + a_{33}z)$$

where *x* is the number of individual or density of the first species which are consider as a prey of the second, *y* is the number of individual of the second species, and third species which is symbolized the variable *z*. The e_i are growing indices of each species and a_{ij} represent the interrelation between the species *i* and *j*.

Our first step is to get an integral which has to say as a variable is express as function of the remained two. Let

$$z = f(x,y) \tag{2}$$

now we are interested in the determination of this function. Then we derive with respect to time and we obtain

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{3}$$

Now replacing from the right to the left in the equation (3) the derivates with respect to time by the expressions given by (1), we obtain a partial differential equation of first order for the function *f*:

$$z(-e_3 + a_{31}x + a_{32}y + a_{33}z) = \frac{\partial f}{\partial x}x(e_1 - a_{12}y - a_{13}z) + \frac{\partial f}{\partial y}y(-e_2 + a_{21}x - a_{23}z) \tag{4}$$

In order to evaluate this last differential equation, we assume the following system:

$$a_{33}z = \frac{\partial f}{\partial x}a_{13}x + \frac{\partial f}{\partial y}a_{23}y \tag{5}$$

and

$$z(-e_3 + a_{31}x + a_{32}y) = \frac{\partial f}{\partial x}x(e_1 - a_{12}y) + \frac{\partial f}{\partial y}y(-e_2 + a_{21}x) \tag{6}$$

It is clear that the f solution of (5) and (6), it is also solution of (4).

Now we will solve the system (5)-(6). Using the standard Lagrange method the solution might be consider as an arbitrary function $F(k_1,k_2) = 0$, where k_1 and k_2 are integration constants of the system

$$\frac{dx}{a_{13}x} = \frac{dy}{a_{23}y} = \frac{dz}{a_{33}z} \tag{7}$$

From the first two equalities of (7), we obtain that:

$$\ln y^{1/a_{23}} = \ln x^{1/a_{13}} + k_1$$

and from the second two, that:

$$\ln z^{1/a_{33}} = \ln y^{1/a_{23}} + k_2$$

In this way:

$$z = y^{a_{33}/a_{23}} F\left(\frac{y^{1/a_{23}}}{x^{1/a_{13}}}\right) \tag{8}$$

where the function *F* is arbitrary and is determined by the equation (6). Introducing the expression of *z* given by (8) in the partial differential equation (6), we have that

$$\begin{aligned} &F(w)(-e_3 + a_{31}x + a_{32}y) = \\ &\quad -\frac{1}{a_{13}}F'(w)w(e_1 - a_{12}y) + \frac{a_{33}}{a_{23}}F(w)(-e_2 + a_{21}x) + \frac{1}{a_{23}}F'(w)w(-e_2 + a_{21}x) \end{aligned} \tag{9}$$

where we have introduced the new variable

$$w = \frac{y^{1/a_{23}}}{x^{1/a_{13}}}$$

Operating in (9) we conclude that *F* has to satisfy the following differential equation of first order:

$$F(w)\left(\left(-e_3+e_2\frac{a_{33}}{a_{23}}\right)+\left(a_{31}-a_{21}\frac{a_{33}}{a_{23}}\right)x+a_{32}y\right)=F'(w)w\left(-\left(\frac{e_2}{a_{23}}+\frac{e_1}{a_{13}}\right)+\frac{a_{21}}{a_{23}}x+\frac{a_{12}}{a_{13}}y\right)$$

calling $d_1=-e_3+e_1\frac{a_{33}}{a_{23}}$, $d_2=a_{31}-a_{21}\frac{a_{33}}{a_{23}}$, $p_1=\frac{-e_2}{a_{23}}-\frac{e_1}{a_{13}}$, $p_2=\frac{a_{21}}{a_{23}}$ and $p_3=\frac{a_{12}}{a_{13}}$.

