

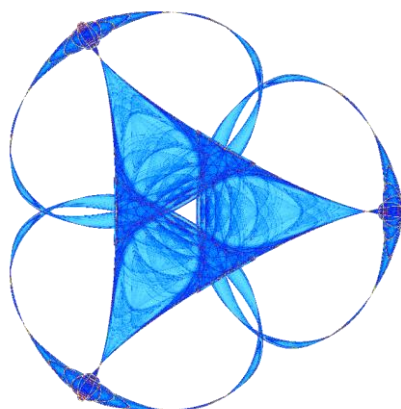
WHAT IS THE RECIPROCAL OF A FUNCTION CONCAVE OR CONVEX?

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## When is the reciprocal of a function concave or convex?

By

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**Abstract:** In this short note we study a simple problem: when the reciprocal of a function is concave or convex. We also study the quotient of two functions.

Key words: convexity – reciprocal functions, optimization.

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## 1. Introduction

In this paper we study when the reciprocal of a function is concave or convex. We do not provide wide bibliography since the concepts that we handle are elemental and consistent.

Consider a real function  $f : K \rightarrow \mathbb{R}$  where the domain is a convex, non-empty and compact set in an euclidean space  $K \subset \mathbb{R}^n$ , where  $\mathbb{R}$  indicates the real numbers set and  $n$  the dimension. For practical purposes we consider  $f$  strictly positive, that is  $f(x) > 0, \forall x \in K$ , therefore it achieve the maximum

and the minimum. Now, the reciprocal function is  $F(x) = \frac{1}{f(x)}, x \in K$ .

## 2. Concavity

If we wish to have it concave, then by definition it must be

$\lambda F(x) + (1-\lambda)F(y) \leq F(\lambda x + (1-\lambda)y)$   
for all  $x, y \in K$  and  $\lambda \in [0; 1]$ . Replacing by  $f$ , we have

$$\lambda \frac{1}{f(x)} + (1-\lambda) \frac{1}{f(y)} \leq \frac{1}{f(\lambda x + (1-\lambda)y)}$$

or multipling

$$\frac{\lambda f(y) + (1-\lambda)f(x)}{f(x)f(y)} \leq \frac{1}{f(\lambda x + (1-\lambda)y)}$$

or

$$[\lambda f(y) + (1-\lambda)f(x)][f(\lambda x + (1-\lambda)y)] \leq f(x)f(y)$$

Since  $f(x)$  and  $f(y)$  are positives.

Now if  $f$  is concave, from the first term and second factor we have

$$\begin{aligned} & [\lambda f(y) + (1-\lambda)f(x)][\lambda f(x) + (1-\lambda)f(y)] \leq \\ & \leq [\lambda f(x) + (1-\lambda)f(y)][f(\lambda y + (1-\lambda)x)] \leq f(x)f(y) \end{aligned}$$

and performing the arrangements and multiplications we have for the different elements by taking the last term and the first one

$$\lambda^2 f(x)f(y) + (1-\lambda)\lambda f^2(y) + \lambda(1-\lambda)f^2(x) + (1-\lambda)^2 f(x)f(y) \leq f(x)f(y)$$

and distributing

$$[\lambda^2 + (1-\lambda)^2]f(x)f(y) + (1-\lambda)\lambda[f^2(x) + f^2(y)] \leq f(x)f(y)$$

or simplifying the terms of  $\lambda$ , it remains

$$-2f(x)f(y) + [f^2(x) + f^2(y)] \leq 0$$

that is

$$[f(x) - f(y)]^2 \leq 0, \quad \forall x, y \in K$$

This is valid for the constant function and no thing else. Or it is impossible for an arbitrary function different from a positive constant.