The last equation might be written in the following way

$$\frac{F'(w)}{F(w)}=\frac{d_1+d_2x+a_{32}y}{w(p_1+p_2x+p_3y)} \tag{10}$$

If the following conditions are satisfied

$$d_2=p_2,\qquad d_1=p_1,\qquad a_{32}=p_3 \tag{11}$$

The previous equation is

$$\frac{F'(w)}{F(w)}=\frac{1}{w} \tag{12}$$

which has the integral

$F(w)=c_0w$

Replacing w we have

$$z=c_0\frac{y^{(a_{33}+1)/a_{23}}}{x^{1/a_{13}}} \tag{13}$$

where the constant c_0 must satisfy the initial conditions of the problem:

$$c_0=\frac{z_0x_0^{1/a_{13}}}{y_0^{(a_{33}+1)/a_{23}}}$$

where x_0, y_0 and z_0 represent the initial value of number of individual of each of the species. The next step is to replace the value of z given by (13) in the first equation of the system (1):

$$\frac{dx}{dt}=x(e_1-a_{12}y-D\,{y^{\alpha}}/{x^{\beta}})$$

$$\frac{dy}{dt}=y(-e_2+a_{21}x-E\,{y^{\alpha}}/{x^{\beta}}) \tag{14}$$

where $D=a_{13}c_0$, $E=a_{23}c_0$, $\alpha={(a_{33}+1)}/{a_{23}}$ and $\beta={1}/{a_{13}}$.

Our next step is the integration of (14). For this we introduced the following constants defined by $a_{12}={-a}/{\beta_1}$, $a_{21}={a}/{\beta_2}$, $q_1={e_2\beta_2}/{a}$ and $q_2={-e_1\beta_1}/{a}$. Now following Montrol et all, we replace the following change of variables:

$$v_1=\ln\left({x}/{q_1}\right)\ \text{and}\ v_2=\ln\left({x}/{q_2}\right) \tag{15}$$

replacing in the first equation of the system (14) we obtain:

$$q_1\exp(v_1)\frac{dv_1}{dt}=$$
$$e_1q_1\exp(v_1)+\frac{a}{\beta_1}q_1\exp(v_1)q_2\exp(v_2)-D\frac{q_2^{\alpha}}{q_1^{\beta-1}}\exp(\alpha v_2-(\beta_1-1)v_1) \tag{16}$$

from here making the simplification of $\left({q_1}/{\beta_1}\right)\exp(v_1)$:

$$\beta_1 \frac{dv_1}{dt} = aq_2\left(\exp(v_2) - 1 - \beta_1 D \frac{q_2^{\alpha-1}}{aq_1^{\beta}} \exp(\alpha v_2 - \beta v_1) \right) \tag{17}$$

In a similar way from the second equation of (14) it is derived:

$$\beta_2 \frac{dv_2}{dt} = aq_1\left(\exp(v_1) - 1 - \beta_2 E \frac{q_2^{\alpha}}{aq_1^{\beta+1}} \exp(\alpha v_2 - \beta v_1) \right) \tag{18}$$

Calling:

$$H = \frac{\beta_1 D q_2^{\alpha-1}}{aq_1^{\beta}} \text{ and } J = -\frac{\beta_2 E q_2^{\alpha}}{aq_1^{\beta+1}}$$

the equations (17) and (18) become:

$$\beta_1 \frac{dv_1}{dt} = aq_2(\exp(v_2) - 1 + H\exp(\alpha v_2 - \beta v_1))$$

$$\beta_2 \frac{dv_2}{dt} = aq_1(\exp(v_1) - 1 + J\exp(\alpha v_2 - \beta v_1)) \tag{19}$$

Cross multiplying both equations in (19), we conclude that

$$\beta_1 \frac{dv_1}{dt} q_1(\exp(v_1) - 1 + J\exp(\alpha v_2 - \beta v_1)) = \beta_2 \frac{dv_2}{dt} q_2(\exp(v_2) - 1 + H\exp(\alpha v_2 - \beta v_1))$$

but:

$$\frac{d}{dt}(\exp(\alpha v_2 - \beta v_1)) = \alpha \exp(\alpha v_2 - \beta v_1)\frac{dv_2}{dt} - \beta \exp(\alpha v_2 - \beta v_1)\frac{dv_1}{dt}$$