### 3. Convexity

On the other hand if  $F$  is convex, therefore

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y)$$

for all  $x, y \in K$  and  $\lambda \in [0; 1]$ , or replacing

$$\frac{1}{f(\lambda x + (1 - \lambda)y)} \leq \lambda \frac{1}{f(x)} + (1 - \lambda) \frac{1}{f(y)}$$

or performing some operators

$$f(x)f(y) \leq f(\lambda x + (1 - \lambda)y)[\lambda f(y) + (1 - \lambda)f(x)]$$

Now if  $f$  is convex

$$\begin{aligned} f(x)f(y) &\leq f(\lambda x + (1 - \lambda)y)[\lambda f(y) + (1 - \lambda)f(x)] \\ &\leq [\lambda f(x) + (1 - \lambda)f(y)][\lambda f(y) + (1 - \lambda)f(x)] \end{aligned}$$

Now from the first and the last term we have making some operations

$$\lambda^2 f(x)f(y) + (1 - \lambda)\lambda f^2(y) + \lambda(1 - \lambda)f^2(x) + (1 - \lambda)^2 f(x)f(y) \geq f(x)f(y)$$

or

$$(1 - \lambda)\lambda[f^2(x) + f^2(y)] + [\lambda^2 + (1 - \lambda)^2]f(x)f(y) \geq f(x)f(y)$$

But  $\lambda^2 + (1 - \lambda)^2 - 1 = \lambda^2 + 1 - 2\lambda + \lambda^2 - 1 = 2\lambda^2 - 2\lambda = 2\lambda(\lambda - 1) = -2\lambda(1 - \lambda)$  and simplifying the terms of  $\lambda$  it remains,

$$f^2(x) + f^2(y) - 2f(x)f(y) \geq 0$$

or

$$[f(x) - f(y)]^2 \geq 0 \quad (1)$$

This condition is always true for any function. Therefore a sufficient condition for  $F$  be convex is that is satisfied for  $f$  convex.

As an example take  $f(x) = \frac{1}{x}$  in  $\left[\frac{1}{2}, 1\right] \subset \mathbb{R}$ . Then

$$[f(x) - f(y)]^2 = \left(\frac{1}{x} - \frac{1}{y}\right)^2 = \left(\frac{y-x}{xy}\right)^2 \geq 0 \quad \forall x, y \in \left[\frac{1}{2}, 1\right]$$

We have to note the asymmetry with the case studied before for concave functions. The fact is that  $f$  is concave then  $-f$  is convex. But it is important to express that if  $f > 0$  then  $-f < 0$  and our study do not apply for it.

### The case of quotient

Consider now  $F(x)$  as the quotient of two functions for example  $f_1(x)$  and  $f_2(x) > 0$ . Assume that  $F(x)$  be concave

$$F(x) = \frac{f_1(x)}{f_2(x)} \quad \text{and}$$

$$\lambda F(x) + (1-\lambda)F(y) \leq F(\lambda x + (1-\lambda)y)$$

or

$$\lambda \frac{f_1(x)}{f_2(x)} + (1-\lambda) \frac{f_1(y)}{f_2(y)} \leq \frac{f_1(\lambda x + (1-\lambda)y)}{f_2(\lambda x + (1-\lambda)y)}$$

and performing some operations, it results

$$[\lambda f_1(x)f_2(y) + (1-\lambda)f_1(y)f_2(x)]f_2(\lambda x + (1-\lambda)y) \leq f_1(\lambda x + (1-\lambda)y)f_2(x)f_2(y)$$

If now we assume that  $f_2$  be concave and convex  $f_1$  therefore we obtain the following inequality

$$\begin{aligned} & [\lambda f_1(x)f_2(y) + (1-\lambda)f_1(y)f_2(x)] [\lambda f_2(x) + (1-\lambda)f_2(y)] \leq \\ & \leq \lambda f_1(x)f_2(x)f_2(y) + (1-\lambda)f_1(y)f_2(x)f_2(y) \end{aligned}$$

Performing the multiplications and arranging the corresponding terms it results

$$\begin{aligned} & (-\lambda + \lambda^2)f_1(x)f_2(x)f_2(y) + (1-\lambda)\lambda f_1(y)f_2^2(x) + \lambda(1-\lambda)f_1(x)f_2^2(y) \\ & + (1-\lambda)^2 f_1(y)f_2(x)f_2(y) \leq (1-\lambda)f_1(y)f_2(x)f_2(y); \\ & -\lambda(1-\lambda)f_1(x)f_2(x)f_2(y) + (1-\lambda)\lambda f_1(y)f_2^2(x) + \lambda(1-\lambda)f_1(x)f_2^2(y) \\ & [(1-\lambda)^2 - (1-\lambda)]f_1(y)f_2(y)f_2(x) \leq 0 \end{aligned}$$

Dividing by  $\lambda(1-\lambda)$  it turns out

$$\begin{aligned} & -f_1(x)f_2(y) + f_1(y)f_2^2(x) + f_1(x)f_2^2(y) - f_1(y)f_2(y)f_2(x) \leq 0 \\ & \text{or} \\ & f_1(y)f_2^2(x) + f_1(x)f_2^2(y) \leq f_1(x)f_2(x)f_2(y) + f_1(y)f_2(y)f_2(x) \end{aligned}$$

or rearranging again

$$\begin{aligned} & f_1(x)[f_2^2(y) - f_2(x)f_2(y)] + f_1(y)[f_2^2(x) - f_2(x)f_2(y)] \leq 0 \\ & \text{or} \\ & [f_2(y) - f_2(x)][f_1(x)f_2(y) - f_1(y)f_2(x)] \leq 0 \quad (2) \end{aligned}$$

Therefore, a necessary and sufficient condition for having  $F$  convex when  $f_2$  is concave and  $f_1$  convex is that the functions fulfills (2).

As two example are  $f_1(x) = x$  and  $f_2(x) = kx$  and any domain.

We indeed have immediately that (2) is now

$$\begin{aligned} & (ky - kx)(xky - xky) \leq 0 \\ & 0 = k^2(y-x)(xy - xy) \leq 0 \end{aligned}$$

As a second example take  $f_1(x) = x^2$  and  $f_2(x) = x$  in the range say  $[1, 2]$ . Then (2) turns to be

$$\begin{aligned} & (y-x)(x^2y - y^2x) \leq 0 \\ & (y-x)xy(x-y) \leq 0 \end{aligned}$$

If  $y > x$  then  $x < y$  and the product is non positive.

Now take the case that  $F(x)$  be convex. Then

$$\lambda F(x) + (1-\lambda)F(y) \geq F(\lambda x + (1-\lambda)y)$$

and replacing the corresponding functions we have

$$\lambda \frac{f_1(x)}{f_2(x)} + (1-\lambda) \frac{f_1(y)}{f_2(y)} \geq \frac{f_1(\lambda x + (1-\lambda)y)}{f_2(\lambda x + (1-\lambda)y)}$$

and then considering that  $f_2$  be convex and  $f_1$  concave then we have following inequality

$$\begin{aligned} & [\lambda f_1(x) f_2(y) + (1-\lambda) f_1(y) f_2(x)] [\lambda f_2(x) + (1-\lambda) f_2(y)] \geq \\ & \geq f_2(x) f_2(y) [\lambda f_1(x) + (1-\lambda) f_1(y)] \end{aligned}$$

and from here factorizing

$$\begin{aligned} & \lambda^2 f_1(x) f_2(x) f_2(y) + \lambda(1-\lambda) f_1(y) f_2^2(x) + \lambda(1-\lambda) f_1(x) f_2^2(y) + \\ & + (1-\lambda)^2 f_1(y) f_2(x) f_2(y) \geq \lambda f_1(x) f_2(x) f_2(y) + (1-\lambda) f_1(y) f_2(x) f_2(y) \\ & \text{or} \\ & (\lambda^2 - \lambda) f_1(x) f_2(x) f_2(y) + [(1-\lambda)^2 - (1-\lambda)] f_1(y) f_2(x) f_2(y) + \\ & + \lambda(1-\lambda) f_1(y) f_2^2(x) + \lambda(1-\lambda) f_1(x) f_2^2(y) \leq 0 \end{aligned}$$

Dividing by  $\lambda(1-\lambda)$  we have

$$- f_1(x) f_2(x) f_2(y) - f_1(y) f_2(x) f_2(y) + f_1(y) f_2^2(x) + f_1(x) f_2^2(y) \geq 0$$

or

$$\begin{aligned} & - f_1(y) f_2(x) [f_2(y) - f_2(x)] + f_1(x) + f_2(y) [f_2(y) - f_2(x)] \geq 0 \\ & \text{or} \\ & f_1(x) f_2(y) [f_2(y) - f_2(x)] - f_1(y) f_2(x) [f_1(y) - f_2(x)] \end{aligned}$$

and therefore

$$[f_2(y) - f_2(x)] [f_1(x) f_2(y) - f_1(y) f_2(x)] \geq 0 \quad (3)$$

And then, in order to have  $F, f_1$  concave and  $f_2$  convex, we have a necessary and sufficient condition is that (3) holds true in the domain  $[\varepsilon, 1]$  with  $\varepsilon > 0$ .