Applying this property in the last equation we obtain that

$$\begin{aligned} &\frac{d}{dt}(\beta_1 q_1(\exp(v_1) - v_1) + \beta_1 q_1 J\exp(\alpha v_2 - \beta v_1))\frac{dv_1}{dt} = \\ &\frac{d}{dt}(\beta_2 q_2(\exp(v_2) - v_2 + H\exp(\alpha v_2 - \beta v_1))) + \beta_2 q_2 H\exp(\alpha v_2 - \beta v_1))\frac{dv_1}{dt} \end{aligned} \tag{20}$$

Imposing the condition that:

$$E = \frac{\beta}{\alpha} D \tag{21}$$

the equation (20) becomes:

$$\beta_1 q_1(\exp(v_1) - v_1) = c_1 + \beta_2 q_2(\exp(v_2) - v_2 + H\exp(\alpha v_2 - \beta v_1)) \tag{22}$$

where c_1 is an integration constant.

Using the proper parameters of the problem, the equation (22) can be written in the following way:

$$\frac{-1}{e_1}(\exp(v_1) - v_1) = K + \frac{1}{e_2}(\exp(v_2) - v_2 + H\exp(\alpha v_2 - \beta v_1)) \tag{23}$$

where $K = \frac{c_1 a}{\beta_1 \beta_2 e_1 e_2}$.

Taking now the old variables, we get:

$$v_1 = \ln(c_1 x) \text{ with } c_1 = 1/q_1 = a_{21}/e_2$$

$$v_2 = \ln(c_2 x) \text{ with } c_2 = 1/q_2 = a_{12}/e_1$$

In this way the first equation (23) might be written in the following form:

$$\frac{-1}{e_1}(c_1 x - \ln(c_1 x)) = K + \frac{1}{e_2}(c_2 y - \ln(c_2 y) + H\frac{c_2 y}{c_1 x} \tag{24}$$

From the last expression we obtain the important relation

$$\left(\frac{c_1 x}{\exp(c_1 x)}\right)^{1/e_1} = P\left(\frac{c_2 y}{\exp\left(c_2 y + M\frac{y^\alpha}{x^\beta}\right)}\right)^{-1/e_2} \qquad (25)$$

Where $P = \exp(K)$ and $M = H^{c_2/c_1}$.

We call ζ_1 the function in the left side of the equality (25) and ζ_2 the function in the right side of the same equality. The function ζ_1 is similar to that obtained by Volterra in the problem of two species, and the graph is shown in the figure 1. Now we remark that the function ζ_2 depends of variable *x*, that is to say is more general that the corresponding obtained by Volterra. The graph of ζ_2 which depends parametrically of *x* is also shown in the figure 1. Graphically it is clearly observed that for a given value for *x* belonging to the interval $[x_0, x_1]$, there are exactly two values for *y* such that

$$\zeta_1(x) = \zeta_2(x, y) \qquad (26)$$

Now we will prove the existence of the points x_0 and x_1. First of all we are looking for the minimum value y_0 of the function ζ_2, given in the function of *x*. From the condition $\frac{\partial \zeta_2}{\partial y} = 0$, we conclude that $\frac{-1}{c_2 y_0} + c_2 + M\alpha \frac{y_0^{\alpha-1}}{x^\beta} = 0$ or equivalently:

$$y_0 + c_2 \frac{x^\beta y_0}{M\alpha} = \frac{x^\beta}{c_2 M\alpha} \qquad (27)$$

We will prove that there exist a unique value y_0 that satisfies the condition (27). That is to say the function ζ_2 has only a unique minimum. For this consider as in the figure 2, the value $x(c_2 M)^{-1}$ in the vertical axes, then the two different functions, are linear with coefficient $c_2 x^\beta (M\alpha)^{-1}$ and the remaining y^α. Adding both functions as in figure 2, we might see that there exists a unique positive value of *y* such that the relationship (27) is true. The function $y_0(x)$ is strictly concave. Its graphic is shown in figure 3. On the other hand, the minimum condition given by (27) might be used in order to obtain the expression of $\zeta_2(x, y)$:

$$\zeta_3(y_0) = \zeta_2(x, y_0) = P\left(\frac{c_2 y_0}{\exp\left(c_2 y_0 + \frac{1}{c_2 \alpha} - \frac{c_2 y_0}{\alpha}\right)}\right)^{-1/e_2}$$