Then condition (3) says for this example

$$(y^2 - x^2)(xy^2 - yx^2) \geq 0$$

or

$$(y^2 - x^2)xy(y - x) \geq 0$$

For  $x < y$  then  $x^2 < y^2$  and the inequality holds true. For  $y < x$ , then  $y^2 < x^2$  and the inequality is valid again.

In this way we finish our presentation. For different ways of products and quotient we might relate the topic studied here with the product of two convex or concave functions as introduced in [3].

### Applications

Now we are going to apply the previous results to some interesting applications.

First we know that for  $f_1$  be convex and  $f_2$  be concave then  $F$  is convex. Now, if  $f_1$  and  $f_2$  satisfy

$$[f_2(y) - f_1(x)][f_1(x)f_2(y) - f_1(y)f_2(x)]$$

by Jensen inequality apply to every convex functions in particular to  $F$  that results

$$E\left(\frac{f_1(x)}{f_2(x)}\right) \geq F(E(X)) = \frac{f_1(E(X))}{f_2(E(X))}$$

where  $X$  is a random variable taking values to the domain of  $F$ . Here  $E$  denotes the mathematical expectation.

We are taking material from Wikipedia. Another consequence is

$$F(a) + F(b) = F(a + b)$$

or

$$\frac{f_1(x)}{f_2(x)} + \frac{f_1(y)}{f_2(y)} = \frac{f_1(x+y)}{f_2(x+y)}$$

Further some similar applications could be made with the first and second results. In particular we have

$$E(F(X)) \geq F(E(X))$$

or

$$E\left(\frac{1}{f(x)}\right) \geq E\left(\frac{1}{F(x)}\right)$$

An analogous results might be applied with the material of second place in the previous paragraph.

Next when  $F$  is convex and  $f$  is convex

$$F(tx) \leq tF(x) \text{ for } t \in (0,1)$$

wider hypotheses that be convex.

Something similar it is possible to derive for the next to the evens.



Now if  $f$  and  $g$  are convex then

$$m(x) = \max\{f(x), g(x)\}$$

is convex and

$$F = \frac{1}{\max\{f(x), g(x)\}} \text{ is convex.}$$

If  $f$  and  $g$  are convex functions and  $g$  is non-increasing then

$$h(x) = g(f(x))$$

is convex and

$$F = \frac{1}{h(x)} = \frac{1}{g(f(x))} \text{ is convex.}$$

On the other hand

$$e^{F(x)} = e^{1/f(x)}$$

is convex.

Now a new result. If  $f$  is concave and  $g$  convex then  $h(x)$  is convex and  $1/h(x) = 1/g(f(x))$  is convex. Convexity is invariant under affine maps, that is if  $f(x)$  is convex with  $x \in \mathbb{R}^n$  then so for  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$

$$g(x) = f(Ax + b)$$

is convex and

$$F(x) = \frac{1}{h(x)} = \frac{1}{f(Ax + b)}$$

is convex.

If  $f(x, y)$  is convex in  $x$  then  $g(x) = \sup_y \{f(x, y)\}$  is convex in  $x$  whose domain is convex and provided  $g(x) > -\infty$  for some  $x$ .

$$F(x) = \frac{1}{g(x)} = \frac{1}{\sup_y \{f(x, y)\}}$$

is convex too.

If  $f(x)$  is convex, then its perspective  $g(x, t) = t f(x/t)$  whose domain is  $\{(x, t) : t \in \text{Dom}(f), t > 0\}$  is convex. From here

$$F(x, t) = \frac{1}{t f(x/t)}$$

is convex in  $x$ .

In this way it is possible to give many other applications at least all those cases included in Wikipedia.

All this material can be studied in Banach spaces. A rather good application could be the Kachinroskii's theorem.

Final remark: This contribution allows to extend the optimization field to a wider class of functions to be developed in the future. This content and study was inspired by the reading and understanding of the transportation and traffic theory provided by the work of the authors all quoted in the bibliography.

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