The curve of ζ_3 as function of y_0 is shown in the figure 4. In this way the composition $\zeta_2(x, y_0(x))$ as function of *x* is shown in figure 5. Now if we compare the function $\zeta_2(x, y_0(x))$ in the figure 5 with the function ζ_1 in the figure 1, we obtain that under same condition of the parameters, in particular P and M, which they cut in two points. Such points are these already mentioned x_0 and x_1. This comparison is shown in the figure 6. Such points are the superior and inferior points where the cycles closed. In this way the cycle is obtained under certain condition of the parameters. Thus our technique provides the existence of a cycle for our three species equations and we have obtained a similar result as the Bendixon-Poincare Theorem for this case.

A particular case

In this section we develop a particular case for a given suitable set of parameters. We first consider $\alpha = 1$ and $\beta = 1/2$. Now the equation (25) takes the form

$$\left(\frac{c_1 x}{\exp(c_1 x)}\right)^{1/e_1} = P\left(\frac{c_2 y}{\exp\left(c_2 y + M\frac{y}{x^{1/2}}\right)}\right)^{-1/e_2}.$$

We evaluated the minimum of ζ_2, then we find that it is obtained at the coordinate

$$y_0 = \frac{x^{1/2}}{c_2\left(c_2 x^{1/2} + M\right)}.$$

The extreme points x_0 and x_1, satisfy the conditions:

$$\left(\frac{c_1 x}{\exp(c_1 x)}\right)^{1/e_1} = P\left(x^{1/2}\exp\left(\frac{x^{1/2}}{\left(c_2 x^{1/2} + M\right)} + \frac{M}{\left(c_2^2 x^{1/2} + c_2 M\right)}\right)\right)^{-1/e_2}\left(c_2 x^{1/2} + M\right)^{1/e_2} =$$

$$P\exp\left(\frac{1}{c_2 e_2}\right)\left(\frac{x^{1/2}}{\left(c_2 x^{1/2} + M\right)}\right)^{-1/e_2} \qquad (28)$$

Such a relation is shown from a graphical point of in the figure 6, where both functions are presented.

Period computation

In this paragraph we study for the particular case already mentioned an approximation in order to determine the period of the fluctuation. From the equation (25), taking logarithm if appears

$$\frac{1}{e_1}\ln(c_1 x) - \frac{1}{e_1 c_1 x} = \ln(P) - \frac{1}{e_2}\ln(c_2 y) + \frac{1}{e_2}\left(c_2 + \frac{c_3}{x^{1/2}}\right)y \qquad (29)$$

where $c_1 = a_{21}/e_2$, $c_2 = a_{12}/e_1$ and $c_3 = H\frac{e_2 a_{12}}{e_1 a_{21}}$.

We take the lineal approximation:

$$\ln(y) = a + by \qquad (30)$$

where the parameters a and b are adjusted for each one of the four parts shown in the figure 7 where the cycle is shown in the plane (x,y), this is the projection of the cycle solution in the phase (x,y,z) on the plane just mentioned. Replacing (30) in (29) with the values already mentioned it is derived

$$y = \frac{A + \frac{1}{e_1}\ln(x) - \frac{c_1}{e_1}x - B}{C + \frac{c_3}{e_2}x^{1/2}} \qquad (31)$$

where the constants are $A = \frac{1}{e_1}\ln(c_1)$, $B = \ln(P) - \frac{a}{e_2} - \frac{1}{e_2}\ln(c_2)$ and $C = \frac{-b}{e_2} + \frac{c_2}{e_2}$.

With the approximations for y in each one of the regions, we replace in the first differential equation of the system (15), therefore

$$\mathrm{dt} = \frac{\mathrm{dx}}{e_1 x - \left(a_{12}x + Dx^{1/2}\right)\dfrac{\left(\frac{1}{e_1}\ln(x) - c_1 e_1 x + A - B\right)}{\frac{c_3}{e_2}x^{1/2} + e}} \tag{32}$$

with the aim to integrate (32) we approximate the logarithm $\ln(x) = g + fx$, where the parameters are adjusted conveniently. In this way we obtain the following expression

$$\mathrm{dt} = \frac{\left(\frac{M}{x^{1/2}} + C\right)\mathrm{dx}}{e_1 x\left(\frac{M}{x^{1/2}} + C\right) - \left(a_{12}x + Dx^{1/2}\right)\left(N + Qx\right)} \tag{33}$$

with $M = \frac{c_3}{e_2}$, $Q = \frac{f}{e_1} - \frac{c_1}{e_1}$ and $N = \frac{g}{e_1} + A - B$.

Calling $H_1 = e_1 M - DN$, $H_2 = e_2 c - a_{12}N$, $H_3 = -DQ$ and $H_4 = -a_{12}Q$, the equation (33) becomes

$$\mathrm{dt} = \frac{\left(\frac{M}{x^{1/2}} + C\right)\mathrm{dx}}{H_1 x^{1/2} + H_2 x + H_3 x^{3/2} + H_4 x^2} \tag{34}$$

and if we take $u = x^{1/2}$, we obtain the expression for the computation of the period, which is easily derived

$$\mathrm{dt} = \frac{2M\mathrm{du}}{u(H_1 + H_2 u + H_3 u^2 + H_4 u^3)} + \frac{2C\mathrm{du}}{(H_1 + H_2 u + H_3 u^2 + H_4 u^3)} \tag{35}$$

Example

Now we present a numerical example of a Lotka-Volterra of three interacting species. This is done following the aims given in the previous sections. As a simple example to be studied the parameters take the following values: $e_1 = 3$, $e_2 = 0.5$, $e_3 = 2$, $a_{12} = 2$, $a_{21} = 1$, $a_{31} = 1$, $a_{13} = 2$, $a_{23} = 1$, $a_{32} = 1$ and $a_{33} = 0$. And the initial values $x(0) = 2$, $y(0) = 1.5$ and $z(0) = 1$. This set of parameters satisfies the imposed conditions by the relations (11) and (21). And with the values $\alpha = 0.5$ and $\beta = 1$. In order to obtain the graphical description of $x(t)$, $y(t)$ and $z(t)$, we utilize a standard general method. Consider as analytic function in the variable t:

$$x(t) = \sum_{k=0} x_k t^k$$

$$y(t) = \sum_{k=0} y_k t^k$$

$$z(t) = \sum_{k=0} z_k t^k$$

Now replacing these last expressions in the system (1) and identificating coefficients, we obtain the following recurrence relations given by

$$x_{k+1} = \frac{1}{k+1}\left(e_1 x_k - a_{12}\sum_{j=0}^{k} x_j y_{k-j} - a_{13}\sum_{j=0}^{k} x_j z_{k-j} \right)$$

$$y_{k+1} = \frac{1}{k+1}\left(-e_2 y_k + a_{21}\sum_{j=0}^{k} y_j x_{k-j} - a_{23}\sum_{j=0}^{k} y_j z_{k-j} \right)$$

$$z_{k+1} = \frac{1}{k+1}\left(\frac{a_{31}}{a_{13}} e_1 x_k - \frac{a_{32}}{a_{23}} e_2 y_k - e_3 z_k + \left(\frac{a_{32}a_{21}}{a_{23}} - \frac{a_{31}a_{12}}{a_{13}} \right)\sum_{j=0}^{k} x_j y_{k-j} - a_{33}\sum_{j=0}^{k} y_j z_{k-j} \right)$$
$$- \frac{a_{31}}{a_{13}} x_{k+1} - \frac{a_{32}}{a_{23}} y_{k+1}$$

Using these last three relations we obtain the curve in figure 8. In the same way, we obtain the corresponding cycle showing in figure 5. In order to obtain the period, we have:

$c_1 = 2 \quad c_2 = 0{,}667 \quad c_3 = 0.628$

$A = 0{,}231 \quad P = 0{,}015 \quad H = 1{,}332$

$M = 1{,}258 \quad D = 1{,}884$

For the interval showing in figure 5, the range of variation of x is $(0{,}378;1{,}606)$ and for y is $(0{,}615;1{,}535)$. We have the next values for:

$g = -1{,}23 \quad f = 1{,}14 \quad a = -1{,}008 \quad b = 0{,}615$

$H_1 = 1{,}52 \quad H_2 = -4{,}202 \quad H_3 = 0{,}541 \quad H_4 = 0{,}574$

The roots in (35) are $u_0 = 0{,}389; u_1 = 2{,}025$ and $u_2 = -3{,}357$. The limits for the integral are $u' = 1{,}267$ and $u'' = 0{,}615$. Integrating expression (35) as a rational function, we obtain the time for the first region $T_A = 0{,}682$. In the same way we have $T_B = 0{,}403; T_C = 0{,}794$ and $T_D = 0{,}814$. Therefore the period is $T = 2{,}693$.

Conclusion

According to that was developed in the present study, we found a simple but very effective method to find cycles in Lotka-Volterra model with three interacting species. We can add that apart from this important result, it can be concluded that this method can be easily extend to n species problem.

Figures

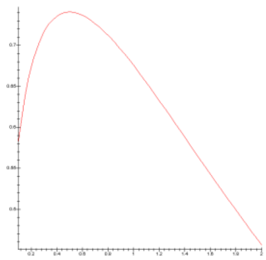


Figure 1

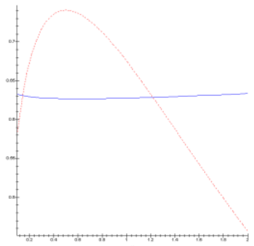


Figure 2

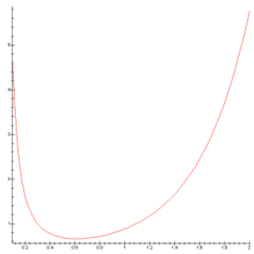


Figure 3

Bibliography

[1] Volterra, Vito, “Variazioni e Fluctuacioni del numero d´individui in specie animali conviventi”. R. Comitato Talassografico Italiano. Nº 442, 1910.

[2] Goel, N., Maitra, S. and Montroll, E. “On the Volterra and other Non-linear Models of Interacting Populations”. Rev. Modern Physics 43, pp 231-276. (1971)

[3] Marchi, E. and Millan, L. “On the exact solutions of differential equations of Hodgkin-Huxley model type”. In Proc. 1º Int. Conf. Mathematical Modeling, Vol. 2, pp 447-458, (1977)

[4] Marchi, E., Millan, L. and Crino, E. “Exact Solution of the rate equation for the Three Energy Level Laser”. Journal of Quantum Electronics, IEEE, Vol. QE-16. N4. (1980)

[5] ZHENGYI Lu and YONG Luo. “Two Limit Cycles in Three-Dimensional Lotka-Volterra Systems”. Computers and Mathematics with Applications 44 (2002) 51-66

[6] ZHENGYI Lu and YONG Luo. “Three Limit Cycles for a Three-Dimensional Lotka-Volterra Competitive System with a Heteroclinic Cycle”. Computers and Mathematics with Applications 46 (2003) 231-238

[7] Uno, T. “Robustness and Global Behavior of 3-dimensional Lotka-Volterra Equations”. Nonlinear Analysis 47 (2001) 1309-1320

[8] Bradshaw, A. T. and Moseley L.L. “Dynamics of competing predator-prey species”. Physica A 261 (1998) 107-114

[9] Puyun Gao and Zhaonan Liu “An indirect method of finding integrals for three-dimensional quadratic homogeneous systems” Physics Letters A 244 (1998) 49-52

[10] Shawagfeh N. T. and Adomian, G. “Non-Perturbative Analytical Solution of the General Lotka-Volterra Three-Species System”. *APPLIED MATHEMATICS AND COMPUTATION* 76:251-2613 (1996)